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ON THE MODULAR FUNCTION j_4 OF LEVEL 4

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ABSTRACT. Since the modular curves $X(N) = \Gamma(N)\backslash\mathfrak{H}^*$ ($N = 1, 2, 3$) have genus 0, we have field isomorphisms $K(X(1)) \approx \mathbb{C}(J)$, $K(X(2)) \approx \mathbb{C}(\lambda)$ and $K(X(3)) \approx \mathbb{C}(j_3)$ where J, λ are the classical modular functions of level 1 and 2, and j_3 can be represented as the quotient of reduced Eisenstein series. When $N = 4$, we see from the genus formula that the curve $X(4)$ is of genus 0 too. Thus the field $K(X(4))$ is a rational function field over $\mathbb{C}$. We find such a field generator $j_4(z) = x(z)/y(z)$ ($x(z) = \theta_3(\frac{z}{2})$, $y(z) = \theta_4(\frac{z}{2})$ Jacobi theta functions). We also investigate the structures of the spaces $M_k(\Gamma(4))$, $S_k(\Gamma(4))$, $M_{\frac{k}{2}}(\widetilde{\Gamma}(4))$ and $S_{\frac{k}{2}}(\widetilde{\Gamma}(4))$ in terms of $x(z)$ and $y(z)$. As its application, we apply the above results to quadratic forms.

0. Introduction

Let $\mathfrak{H}$ be the complex upper half plane. Then $SL_2(\mathbb{Z})$ acts on $\mathfrak{H}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \tau = \frac{a\tau+b}{c\tau+d}$ for $\tau \in \mathfrak{H}$. Let $\Gamma(N)$ ($N = 1, 2, 3, \cdots$) be the principal congruence subgroups of $SL_2(\mathbb{Z})$ of level N and let $\mathfrak{H}^*$ be the union of $\mathfrak{H}$ and $\mathbb{P}^1(\mathbb{Q})$. The modular curve $\Gamma(N)\backslash\mathfrak{H}^*$ is a projective closure of smooth affine curve $\Gamma(N)\backslash\mathfrak{H}$, which we denote by $X(N)$, with genus g_N. We identify the function field $K(X(N))$ on the modular curve $X(N)$ with the field of modular functions of level N. By the genus formula ([11] Ch. IV §7, or [14] Proposition 1.40), the curves $X(1), X(2)$ and $X(3)$ have genus 0. Theoretically, we then have field isomorphisms $K(X(1)) \approx \mathbb{C}(J)$, $K(X(2)) \approx \mathbb{C}(\lambda)$ and $K(X(3)) \approx \mathbb{C}(j_3)$ where J, λ are the classical modular functions of level 1 and 2, respectively and j_3

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can be represented as the quotient of reduced Eisenstein series ([11] Ch. VII §1.2). Since the curve $X(4)$ is of genus 0 too, the field $K(X(4))$ is a rational function field over $\mathbb{C}$. In this case we shall find such a field generator j_4 (§2, Theorem 7) by means of theory of half integral modular forms. For generalities of half integral forms, we refer to [3] and [15].

In §1 we shall show, for later use, the generators and the cusps of the inhomogeneous group $\overline{\Gamma}(4)$. In §3 we shall investigate the generators of the spaces $M_k(\Gamma(4))$, $S_k(\Gamma(4))$, $M_{\frac{k}{2}}(\widetilde{\Gamma}(4))$ (the space of half integral modular forms of level 4) and $S_{\frac{k}{2}}(\widetilde{\Gamma}(4))$ (the space of half integral cusp forms of level 4) in terms of Jacobi theta functions. Also, we shall prove in Theorem 16 that the normalized field generator $N(j_4)(z)$ is an algebraic integer for $z \in \mathfrak{H} \cap \mathbb{Q}(\sqrt{-d})$ $(d > 0)$ (for notations, refer to [1]). In §4 we shall express j_4 as the quotient of reduced $\wp$-division values $\wp_{N,\vec{a}}{}^*$ where $\wp$ is the Weierstrass $\wp$-function. And we shall show in Theorem 18 that $\mathbb{Q}(j_4)$ is none other than the field of all the modular functions of level 4 whose Fourier expansions with respect to q_4 $(= e^{\pi i z/2})$ have rational coefficients.

In §5 we shall apply the result that $K(X(4))$ is equal to $\mathbb{C}(j_4)$ to quadratic forms. Let $Q(n,1)$ be the set of even unimodular positive definite integral quadratic forms in n-variables. For $A[X]$ in $Q(n,1)$, the theta series $\theta_A(z) = \sum_{X\in\mathbb{Z}^n} e^{\pi i z A[X]}$ $(z \in \mathfrak{H})$ is a modular form of weight $\frac{n}{2}$. If $n \geq 24$ and $A[X], B[X] \in Q(n,1)$, then the quotient $\frac{\theta_A(z)}{\theta_B(z)}$ is a modular function of level N. We shall extend the results in [5] to the case $N = 4$. In other words, since $\frac{\theta_A(z)}{\theta_B(z)}$ is also a modular function of level 4, we can write it as a rational function of j_4 (Theorem 21). In case $n = 24$, we shall be successful in §6 and Appendix B in completely determining the theta series $\theta_A(z)$ as symmetric polynomials over $\mathbb{Q}$ in $\theta_3(\frac{z}{2})$ and $\theta_4(\frac{z}{2})$ where θ_3, θ_4 are the Jacobi theta functions.

Through this article we adopt the following notations:

$\mathfrak{H}^*$ the extended complex upper half plane

$\Gamma(N) = \{\gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv I \mod N\}$

$\Gamma_0(N)$ the Hecke subgroup $\{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma(1) \mid c \equiv 0 \mod N\}$

$X(N) = \Gamma(N)\backslash\mathfrak{H}^*$

$X_0(N) = \Gamma_0(N)\backslash\mathfrak{H}^*$

$\overline{\Gamma}$ the inhomogeneous group of $\Gamma (= \Gamma/\pm I)$

$q_h = e^{2\pi i z/h}$, $z \in \mathfrak{H}$

$M_k(\Gamma(N))$ the space of modular forms of weight k with respect to

the group $\Gamma(N)$
$a \sim b$ means that a is equivalent to b
$z \to i\infty$ denotes that z goes to $i\infty$.

We shall always take the branch of the square root having argument in $(-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus, $\sqrt{z}$ is a holomorphic function on the complex plane with the negative real axis $(-\infty, 0]$ removed. For any integer k, we define $z^{\frac{k}{2}}$ to mean $(\sqrt{z})^k$.

1. Generators and cusps of $\overline{\Gamma}(4)$

Let Γ_1 and Γ_2 be two congruence subgroups of $\Gamma(1)$ such that $\Gamma_2 \subseteq \Gamma_1$. A subset $\mathcal{F}_1$ of the extended upper half plane $\mathfrak{H}^*$ is called a *fundamental set* for the group $\overline{\Gamma}_1$ if it contains exactly one representative of each class of points of $\mathfrak{H}^*$ equivalent under $\overline{\Gamma}_1$. A set $\mathcal{F}_1$ is called a *fundamental region* if $\mathcal{F}_1$ contains a fundamental set and if $z \in \mathcal{F}_1, \gamma z \in \mathcal{F}_1$ and $\gamma(\neq I) \in \overline{\Gamma}_1$ imply that z is a boundary point of $\mathcal{F}_1$.

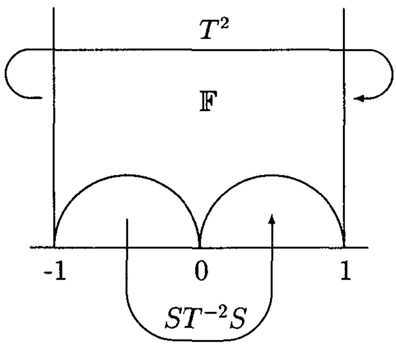
PROPOSITION 1. *If $\overline{\Gamma}_1 = \cup_{\nu=1}^{\mu} \overline{\Gamma}_2 \alpha_\nu$ is a coset decomposition of $\overline{\Gamma}_1$ and $\mathcal{F}_1$ is a fundamental region for $\overline{\Gamma}_1$, then $\mathcal{F}_2 = \cup_{\nu=1}^{\mu} \alpha_\nu(\mathcal{F}_1)$ is a fundamental region for $\overline{\Gamma}_2$.*

Proof. Theorem 2.3.5 [10]. □

THEOREM 2. *Let $\overline{\Gamma}_1$ be a congruence subgroup of $\overline{\Gamma}(1)$ of finite index and $\mathcal{F}$ be a fundamental region for $\overline{\Gamma}_1$. Then the sides of $\mathcal{F}$ can be grouped into pairs $\lambda_j, \lambda_j' (j = 1, 2, \ldots, s)$ in such a way that $\lambda_j \subseteq \mathcal{F}$ and $\lambda_j' = \gamma_j \lambda_j$ where $\gamma_j \in \overline{\Gamma}_1 (j = 1, 2, \ldots, s)$. γ_j's are called boundary substitutions of $\mathcal{F}$. Furthermore, $\overline{\Gamma}_1$ is generated by the boundary substitutions $\gamma_1, \ldots, \gamma_s$.*

Proof. For the first part, one is referred to [10], p. 58. For any $\gamma \in \overline{\Gamma}_1$, suppose there exists a sequence of images of $\mathcal{F}$; $\mathcal{F}, S_1\mathcal{F}, S_2\mathcal{F}, \ldots, S_n\mathcal{F} = \gamma\mathcal{F}$ $(S_j \in \overline{\Gamma}_1)$, each adjacent to its successor. Let $\mathcal{F} \cap S_1\mathcal{F} \supseteq \lambda_j'$. Since $\gamma_j\lambda_j = \lambda_j'$ and $\gamma_j\mathcal{F}$ is another fundamental region, $\gamma_j\mathcal{F} = S_1\mathcal{F}$, that is, $S_1 = \gamma_j$. Then, $\gamma_j\lambda_i, \gamma_j\lambda_i' (i = 1, 2, \ldots, s)$ form the sides of $S_1\mathcal{F}$. And $(\gamma_j\gamma_i\gamma_j^{-1})\gamma_j\lambda_i = \gamma_j\lambda_i'$, i.e., $\gamma_j\gamma_i\gamma_j^{-1} (i = 1, \ldots, s)$ are boundary substitutions of $S_1\mathcal{F}$. Now, we will use induction on n to show that $S_n(= \gamma)$ is generated by $\gamma_1, \ldots, \gamma_s$ and boundary substitutions are also generated by them. The case $n = 1$ has been done. Now, denote the sides of $S_{n-1}\mathcal{F}$ by μ_i, μ_i' $(i = 1, 2, \ldots, s)$. Let $L_i\mu_i = \mu_i'$ for $i = 1, \ldots, s$. Then, by induction hypothesis, S_{n-1} and L_i $(i = 1, \ldots, s)$ are generated by $\gamma_1, \ldots, \gamma_s$. If $S_{n-1}\mathcal{F} \cap S_n\mathcal{F} \supseteq \mu_j'$, then $L_j\mu_j = \mu_j'$ implies that $L_jS_{n-1}\mathcal{F} = S_n\mathcal{F}$, i.e., $S_n = L_jS_{n-1}$. Hence, it is generated by $\gamma_1, \ldots, \gamma_s$. Also, the set of all points in $\mathfrak{H}$ belonging to the region $S_n\mathcal{F}$ that can be reached by such sequences is open, and so also is its complement in $\mathfrak{H}$ which must therefore be empty by connectedness of $\mathfrak{H}$. This completes the proof of the theorem. □

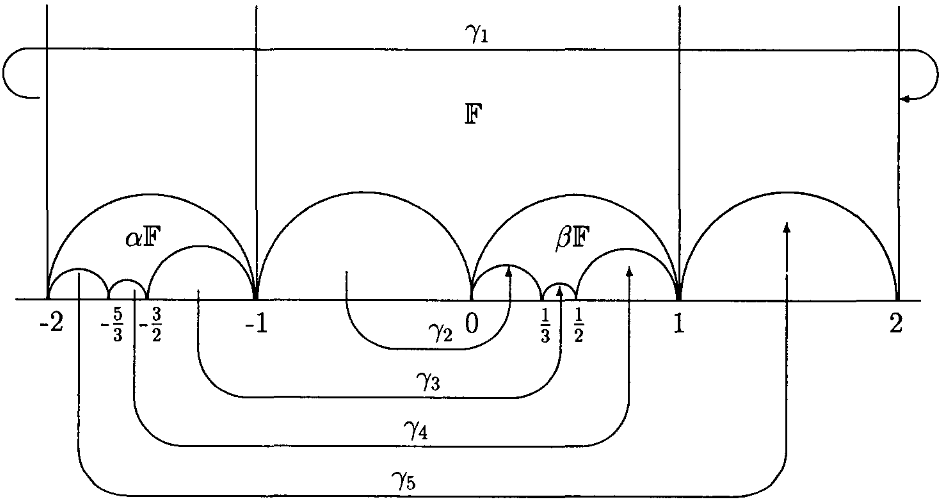
Now, we will find the generators of the group $\overline{\Gamma}(4)$ by means of Proposition 1 and Theorem 2. It is well known that the fundamental region for $\overline{\Gamma}(2)$ is given by the figure ([11], p. 84) where $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.



On the other hand, $\overline{\Gamma}(2)$ has the following right coset decomposition

$$\overline{\Gamma}(2) = \overline{\Gamma}(4) \cup \overline{\Gamma}(4)T^2 \cup \overline{\Gamma}(4)\alpha \cup \overline{\Gamma}(4)\beta$$

where $\alpha = ST^{-2}S$ and $\beta = (ST)^{-1}T^2ST$. Then Proposition 1 gives rise to the following fundamental region for $\overline{\Gamma}(4)$.



Note that $-2 \sim 2$, $-1 \sim \frac{1}{3}$, $-\frac{3}{2} \sim \frac{1}{2}$ and $-\frac{5}{3} \sim 1$ in $\Gamma(4)\backslash\mathfrak{H}^*$, which illustrates that there are six $\Gamma(4)$-inequivalent cusps $\infty, 0, 1, -1, -2, \frac{1}{2}$. Now, we will choose appropriate elements from $\Gamma(4)$ which describe the above equivalences. The proof of Lemma 1.41 in [14] provides the idea of explicit construction of them. Based on it, one can have

$$\begin{array}{ll} \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right) \cdot 0 = 0 & \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right) \cdot (-1) = 1/3 \\ \left(\begin{smallmatrix} -3 & -4 \\ -8 & -11 \end{smallmatrix}\right) \cdot (-1) = 1/3 & \left(\begin{smallmatrix} -3 & -4 \\ -8 & -11 \end{smallmatrix}\right) \cdot (-3/2) = 1/2 \\ \left(\begin{smallmatrix} 5 & 8 \\ 8 & 13 \end{smallmatrix}\right) \cdot (-3/2) = 1/2 & \left(\begin{smallmatrix} 5 & 8 \\ 8 & 13 \end{smallmatrix}\right) \cdot (-5/3) = 1 \\ \left(\begin{smallmatrix} -7 & -12 \\ -4 & -7 \end{smallmatrix}\right) \cdot (-5/3) = 1 & \left(\begin{smallmatrix} -7 & -12 \\ -4 & -7 \end{smallmatrix}\right) \cdot (-2) = 2\ . \end{array}$$

Now, put $\gamma_1 = \left(\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}\right)$, $\gamma_2 = \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$, $\gamma_3 = \left(\begin{smallmatrix} -3 & -4 \\ -8 & -11 \end{smallmatrix}\right)$, $\gamma_4 = \left(\begin{smallmatrix} 5 & 8 \\ 8 & 13 \end{smallmatrix}\right)$, and $\gamma_5 = \left(\begin{smallmatrix} -7 & -12 \\ -4 & -7 \end{smallmatrix}\right)$. Then, as described in the above figure, γ_i sends boundaries to boundaries for $i = 1, \ldots, 5$ because a linear fractional transformation maps a semicircle to a semicircle.

For the sake of convenience in use, we will express γ_i's as a combination of S and T^2. Obviously, $\gamma_1 = T^4$. Now, consider the case of γ_2. $\gamma_2\infty = \frac{1}{4}$, $S(\gamma_2\infty) = -4$, $T^4S\gamma_2\infty = 0$. By computing $T^4S\gamma_2$, one gets $T^4S\gamma_2 = S$. Hence $\gamma_2 = S^{-1}T^{-4}S$.
Next, consider the case of γ_3. $\gamma_3\infty = \frac{3}{8}$, $S\gamma_3\infty = -\frac{8}{3}$, $T^2S\gamma_3\infty = -\frac{2}{3}$, $ST^2S\gamma_3\infty = \frac{3}{2}$, $T^{-2}ST^2S\gamma_3\infty = -\frac{1}{2}$, $ST^{-2}ST^2S\gamma_3\infty = 2$,

$T^{-2}ST^{-2}ST^2S\ \gamma_3\infty = 0,\ ST^{-2}ST^{-2}ST^2S\ \gamma_3\ \infty = \infty$. By computing $ST^{-2}ST^{-2}ST^2S\gamma_3$, one gets $ST^{-2}ST^{-2}ST^2S\gamma_3 = T^2$. Hence,

$$\begin{aligned}\gamma_3 &= S^{-1}T^{-2}S^{-1}T^2S^{-1}T^2S^{-1}T^2\\ &= ST^{-2}ST^2ST^2ST^2 \quad \text{since } S^{-1} = S.\end{aligned}$$

By a similar computation, one has

$$\begin{aligned}\gamma_4 &= ST^{-2}ST^{-2}ST^2ST^2\\ \gamma_5 &= T^2S^{-1}T^4ST^2.\end{aligned}$$

2. Hauptfunktionen of level 4 as a quotient of Jacobi theta functions

For $\mu, \nu \in \mathbb{R}$ and $z \in \mathfrak{H}$, put

$$\Theta_{\mu,\nu}(z) = \sum_{n\in\mathbb{Z}} \exp\left\{\pi i\left(n + \frac{1}{2}\mu\right)^2 z + \pi i n\nu\right\}.$$

This series uniformly converges for $\mathrm{Im}(z) \geq \eta > 0$, and hence defines a holomorphic function on $\mathfrak{H}$.

THEOREM 3. *If $z \in \mathfrak{H}$, then* $\Theta_{\mu,\nu}(z) = \frac{e^{-\frac{1}{2}\pi i\mu\nu}}{(-iz)^{\frac{1}{2}}}\Theta_{\nu,-\mu}(-1/z)$.

Proof. Theorem 7.1.1 [10]. □

We recall the Jacobi theta functions $\theta_2, \theta_3, \theta_4$ defined by

$$\begin{aligned}\theta_2(z) &:= \Theta_{1,0}(z) = \sum_{n\in\mathbb{Z}} q_2^{\left(n+\frac{1}{2}\right)^2}\\ \theta_3(z) &:= \Theta_{0,0}(z) = \sum_{n\in\mathbb{Z}} q_2^{n^2}\\ \theta_4(z) &:= \Theta_{0,1}(z) = \sum_{n\in\mathbb{Z}} (-1)^n q_2^{n^2}.\end{aligned}$$

Then we have the following transformation formulas.

THEOREM 4. *For all* $z \in \mathfrak{H}$,

$$(i)\quad \theta_2(z+1) = e^{\frac{1}{4}\pi i}\theta_2(z) \qquad (ii)\quad \theta_2(-1/z) = (-iz)^{\frac{1}{2}}\theta_4(z)$$
$$\theta_3(z+1) = \theta_4(z) \qquad \theta_3(-1/z) = (-iz)^{\frac{1}{2}}\theta_3(z)$$
$$\theta_4(z+1) = \theta_3(z) \qquad \theta_4(-1/z) = (-iz)^{\frac{1}{2}}\theta_2(z).$$

Proof. Theorem 7.1.2 [10]. □

Let $x(z) = \theta_3(\frac{z}{2})$ and $y(z) = \theta_4(\frac{z}{2})$. We then readily have the transformation formulas using the above theorem.

COROLLARY 5. *For all* $z \in \mathfrak{H}$,

$$(i)\quad \theta_2(z+4) = -\theta_2(z) \qquad (ii)\quad \theta_2(-2/z) = (-iz/2)^{\frac{1}{2}}y(z)$$
$$x(z+2) = y(z) \qquad x(-1/z) = (-2iz)^{\frac{1}{2}}x(4z)$$
$$x(z+4) = x(z) \qquad x(-4/z) = (-iz/2)^{\frac{1}{2}}x(z)$$
$$y(z+2) = x(z),\ y(z+4) = y(z) \qquad y(-1/z) = (-2iz)^{\frac{1}{2}}\theta_2(2z).$$

THEOREM 6. $x(z), y(z) \in M_{\frac{1}{2}}(\widetilde{\Gamma}(4))$.

Proof. First, we will show the slash operator invariance by making use of the idea from [3], p. 148. For $\gamma' \in \Gamma_0(4)$ and $z \in \mathfrak{H}$,

$$\Theta(\gamma' z) = j(\gamma', z)\Theta(z) \tag{2.1}$$

where $\Theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2} (q = q_1)$ and $j(\gamma', z)$ is the automorphy factor for $\Gamma_0(4)$. Then $x(z) = \Theta(\frac{z}{4})$ for any $z \in \mathfrak{H}$.
For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(4)$, put $\gamma' = \begin{pmatrix} a & \frac{b}{4} \\ 4c & d \end{pmatrix}$. Note that $\gamma' \in \Gamma_0(4)$ and $\gamma' \cdot \frac{z}{4} = \frac{\gamma z}{4}$. Then $x(\gamma z) = \Theta(\frac{\gamma z}{4}) = \Theta(\gamma' \cdot \frac{z}{4})$ and, by (2.1),

$$\begin{aligned}\Theta\left(\gamma' \cdot \frac{z}{4}\right) &= j\left(\gamma', \frac{z}{4}\right)\Theta\left(\frac{4}{z}\right)\\ &= \left(\frac{4c}{d}\right)\varepsilon_d^{-1}\sqrt{4c\cdot\frac{z}{4} + d}\cdot x(z)\\ &= \left(\frac{c}{d}\right)\sqrt{cz+d}\cdot x(z) \text{ since } d \equiv 1 \bmod 4\\ &= j(\gamma, z)x(z).\end{aligned}$$

This implies

$$x(\gamma z) = j(\gamma, z)x(z), \tag{2.2}$$

which means that $x|_{[\gamma]_{\frac{1}{2}}} = x(z)$ for any $\gamma \in \Gamma(4)$. For the case $y(z)$, put $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ as usual. Then by Corollary 5, $y(z) = x(z+2) = x(T^2 z)$. Since $\Gamma(4)$ is a normal subgroup of $\Gamma(1)$, one has $T^{-2}\Gamma(4)T^2 = \Gamma(4)$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(4)$, put $\gamma' = T^2\gamma T^{-2} = \left(\begin{smallmatrix} * & * \\ c & d-2c \end{smallmatrix}\right) \in \Gamma(4)$. Then,

$$\begin{aligned} y(\gamma z) &= x(T^2\gamma z) \\ &= x(T^2(T^{-2}\gamma' T^2)z) = x(\gamma'(T^2 z)) \\ &= j(\gamma', T^2 z)x(T^2 z) \text{ by (2.2)} \\ &= \left(\frac{c}{d-2c}\right)\sqrt{c(z+2)+d-2c}\cdot y(z) \\ &= \left(\frac{c}{d-2c}\right)\sqrt{cz+d}\cdot y(z). \end{aligned}$$

To get the identity $y(\gamma z) = j(\gamma, z)y(z)$, it remains to check that

$$\left(\frac{c}{d-2c}\right) = \left(\frac{c}{d}\right) \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(4). \tag{2.3}$$

Write $c = (-1)^{sgn(c)}2^n \cdot c'$ where c' is not divisible by 2 and $c' > 0$. Since $d - 2c \equiv d \bmod 8$, we have $\left(\frac{-1}{d-2c}\right) = \left(\frac{-1}{d}\right)$ and $\left(\frac{2}{d-2c}\right) = \left(\frac{2}{d}\right)$. Thus it suffices to show $\left(\frac{c'}{d-2c}\right) = \left(\frac{c'}{d}\right)$. From the generalized quadratic reciprocity law ([3], p. 153), we recall that $\left(\frac{d}{c}\right) = (-1)^{\frac{c-1}{2}\cdot\frac{d-1}{2}}\left(\frac{c}{d}\right)$ if c or d is positive. Indeed, since $d - 2c \equiv 1 \bmod 4$ implies $\left(\frac{c'}{d-2c}\right) = \left(\frac{d-2c}{c'}\right)$ $= \left(\frac{d}{c'}\right) = \left(\frac{c'}{d}\right)$. Next, we check the cusp conditions. We saw in §1 that there are six $\Gamma(4)$-inequivalent cusps $\infty, 0, 1, -1, -2, \frac{1}{2}$.

(i) $s = \infty$:

From the definitions of $x(z)$ and $y(z)$

$$\begin{aligned} x(z) &= \sum_{n\in\mathbb{Z}} q_4^{n^2} = 1 + 2q_4 + 2q_4^4 + 2q_4^9 + \dots \\ y(z) &= \sum_{n\in\mathbb{Z}} (-1)^n q_4^{n^2} = 1 - 2q_4 + 2q_4^4 - 2q_4^9 + \dots, \end{aligned}$$

and so $x(\infty) = y(\infty) = 1$.

(ii) $s = 0$:

Take $\xi = (S, \sqrt{z})$ with $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Observe that $\xi\infty = 0$. Then

$$\begin{aligned} x|_{[\xi]_{\frac{1}{2}}} &= x(Sz)\sqrt{z}^{\,-1} \\ &= (-2iz)^{\frac{1}{2}} z^{-\frac{1}{2}} x(4z) \text{ by Corollary 5} \\ &= (-2i)^{\frac{1}{2}} x(4z) \end{aligned}$$

so that we conclude

$$x(0) = \lim_{z \to i\infty} x|_{[\xi]_{\frac{1}{2}}} = (-2i)^{\frac{1}{2}}.$$

Similarly

$$\begin{aligned} y|_{[\xi]_{\frac{1}{2}}} &= y(Sz)\sqrt{z}^{\,-1} \\ &= (-2iz)^{\frac{1}{2}} z^{-\frac{1}{2}} \theta_2(2z) \text{ by Corollary 5} \\ &= (-2i)^{\frac{1}{2}} \sum_{n\in\mathbb{Z}} q_4^{4(n+\frac{1}{2})^2} \\ &= (-2i)^{\frac{1}{2}} \sum_{n\in\mathbb{Z}} q_4^{(2n+1)^2} \\ &= (-2i)^{\frac{1}{2}} (2q_4 + 2q_4^9 + 2q_4^{25} + \dots), \end{aligned}$$

hence y has a zero of order 1 at 0.

(iii) $s = 1$:

Take $\xi = (ST^{-1}S, \sqrt{-z-1})$. Then $\xi\infty = 1$ and $x|_{[\xi]_{\frac{1}{2}}} = x(ST^{-1}Sz) \cdot \sqrt{-z-1}^{\,-1}$. It follow from Corollary 5 that $x(Sz) = (-2iz)^{\frac{1}{2}} x(4z)$, $x(ST^{-1}z) = (-2iz+2i)^{\frac{1}{2}} x(4z)$, and $x(ST^{-1}Sz) = (2i\frac{1}{z} + 2i)^{\frac{1}{2}} (-i\frac{z}{2})^{\frac{1}{2}} x(z) = (1+z)^{\frac{1}{2}} x(z)$. Hence, $x|_{[\xi]_{\frac{1}{2}}} = (1+z)^{\frac{1}{2}} i^{-1}\ (1+z)^{-\frac{1}{2}} x(z)$. As $z \to i\infty$, we have that $x(1) = -i$. On the other hand, we have $y|_{[\xi]_{\frac{1}{2}}} = y(ST^{-1}Sz)\sqrt{-z-1}^{\,-1}$. Meanwhile, we know again by Corollary 5 that $y(Sz) = (-2iz)^{\frac{1}{2}} \theta_2(2z)$, $y(ST^{-1}z) = (-2iz+2i)^{\frac{1}{2}} \theta_2(2z-2) = (-2iz + 2i)^{\frac{1}{2}} (-i)\theta_2(2z)$, and $y(ST^{-1}Sz) = (-i)(1+z)^{\frac{1}{2}} y(z)$. Thus we come up with $y|_{[\xi]_{\frac{1}{2}}} = (-i)(1+z)^{\frac{1}{2}} i^{-1} (1+z)^{-\frac{1}{2}} y(z)$. As $z \to i\infty$, $y(1) = -1$.

(iv) $s = -1$:

Take $\xi = (STS, \sqrt{z-1})$. Then $\xi\infty = -1$ and $x(STz) = (-2iz - 2i)^{\frac{1}{2}} x(4z)$, $x(STSz) = (2i\frac{1}{z} - 2i)^{\frac{1}{2}} (-i\frac{z}{2})^{\frac{1}{2}} x(z) = (1-z)^{\frac{1}{2}} x(z)$. Therefore, $x|_{[\xi]_{\frac{1}{2}}} = i^{-1} x(z)$. As $z \to i\infty$, $x(-1) = -i$. Similarly, $y(STz) = (-2iz -$

$2i)^{\frac{1}{2}} i\theta_2(2z)$, and $y(STSz) = i(1-z)^{\frac{1}{2}} y(z)$. Hence, $y|_{[\xi]_{\frac{1}{2}}} = y(z)$. As $z \to i\infty$, $y(-1) = 1$.

(v) $s = -2$:

Take $\xi = (T^{-2}S, \sqrt{z})$. Then $\xi\infty = -2$ and $x(T^{-2}z) = x(z-2) = y(z)$, $x(T^{-2}Sz) = y(-\frac{1}{z}) = (-2iz)^{\frac{1}{2}}\theta_2(2z)$. Therefore, $x|_{[\xi]_{\frac{1}{2}}} = x(T^{-2}Sz)z^{-\frac{1}{2}}$ $= (-2iz)^{\frac{1}{2}} z^{-\frac{1}{2}}\theta_2(2z) = (-2i)^{\frac{1}{2}}\ \theta_2(2z) = (-2i)^{\frac{1}{2}}(2q_4 + 2q_4^9 + 2q_4^{25} + \dots)$. It then follows that x has a zero of order 1 at -2. In a similar way, $y(T^{-2}z) = y(z-2) = x(z)$ and $y(T^{-2}Sz) = x(-\frac{1}{z}) = (-2iz)^{\frac{1}{2}} x(4z)$. This yields that $y|_{[\xi]_{\frac{1}{2}}} = (-2i)^{\frac{1}{2}} x(4z)$. As $z \to i\infty$, we have $y(-2) = (-2i)^{\frac{1}{2}}$.

(vi) $s = \frac{1}{2}$:

Take $\xi = (ST^{-2}S, \sqrt{-2z-1})$. Then $x(ST^{-2}z) = (-2iz+4i)^{\frac{1}{2}} x(4z)$ and $x(ST^{-2}S\ z) = (2i\frac{1}{z}+4i)^{\frac{1}{2}}\ (-i\frac{z}{2})^{\frac{1}{2}} x(z) = (1+2z)^{\frac{1}{2}} x(z)$; hence $x|_{[\xi]_{\frac{1}{2}}} = i^{-1}x(z)$. As $z \to i\infty$, $x(\frac{1}{2}) = -i$. In like manner, $y(ST^{-2}z) = (-2iz + 4i)^{\frac{1}{2}}\theta_2(2z-4) = (-2iz+4i)^{\frac{1}{2}}(-1)\theta_2(2z)$, $y(ST^{-2}Sz) = -(1+2z)^{\frac{1}{2}} y(z)$. Therefore $y|_{[\xi]_{\frac{1}{2}}} = -i^{-1}y(z)$. As $z \to i\infty$, we have $y(\frac{1}{2}) = i$. □

Put

$$\begin{aligned} j_4(z) &= \frac{x(z)}{y(z)} \\ &= 1 + 4q_4 + 8{q_4}^2 + 16{q_4}^3 + 32{q_4}^4 + 56{q_4}^5 + 96{q_4}^6 + 160{q_4}^7 + \cdots . \end{aligned}$$

THEOREM 7. *$K(X(4)) = \mathbb{C}(j_4)$ and j_4 has the following value at each cusp: $j_4(\infty) = 1$, $j_4(0) = \infty$ (a simple pole), $j_4(1) = i$, $j_4(-1) = -i$, $j_4(-2) = 0$ (a simple zero), $j_4(\frac{1}{2}) = -1$.*

Proof. First, we claim that for $f(z) \in M_{\frac{1}{2}}(\widetilde{\Gamma}(4))$, $f^2(z) \in M_1(\Gamma(4))$. In fact, if $\gamma \in \Gamma(4)$ then we have $f|_{[\widetilde{\gamma}]_{\frac{1}{2}}} = f(z)$. This is equivalent to $f(\gamma z) = f(z)j(\gamma, z)$, that is, $f(\gamma z) = f(z)\left(\frac{c}{d}\right)\varepsilon_d^{-1}\sqrt{cz+d} = f(z)\left(\frac{c}{d}\right)$ $\sqrt{cz+d}$ since $d \equiv 1 \bmod 4$. Squaring both sides, we have $f^2(\gamma z) = f^2(z)\cdot(cz+d)$ for any $\gamma \in \Gamma(4)$. Therefore $f^2 \in M_1(\Gamma(4))$. Thus by Theorem 6 $x^2(z), y^2(z) \in M_1(\Gamma(4))$. Meanwhile, we saw in the proof of Theorem 6 that each of $x(z)$ and $y(z)$ has a simple zero at only one cusp. Observe that for $f \in M_k(\Gamma(N))$, the sum of zeros is $\nu_0(f) = \frac{\mu_N \cdot k}{12}$ where $\mu_N = [\overline{\Gamma} : \overline{\Gamma}(N)]$. It then follows that $\nu_0(x^2) = \nu_0(y^2) = \frac{\mu_4 \cdot 1}{12} = 2$. Since x^2 and y^2 already have a zero of order 2 at cusps, they have no zero in $\mathfrak{H}$. This asserts that $\deg(j_4)_0 = 1$, and hence $[K(X(4)) :$

$\mathbb{C}(j_4)]$ $=\deg(j_4)_0=1$. The second part is immediate by definition and Theorem 6. □

PROPOSITION 8. *The cusps of* $\Gamma(4)$ *are regular in the sense of half integral weight forms. (for definitions and notations, refer to* [3], Ch. IV)

Proof. We know that if $f(z) \in M_{\frac{k}{2}}(\widetilde{\Gamma}(4))$, then $f(s) = 0$ for a k-irregular cusp s. Since $x(z)$ and $y(z)$ belong to $M_{\frac{1}{2}}(\widetilde{\Gamma}(4))$, if a 1-irregular cusp s exists then we must have $x(s) = y(s) = 0$. We saw, however, in the proof of Theorem 6 that such a cusp does not exist. □

Alternative proof of Proposition 8. At ∞, we readily see that $\xi = 1$, $h = 1$ and $t = 1$. At 0, take $\xi = ((\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right), \sqrt{z})$ so that $\xi^{-1} = ((\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right), -i\sqrt{z})$. We need $\widetilde{\Gamma}(4) \ni \xi\,((\left(\begin{smallmatrix}1&h\\0&1\end{smallmatrix}\right), t)\xi^{-1} = ((\left(\begin{smallmatrix}1&0\\-h&1\end{smallmatrix}\right), -it\sqrt{hz-1})$, which is valid when $h = 4$ and $t = 1$.
At the cusp 1, take $\alpha = \left(\begin{smallmatrix}-1&0\\-1&-1\end{smallmatrix}\right)$ and $\xi = ((\left(\begin{smallmatrix}-1&0\\-1&-1\end{smallmatrix}\right), \sqrt{-z-1})$ so that $\xi^{-1} = ((\left(\begin{smallmatrix}-1&0\\1&-1\end{smallmatrix}\right), \sqrt{z-1})$. One must choose $h = 4$ to obtain $\alpha\left(\begin{smallmatrix}1&h\\0&1\end{smallmatrix}\right)\alpha^{-1} = \left(\begin{smallmatrix}1-h&h\\-h&1+h\end{smallmatrix}\right) \in \Gamma(4)$. To find t we compute $\xi((\left(\begin{smallmatrix}1&4\\0&1\end{smallmatrix}\right), t)\xi^{-1} = ((\left(\begin{smallmatrix}-3&4\\-4&5\end{smallmatrix}\right), t\sqrt{-4z+5})$, which implies $j((\left(\begin{smallmatrix}-3&4\\-4&5\end{smallmatrix}\right), z) = t\sqrt{-4z+5}$ provided that $t = 1$. Therefore 1 is regular.
At the cusp -1, take $\alpha = \left(\begin{smallmatrix}-1&0\\1&-1\end{smallmatrix}\right)$ and $\xi = ((\left(\begin{smallmatrix}-1&0\\1&-1\end{smallmatrix}\right), \sqrt{z-1})$ so that $\xi^{-1} = ((\left(\begin{smallmatrix}-1&0\\-1&-1\end{smallmatrix}\right), \sqrt{-z-1})$. To get $\alpha\left(\begin{smallmatrix}1&h\\0&1\end{smallmatrix}\right)\alpha^{-1} = \left(\begin{smallmatrix}1+h&h\\-h&1-h\end{smallmatrix}\right) \in \Gamma(4)$, one must take $h = 4$. For t, compute $\xi((\left(\begin{smallmatrix}1&4\\0&1\end{smallmatrix}\right), t)\xi^{-1} = ((\left(\begin{smallmatrix}5&4\\-4&-3\end{smallmatrix}\right), t\sqrt{-4z-3})$, which gives $j((\left(\begin{smallmatrix}5&4\\-4&-3\end{smallmatrix}\right), z) = t\sqrt{-4z-3}$ provided that $t = 1$. Thus -1 is regular.
At the cusp -2, take $\alpha = \left(\begin{smallmatrix}-2&-1\\1&0\end{smallmatrix}\right)$ and $\xi = ((\left(\begin{smallmatrix}-2&-1\\1&0\end{smallmatrix}\right), \sqrt{z})$; hence $\xi^{-1} = ((\left(\begin{smallmatrix}0&1\\-1&-2\end{smallmatrix}\right), \sqrt{-z-2})$. To have $\alpha\left(\begin{smallmatrix}1&h\\0&1\end{smallmatrix}\right)\alpha^{-1} = \left(\begin{smallmatrix}1+2h&4h\\-h&1-2h\end{smallmatrix}\right) \in \Gamma(4)$, one is to take $h = 4$. For t, compute $\xi((\left(\begin{smallmatrix}1&4\\0&1\end{smallmatrix}\right), t)\xi^{-1} = ((\left(\begin{smallmatrix}9&16\\-4&-7\end{smallmatrix}\right), t\sqrt{-4z-7})$, which implies $j((\left(\begin{smallmatrix}9&16\\-4&-7\end{smallmatrix}\right), z) = t\sqrt{-4z-7}$ provided that $t = 1$. Hence -2 is regular.
Finally at the cusp $\frac{1}{2}$, take $\alpha = \left(\begin{smallmatrix}-1&0\\-2&-1\end{smallmatrix}\right)$ and $\xi = ((\left(\begin{smallmatrix}-1&0\\-2&-1\end{smallmatrix}\right), \sqrt{-2z-1})$ so that $\xi^{-1} = ((\left(\begin{smallmatrix}-1&0\\2&-1\end{smallmatrix}\right), \sqrt{2z-1})$. To have $\alpha\left(\begin{smallmatrix}1&h\\0&1\end{smallmatrix}\right)\alpha^{-1} = \left(\begin{smallmatrix}1-2h&h\\-4h&1+2h\end{smallmatrix}\right) \in \Gamma(4)$, again one choose $h = 4$. To find t, compute $\xi((\left(\begin{smallmatrix}1&4\\0&1\end{smallmatrix}\right), t)\xi^{-1} = ((\left(\begin{smallmatrix}-7&4\\-16&9\end{smallmatrix}\right), t\sqrt{-16z+9})$. This gives $j((\left(\begin{smallmatrix}-7&4\\-16&9\end{smallmatrix}\right), z) = t\sqrt{-16z+9}$ when $t = 1$, which amounts to say that $\frac{1}{2}$ is regular.

3. Structures of $M_k(\Gamma(4))$ and $S_k(\Gamma(4))$

We recall from [10] and [14] the following facts:

FACT 1. *For $k \geq 2$ and Γ' a congruence subgroup of $\Gamma(1)$, we have*

$$\dim M_k(\Gamma') = \begin{cases} g + \sigma_\infty(\Gamma') - 1 & (k = 2) \\ (k-1)(g-1) + \frac{k}{2} \cdot \sigma_\infty(\Gamma') + \sum_{i=1}^{r} \left[\frac{k(e_i - 1)}{2e_i}\right] & (k \text{ even}) \\ (k-1)(g-1) + \frac{uk}{2} + \frac{u'(k-1)}{2} + \sum_{i=1}^{r} \left[\frac{k(e_i-1)}{2e_i}\right] & (k \text{ odd}, -1 \notin \Gamma') \end{cases}$$

where g is the genus of $\Gamma' \backslash \mathfrak{H}^*$, $\sigma_\infty(\Gamma')$ the number of Γ'-inequivalent cusps, $e_1, \ldots, e_r$ the orders of inequivalent elliptic elements of Γ' and u (resp. u') the number of inequivalent regular (resp. irregular) cusps of Γ'.

$$\dim S_k(\Gamma') = \begin{cases} \dim M_k(\Gamma') - \sigma_\infty(\Gamma') & \text{if } k > 2 \\ g & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

For $k = 1$ and $\Gamma' = \Gamma(N)$,

$$\dim M_1(\Gamma(N)) = \frac{\mu_N}{2N} \quad \text{with } \mu_N = [\overline{\Gamma}(1) : \overline{\Gamma}(N)], \quad \text{if } u \geq 2g - 2$$

$$\dim S_1(\Gamma(N)) = 0 \quad \text{for } 3 \leq N \leq 11.$$

FACT 2. *Let $X(z) = \theta_2{}^4(z)$, $Y(z) = \theta_3{}^4(z)$ and $\lambda(z) = \frac{X(z)}{Y(z)}$. Then $X, Y \in M_2(\Gamma(2))$ and $K(X(2)) = \mathbb{C}(\lambda)$.*

THEOREM 9. (i) *For $k \geq 1$, $\dim M_{2k}(\Gamma(2)) = k+1$ and $\dim S_{2k}(\Gamma(2)) = k - 2$ if $k \geq 3$.*

(ii) *$M_{2k}(\Gamma(2))$ is spanned over $\mathbb{C}$ by $k+1$ functions $X^k, X^{k-1}Y, \ldots, Y^k$.*

(iii) *$S_{2k}(\Gamma(2))$ is spanned by $k-2$ functions $\Delta_2 X^{k-3}, \Delta_2 X^{k-2} Y, \ldots, \Delta_2 Y^{k-3}$ where $\Delta_2 = XY(X - Y) \in S_6(\Gamma(2))$ and $k \geq 3$.*

Proof. If $\Gamma' = \Gamma(2)$, we have $g = 0, \sigma_\infty = 3$, $e_i = 0$ for all i, $u = 3$, and $\mu_2 = 6$. Then (i) follows from Fact 1. Now, consider (iii). Note that $\lambda(\infty) = 0$, $\lambda(1) = \infty$, and $\lambda(0) = 1$ imply that Δ_2 is a cusp form. For any $f \in M_6(\Gamma(2))$, the number of zeros of f is

$$\nu_0(f) = \frac{\mu_2 \cdot 6}{12} = \frac{6 \cdot 6}{12} = 3. \tag{3.1}$$

Since Δ_2 is a cusp form, $\nu_0(\Delta_2) \geq 3$. But, it follows by (3.1) that $\nu_0(\Delta_2) = 3$. Also, all zeros of Δ_2 appear at the cusps, which means that $\Delta_2(z) \neq 0$ on $\mathfrak{H}$. Observe that each function stated in (iii) is in $S_{2k}(\Gamma(2))$ and the cardinality is the same as dim $S_{2k}(\Gamma(2))$. Therefore it is necessary to check their independency to justify (iii). Suppose that

$$\sum_{i=0}^{k-3} c_i \Delta_2 X^{k-3-i} Y^i = 0 \text{ for } c_i \in \mathbb{C}.$$

Since $\Delta_2 Y^{k-3}$ never vanishes in $\mathfrak{H}$, dividing the above by $\Delta_2 Y^{k-3}$, we have

$$\sum_{i=0}^{k-3} c_i \lambda^{k-3-i} = 0.$$

Since λ is transcendental over $\mathbb{C}$, $c_i = 0$ for all i. (ii) can be proved in a similar fashion. □

THEOREM 10. (i) *dim* $M_k(\Gamma(4)) = 2k+1$ *for* $k \geq 1$, *dim* $S_k(\Gamma(4)) = 2k-5$ *for* $k \geq 3$.

(ii) $M_k(\Gamma(4))$ *is spanned over* $\mathbb{C}$ *by the functions* $x^{2k}, x^{2k-1}y, \ldots, y^{2k}$.

(iii) *Let* $\Delta_4 = xy(x^4 - y^4)$. *Then* $\Delta_4 \in S_3(\Gamma(4))$ *and for* $k \geq 3$, $S_k(\Gamma(4))$ *is spanned by* $\Delta_4 x^{2k-6}, \Delta_4 x^{2k-7}y, \ldots, \Delta_4 y^{2k-6}$.

Proof. If $\Gamma' = \Gamma(4)$, we have $g = 0$, $\sigma_\infty = 6$, $e_i = 0$ for all i, $u = 6$, and $\mu_4 = 24$. Then (i) is immediate by Fact 1. We consider (iii) because (ii) can be handled in a similar way. By Theorem 6, the functions mentioned in (iii) belong to $M_k(\Gamma(4))$. Since $y(0) = 0$ and $x(-2) = 0$, $\Delta_4(0) = \Delta_4(-2) = 0$. And

$$\frac{\Delta_4}{y^6} = j_4(j_4^4 - 1). \tag{3.2}$$

If $s \neq 0, -2$ then $j_4(s)$ is a 4-th root of unity. Also, for $s \neq 0$, $y(s) \neq 0$. Hence, by (3.2), Δ_4 is a cusp form. For any $f \in M_3(\Gamma(4))$, the number of zeros is

$$\nu_0(f) = \frac{\mu_4 \cdot 3}{12} = 6. \tag{3.3}$$

Since Δ_4 is a cusp form, $\nu_0(\Delta_4) \geq 6$. But, by (3.3), $\nu_0(\Delta_4) = 6$ so that Δ_4 never vanishes on $\mathfrak{H}$. Now, all functions in (iii) are in $S_k(\Gamma(4))$. It remains to check that they are linearly independent because the cardinality is equal to the dimension of $S_k(\Gamma(4))$. Suppose that

$$\sum_{i=0}^{2k-6} c_i \Delta_4 x^{2k-6-i} y^i = 0 \text{ for } c_i \in \mathbb{C}.$$

Since $\Delta_4 y^{2k-6}$ never vanishes on $\mathfrak{H}$, dividing the above by $\Delta_4 y^{2k-6}$, we have

$$\sum_{i=0}^{2k-6} c_i j_4^{2k-6-i} = 0. \tag{3.4}$$

Here we have to show that j_4 is transcendental over $\mathbb{C}$. Choose any $c \in \mathbb{C}$ and consider $j_4 - c$. Since $j_4 - c$ is a nonconstant modular function, it has at least one zero. This implies that the image of j_4 is all of $\mathbb{C}$. But if we had an algebraic equation satisfied by j_4, then the image of j_4 would be mapped into the set of solutions of the algebraic equation which is at most a finite set. This is impossible. Therefore $c_i = 0$ for all i in (3.4). □

REMARK. For any $\frac{k}{2} \in \mathbb{N}$, $M_{\frac{k}{2}}(\widetilde{\Gamma}(4)) = M_{\frac{k}{2}}(\Gamma(4))$. Indeed, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(4)$,

$$j(\gamma, z) = \left(\frac{c}{d}\right)\sqrt{cz+d} \quad \text{since } d \equiv 1 \bmod 4.$$

Since k is even, $j(\gamma, z)^k = (cz+d)^{\frac{k}{2}}$, that is, $M_{\frac{k}{2}}(\widetilde{\Gamma}(4))$ has the same automorphy factor as that of $M_{\frac{k}{2}}(\Gamma(4))$.

Before going further we will show the algebraic independency of $x(z)$ and $y(z)$. To this end, we need the following lemma.

LEMMA 11. *If $f_k + f_{k-1} + \cdots + f_0 = 0$ where $k \in \mathbb{N}$ and $f_i \in M_{\frac{i}{2}}(\widetilde{\Gamma}(4))$ for $i = 0, \ldots, k$, then $f_i = 0$ for all $i = 0, \ldots, k$.*

Proof. Fix an arbitrary point $z \in \mathfrak{H}$. Put $\gamma_i = \left(\begin{smallmatrix} 1 & 0 \\ 4i & 1 \end{smallmatrix}\right)$ for $i = 0, \ldots, k+1$. Then $j(\gamma_i, z) = \left(\frac{4i}{1}\right)\sqrt{4iz+1} = \sqrt{4iz+1}$ are distinct. By the assumption,

$$f_k(\gamma_i z) + f_{k-1}(\gamma_i z) + \cdots + f_0(\gamma_i z) = 0.$$

Since $f_i \in M_{\frac{i}{2}}(\widetilde{\Gamma}(4))$ for $i = 0, \ldots, k$, we have

$$j(\gamma_i, z)^k f_k(z) + j(\gamma_i, z)^{k-1} f_{k-1}(z) + \cdots + f_0(z) = 0$$

for $i = 0, \ldots, k+1$. This gives rise to the following linear system

$$\begin{pmatrix} j(\gamma_1, z)^k & \cdots & j(\gamma_1, z) & 1 \\ j(\gamma_2, z)^k & \cdots & j(\gamma_2, z) & 1 \\ \vdots & \ddots & \vdots & \vdots \\ j(\gamma_{k+1}, z)^k & \cdots & j(\gamma_{k+1}, z) & 1 \end{pmatrix} \begin{pmatrix} f_k(z) \\ f_{k-1}(z) \\ \vdots \\ f_0(z) \end{pmatrix} = 0.$$

Note that the determinant of the above system is the well-known Vandermonde determinant, which is nonzero because $j(\gamma_i, z)$'s are all distinct. Hence, $f_i(z) = 0$ for each i. Since z is arbitrary, $f_i = 0$ for any i. □

Now, suppose that there exists a polynomial $F \in \mathbb{C}[X_1, X_2]$ which is satisfied by $x(z)$ and $y(z)$. By Theorem 6 and Lemma 11, we may assume that F is homogeneous. Let $\deg F = n$. Then,

$$\frac{F(x,y)}{y^n} = \sum_{k=0}^{n} a_k {j_4}^k = 0$$

for $a_k \in \mathbb{C}$. Since j_4 is transcendental over $\mathbb{C}$, it follows that $a_k = 0$ for any k; hence $F = 0$. This guarantees the algebraic independency of x and y.

THEOREM 12.

$$X(z) = \theta_2(z)^4 = \frac{1}{4}(x^4 - 2x^2y^2 + y^4)$$
$$Y(z) = \theta_3(z)^4 = \frac{1}{4}(x^4 + 2x^2y^2 + y^4).$$

Proof. Note that ∞ is equivalent to $\frac{1}{2}$, $1 \sim -1$ and $0 \sim -2$ in the curve $\Gamma(2)\backslash\mathfrak{H}^*$. Thus ${\theta_2}^4(\infty) = 0$ implies ${\theta_2}^4(\frac{1}{2}) = 0$. Also, ${\theta_3}^4(1) = 0$ implies ${\theta_3}^4(-1) = 0$. Considering the values of x and y at the cusps, we obtain

$$(x^4 - 2x^2y^2 + y^4)(\infty) = 0 \qquad (x^4 - 2x^2y^2 + y^4)(\frac{1}{2}) = 0$$
$$(x^4 - 2x^2y^2 + y^4)(1) = 0 \qquad (x^4 - 2x^2y^2 + y^4)(-1) = 0.$$

Let us recall that for $f \in M_2(\Gamma(4))$, the number of zeros is

$$\nu_0(f) = \frac{\mu_4 \cdot 2}{12} = 4. \tag{3.5}$$

Since ${\theta_2}^4$ (resp. ${\theta_3}^4$) has a zero of order 1 at ∞ (resp. at 1) in q_2 expansion, it has a zero of order 2 in q_4 expansion. Meanwhile, (3.5) shows that $\nu_0({\theta_2}^4) = \nu_0({\theta_3}^4) = 4$. Hence it turns out that they have no other zeros except those mentioned above. On the other hand, it follows from the equality $(x^4 \pm 2x^2y^2 + y^4) = (x^2 \pm y^2)^2$ that they have zeros of even order. Again by (3.5), they have no other zeros except those. Therefore $\dfrac{{\theta_2}^4}{x^4 - 2x^2y^2 + y^4}$ has no zeros and no poles, which claims that the quotient is a constant. We use the transformation formula for θ_2 in Theorem 4 and Theorem 6 to get that ${\theta_2}^4(0) = -1$ and $(x^4 - 2x^2y^2 + y^4)(0) = ((-2i)^{\frac{1}{2}})^4 = -4$. Hence, the constant should be $\frac{1}{4}$. Likewise, we can show the other case. □

THEOREM 13 (Extended Version of Theorem 10). (i) *For* $k \geq 1$, *dim* $M_{\frac{k}{2}}(\widetilde{\Gamma}(4)) = k+1$ *and* $M_{\frac{k}{2}}(\widetilde{\Gamma}(4))$ *is spanned by* $x^k, x^{k-1}y, \ldots, y^k$, *that is, it is the space of all polynomials in* $\mathbb{C}[x,y]$ *having pure weight* $\frac{k}{2}$.

(ii) *For* $k \geq 6$, *dim* $S_{\frac{k}{2}}(\widetilde{\Gamma}(4)) = k-5$ *and* $S_{\frac{k}{2}}(\widetilde{\Gamma}(4))$ *is generated by* $\Delta_4 x^{k-6}$, $\Delta_4 x^{k-7}y, \ldots, \Delta_4 y^{k-6}$ *with* Δ_4 *as in* Theorem 10.

Proof. For (i), it is enough to consider the case $\frac{k}{2} \notin \mathbb{N}$. Note that $x^k, x^{k-1}y, \ldots, y^k$ are linearly independent and belong to $M_{\frac{k}{2}}(\widetilde{\Gamma}(4))$ due to Theorem 6. Let $\alpha \in M_{\frac{k}{2}}(\widetilde{\Gamma}(4))$. Then, $\alpha \cdot x \in M_{\frac{k+1}{2}}(\widetilde{\Gamma}(4))$. Since $\frac{k+1}{2} \in \mathbb{N}$, by Theorem 10, we obtain

$$\alpha \cdot x = c_0 x^{k+1} + c_1 x^k y + \cdots + c_{k+1} y^{k+1} \tag{3.6}$$

for $c_i \in \mathbb{C}$. Now, evaluate the above at the cusp $s = -2$. Then $x(-2) = 0$ and $y(-2) \neq 0$ give $c_{k+1} = 0$. Since $x(z) \neq 0$ on $\mathfrak{H}$, we can divide the both sides in (3.6) by x. Then $\alpha \in \mathbb{C}x^k + \cdots + \mathbb{C}y^k$, from which (i) follows. (ii) can be similarly proved. The only nontrivial part is that $\Delta_4 x^{k-6}, \Delta_4 x^{k-7}y, \ldots, \Delta_4 y^{k-6}$ span $S_{\frac{k}{2}}(\widetilde{\Gamma}(4))$. Let $\beta \in S_{\frac{k}{2}}(\widetilde{\Gamma}(4))$. Then $\beta \cdot x \in M_{\frac{k+1}{2}}(\widetilde{\Gamma}(4))$. Since $\frac{k+1}{2}$ is an integer, it turns out that

$$\beta \cdot x = c_0 \Delta_4 x^{k-5} + \cdots + c_{k-5} \Delta_4 y^{k-5} \tag{3.7}$$

for $c_i \in \mathbb{C}$. Comparing the order of zero at -2, we see that all terms except $c_{k-5}\Delta_4 y^{2k-6}$ have the orders greater than or equal to 2. But the term $c_{k-5}\Delta_4 y^{k-5}$ has the order 1 at -2, which forces us to have $c_{k-5} = 0$. Dividing the both sides of (3.7) by x, we come up with $\beta \in \mathbb{C}\Delta_4 x^{k-6} + \cdots + \mathbb{C}\Delta_4 y^{k-6}$. □

EXAMPLE. Define

$$\Theta(z) := \sum_{n\in\mathbb{Z}} q^{n^2} \quad (z \in \mathfrak{H}).$$

Then $\Theta \in M_{\frac{1}{2}}(\widetilde{\Gamma}_0(4))$ ([3], p. 184). Hence, $\Theta \in M_{\frac{1}{2}}(\widetilde{\Gamma}(4))$ and, by Theorem 13, it can be written as a linear combination of x and y

$$\Theta = ax + by \tag{3.8}$$

for some $a, b \in \mathbb{C}$. Observe that $\Theta(\infty) = 1$ and $\Theta(\frac{1}{2}) = 0$ because $\frac{1}{2}$ is a 1-irregular cusp of $\Gamma_0(4)$. Evaluating (3.8) at the cusps ∞ and $\frac{1}{2}$, we get $a = b = \frac{1}{2}$. Therefore the result is

$$\Theta = \frac{1}{2}x + \frac{1}{2}y.$$

Before closing this section we try to find the relations between j_4 and the classical modular functions J and λ.

THEOREM 14. (i) *We have*

$$\lambda = \frac{{j_4}^4 - 2{j_4}^2 + 1}{{j_4}^4 + 2{j_4}^2 + 1}$$

and the irreducible polynomial of j_4 *is* $Z^4 + 2\frac{\lambda+1}{\lambda-1}Z^2 + 1 \in \mathbb{C}(\lambda)[Z]$ *over* $\mathbb{C}(\lambda)(= K(X(2)))$.

(ii) *Let* J *be the classical modular function of level* 1 *with* $J(i) = 1$. *Then one has*

$$J = \frac{1}{108}\frac{({j_4}^8 + 14{j_4}^4 + 1)^3}{({j_4}^5 - j_4)^4}$$

and the irreducible polynomial of j_4 *over* $\mathbb{C}(J)$ *is* $(Z^8 + 14Z^4 + 1)^3 - 108J(Z^5 - Z)^4 \in \mathbb{C}(J)\ [Z]$.

Proof. In (i), the equality of λ follows from Theorem 12. Observe that

$$[K(X(4)) : K(X(2))] = [\overline{\Gamma}(2) : \overline{\Gamma}(4)] = 4.$$

Hence, $\deg(\mathrm{Irr}(j_4, \mathbb{C}(\lambda))) = 4$. Clearly, j_4 satisfies $Z^4 + 2\frac{\lambda+1}{\lambda-1}Z^2 + 1$. Thus, the two polynomials are the same. Since

$$J = \frac{4}{27}\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$$

([10], p. 228), plugging (i) into the above we come up with the equality in (ii). Now, $K(X(1)) = \mathbb{C}(J)$, $K(X(4)) = \mathbb{C}(j_4)$ and

$$[K(X(4)) : K(X(1))] = [\overline{\Gamma}(1) : \overline{\Gamma}(4)] = 24.$$

And $\deg(\mathrm{Irr}(j_4, \mathbb{C}(J))) = 24$. By the same reason as in (i), the second part of (ii) follows. □

For $z \in \mathbb{Q}(\sqrt{-d}) \cap \mathfrak{H}$ $(d > 0)$, it is well-known that $j(z)$ $(= 1728J(z))$ is an algebraic integer ([6], [14]). For algebraic proofs, see [2], [7], [13] and [16]. Therefore it is natural to ask whether $j_4(z)$ is so or not. Although we have Theorem 14 at hand, the answer for the above question seems to be negative because the modular function $J(z)$ has no Fourier expansion of the form $q^{-1}(1 + \sum_{n\geq 1} a_n q^n)$. To support the above claim, let us find a counter example as follows. Observe that

$$\theta_2(2z) = \frac{1}{2}\left(\theta_3\left(\frac{z}{2}\right) - \theta_4\left(\frac{z}{2}\right)\right), \tag{3.9}$$

$$\theta_3(2z) = \frac{1}{2}\left(\theta_3\left(\frac{z}{2}\right) + \theta_4\left(\frac{z}{2}\right)\right). \tag{3.10}$$

LEMMA 15. (i) *For* $x \in \mathbb{R}_+$, $j_4(xi) > 0$.
(ii) *For* $z \in \mathfrak{H}$, $j_4(2z)^2 = \frac{1}{2}(j_4(z) + j_4(z)^{-1})$.
(iii) $j_4(\frac{i}{2^n}) = \dfrac{j_4(2^n i) + 1}{j_4(2^n i) - 1}$ *for* $n \in \mathbb{N} \cup \{0\}$.
(iv) $j_4(2z)^4 = \frac{1}{1-\lambda(z)}$.

Proof. It follows from the definition that $\theta_3(\frac{xi}{2}) = \sum_{n\in\mathbb{Z}} e^{\pi i(\frac{xi}{2})n^2} = \sum_{n\in\mathbb{Z}} e^{\frac{-\pi x n^2}{2}} > 0$. And by Theorem 4 (ii) and (3.9), $\theta_4\left(\frac{xi}{2}\right) = \theta_4\left(-\frac{x}{2i}\right) = \left(-i\frac{2i}{x}\right)^{\frac{1}{2}} \theta_2\left(\frac{2i}{x}\right) = \sqrt{\frac{2}{x}}\,\frac{1}{2}\left(\theta_3\left(\frac{i}{2x}\right) - \theta_4\left(\frac{i}{2x}\right)\right) > 0$. This implies (i). For the second, we readily get that

$$\begin{aligned} j_4(2z)^2 &= \frac{\theta_3(z)^2}{\theta_4(z)^2} = \frac{\theta_3(\frac{z}{2})^2 + \theta_4(\frac{z}{2})^2}{2\,\theta_3(\frac{z}{2})\,\theta_4(\frac{z}{2})} \quad \text{by [10], Theorem 7.1.8} \\ &= \frac{1}{2}\,(j_4(z) + j_4(z)^{-1}). \end{aligned}$$

Finally, for $n \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} j_4\left(\frac{i}{2^n}\right) &= \frac{\theta_3(\frac{i}{2^{n+1}})}{\theta_4(\frac{i}{2^{n+1}})} = \frac{\theta_3(2^{n+1}i)}{\theta_2(2^{n+1}i)} \quad \text{by Theorem 4 (ii)} \\ &= \frac{\theta_3(2^{n-1}i) + \theta_4(2^{n-1}i)}{\theta_3(2^{n-1}i) - \theta_4(2^{n-1}i)} \quad \text{by (3.9) and (3.10)} \\ &= \frac{j_4(2^n i) + 1}{j_4(2^n i) - 1}. \end{aligned}$$

Also, $j_4(2z)^4 = \frac{\theta_3(z)^4}{\theta_4(z)^4} = \frac{\theta_3(z)^4}{\theta_3(z)^4-\theta_2(z)^4} = \frac{1}{1-\lambda(z)}$. This completes the lemma. □

In Lemma 15 (iii), let us take $n = 0$. Then we come up with $j_4(i) = 1 \pm \sqrt{2}$. By Lemma 15 (i), $j_4(i) > 0$ and so

$$j_4(i) = 1 + \sqrt{2}. \tag{3.11}$$

Applying again Lemma 15 (i) and (ii) we obtain that $j_4(2i) = \sqrt[4]{2}$ and $j_4(4i) = \sqrt{\frac{\sqrt[4]{2}+\sqrt[4]{2}^{-1}}{2}}$. We claim at this stage that $j_4(4i)$ cannot be an algebraic integer. Suppose that $j_4(4i)$ belongs to the ring $\mathfrak{O}$ of algebraic integers. Then

$$\frac{\sqrt{\sqrt{2}+1}}{\sqrt[8]{32}} \in \mathfrak{O}, \text{ which implies } \frac{1}{\sqrt[8]{32}} \in \mathfrak{O} \text{ because } \sqrt{\sqrt{2}+1} \in \mathfrak{O}^\times.$$

We conclude from the above that

$$\left(\frac{1}{\sqrt[8]{32}}\right)^8 = \frac{1}{32} \in \mathfrak{O},$$

which is a contradiction. Therefore $j_4(4i)$ is not an algebraic integer. In order to overcome this obstacle we borrow the notion of normalized series from Conway-Norton's paper ([1]).

THEOREM 16. *Let* $N(j_4)(z) = \frac{4}{j_4(z)-1}+2 = \frac{1}{q_4}+0+2{q_4}^3-{q_4}^7-2{q_4}^{11}+3{q_4}^{15}+2{q_4}^{19}+\cdots$ *be the normalized generator of* $K(X(4))$. *Then for* $\tau \in \mathbb{Q}(\sqrt{-d}) \cap \mathfrak{H}$, $N(j_4)(\tau)$ *is an algebraic integer.*

Proof. Let j be the modular function whose Fourier expansion with respect to q is $\frac{1}{q} + 744 + 196884q + \cdots$. Then $j(\tau)$ is an algebraic integer for such τ. Note that $J = \frac{j}{1728}$. By Theorem 14, we see that $J = \frac{1}{108}\frac{({j_4}^8+14{j_4}^4+1)^3}{({j_4}^5-j_4)^4}$. Hence, substituting $\frac{4}{N-2}+1$ for j_4, we obtain that

$$\begin{aligned} j &= 2^4 \cdot \frac{({j_4}^8+14{j_4}^4+1)^3}{({j_4}^5-j_4)^4} \\ &= \frac{(N^8+224N^4+256)^3}{(N-2)^4N^4(N^3+2N^2+4N+8)^2} \quad \text{where} \quad N = N(j_4)(\tau). \end{aligned}$$

This implies that $N(j_4)(\tau)$ is integral over $\mathbb{Z}[j(\tau)]$ and hence $N(j_4)(\tau)$ is integral over $\mathbb{Z}$. □

4. Examples

Theorem 13 implies that any f in $M_{\frac{k}{2}}(\widetilde{\Gamma}(4))$ is a homogeneous polynomial in x and y whose degree is k. Furthermore the polynomial expression is unique due to the algebraic independency of x and y. In this section we will describe the modular function j_4 in terms of $\wp$-division values and Fricke functions. First, we recall the definition of the N-*th division values* of $\wp$:

$$\wp_{N,\vec{a}}(\tau) := \wp\left(\frac{a_1\tau + a_2}{N}; L_\tau\right)$$

where $\vec{a} = \left(\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right)$, $L_\tau = \mathbb{Z}\tau + \mathbb{Z}$ and $\wp$ is the Weierstrass $\wp$-function. Since $\wp(u;L) = \wp(v;L)$ if and only if $u \equiv \pm v \mod L$, we see that

$$\wp_{N,\vec{a}} = \wp_{N,\vec{b}} \Leftrightarrow \vec{a} \equiv \vec{b} \mod N\mathbb{Z}^2. \tag{4.1}$$

Now, define the *reduced* $\wp$-*division value* $\wp_{N,\vec{a}}{}^*$ by

$$\wp_{N,\vec{a}}{}^* := \sum_{t \bmod N} \sum_{d>0,\ dt\equiv 1 \bmod N} \frac{\mu(d)}{d^2}\wp_{N,t\vec{a}} \tag{4.2}$$

where $\vec{a}$ runs mod N with $(a_1, a_2) = 1$ and μ is the Möbius function. Then we have $\wp_{N,\vec{a}}{}^* \in M_2(\Gamma(N))$ and at the cusps of $\Gamma(N)$

$$\wp_{N,\vec{a}}{}^*\left(-\tfrac{d}{c}\right) = \begin{cases} N^2 - \frac{N^2}{\sigma_\times(N)} & \text{if } -\frac{d}{c} \text{ is } \Gamma(N)\text{-equivalent to } -\frac{a_2}{a_1}, \\ -\frac{N^2}{\sigma_\times(N)} & \text{otherwise.} \end{cases} \tag{4.3}$$

For the standard facts mentioned above, we refer to [11], p. 171. By theorem 13, $\wp_{4,\vec{a}}{}^*$ is a homogeneous polynomial in x and y of degree 4. Thus we can write $\wp_{4,\vec{a}}{}^*$ as follows:

$$\wp_{4,\vec{a}}{}^* = c_0x^4 + c_1x^3y + c_2x^2y^2 + c_3xy^3 + c_4y^4$$

for $c_i \in \mathbb{C}$. Using (4.3) and the values of x and y at the cusps of $\Gamma(4)$ (see Theorem 6), we can determine the coefficients c_i. In fact,

$$\begin{aligned}
c_0 &= -\frac{1}{4}\wp_{4,\vec{a}}{}^*(0)\\
c_1 &= \frac{1}{4}\wp_{4,\vec{a}}{}^*(\infty) - \frac{1}{4}\wp_{4,\vec{a}}{}^*\left(\tfrac{1}{2}\right) - \frac{1}{4}i\wp_{4,\vec{a}}{}^*(-1) + \frac{1}{4}i\wp_{4,\vec{a}}{}^*(1)\\
c_2 &= \frac{1}{2}\wp_{4,\vec{a}}{}^*(\infty) + \frac{1}{2}\wp_{4,\vec{a}}{}^*\left(\tfrac{1}{2}\right) + \frac{1}{4}\wp_{4,\vec{a}}{}^*(0) + \frac{1}{4}\wp_{4,\vec{a}}{}^*(-2)\\
c_3 &= \frac{1}{4}\wp_{4,\vec{a}}{}^*(\infty) - \frac{1}{4}\wp_{4,\vec{a}}{}^*\left(\tfrac{1}{2}\right) + \frac{1}{4}i\wp_{4,\vec{a}}{}^*(-1) - \frac{1}{4}i\wp_{4,\vec{a}}{}^*(1)\\
c_4 &= -\frac{1}{4}\wp_{4,\vec{a}}{}^*(-2)
\end{aligned}$$

with $i = \sqrt{-1}$. Recall that there are 6 distinct reduced $\wp$-division values which correspond to the cusps of $\Gamma(4)$. They are as follows:

$$\begin{aligned}
s &= \infty, & \wp_{4,\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}{}^* &= \frac{2}{3}x^4 + 4x^3y + 4x^2y^2 + 4xy^3 + \frac{2}{3}y^4\\
s &= 0, & \wp_{4,\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}{}^* &= -\frac{10}{3}x^4 + \frac{2}{3}y^4\\
s &= 1, & \wp_{4,\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)}{}^* &= \frac{2}{3}x^4 + 4ix^3y - 4x^2y^2 - 4ixy^3 + \frac{2}{3}y^4\\
s &= -1, & \wp_{4,\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)}{}^* &= \frac{2}{3}x^4 - 4ix^3y - 4x^2y^2 + 4ixy^3 + \frac{2}{3}y^4\\
s &= -2, & \wp_{4,\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)}{}^* &= \frac{2}{3}x^4 - \frac{10}{3}y^4\\
s &= \frac{1}{2}, & \wp_{4,\left(\begin{smallmatrix}-2\\1\end{smallmatrix}\right)}{}^* &= \frac{2}{3}x^4 - 4x^3y + 4x^2y^2 - 4xy^3 + \frac{2}{3}y^4.
\end{aligned}$$

Using the above result, we get

$$\begin{aligned}
\frac{\wp_{4,\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}{}^* - \wp_{4,\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}{}^*}{\wp_{4,\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)}{}^* - \wp_{4,\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}{}^*} &= \frac{-4x^4 - 4x^3y - 4x^2y^2 - 4xy^3}{-4x^3y - 4x^2y^2 - 4xy^3 - 4y^4}\\
&= \frac{-4x(x^3 + x^2y + xy^2 + y^3)}{-4y(x^3 + x^2y + xy^2 + y^3)}\\
&= \frac{x}{y} = j_4.
\end{aligned}$$

In this way, one can have a field generator of $K(X(4))$ in terms of reduced $\wp$-division values.

REMARK. (Generation of j_4 with Fricke functions) Recall the definition of Fricke function f_{a_1,a_2} where $(a_1,a_2)\in\mathbb{Z}^2$ and both a_1 and a_2 are not multiple of N ([6], [14]). Then,

$$f_{a_1,a_2} = -2^7\cdot 3^5\frac{G_4G_6}{\Delta}\wp_{N,\vec{a}} \text{ with } \vec{a}=\left(\begin{smallmatrix}a_1\\a_2\end{smallmatrix}\right).$$

From the equality $j_4=\dfrac{\wp_{4,\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}^{*}-\wp_{4,\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}^{*}}{\wp_{4,\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)}^{*}-\wp_{4,\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}^{*}}$ and (4.2), it follows that

$$j_4=\frac{\sum_{t \bmod 4}\left(\sum_{d>0,dt\equiv 1 \bmod 4}\frac{\mu(d)}{d^2}\right)(f_{t,0}-f_{0,t})}{\sum_{t \bmod 4}\left(\sum_{d>0,dt\equiv 1 \bmod 4}\frac{\mu(d)}{d^2}\right)(f_{t,2t}-f_{0,t})}. \tag{4.4}$$

In the above, consider the summation $\sum_{d>0,\ dt\equiv 1 \bmod 4}\frac{\mu(d)}{d^2}$. Note that when $t=0,2$ there is no d satisfying the congruence equation $dt\equiv 1 \bmod 4$. Now put $a=\sum_{d>0,\ dt\equiv 1 \bmod 4}\frac{\mu(d)}{d^2}$ and $b=\sum_{d>0,\ dt\equiv 3 \bmod 4}\frac{\mu(d)}{d^2}$. Then in (4.4),

$$\begin{aligned} j_4 &= \frac{a\,(f_{1,0}-f_{0,1})+b\,(f_{3,0}-f_{0,3})}{a\,(f_{1,2}-f_{0,1})+b\,(f_{3,6}-f_{0,3})}\\ &= \frac{a\,(f_{1,0}-f_{0,1})+b\,(f_{1,0}-f_{0,1})}{a\,(f_{1,2}-f_{0,1})+b\,(f_{1,2}-f_{0,1})} \quad \text{by (4.1)}\\ &= \frac{f_{1,0}-f_{0,1}}{f_{1,2}-f_{0,1}}. \end{aligned}$$

LEMMA 17. *For n even, let $f\in M_{\frac{n}{2}}(\Gamma(4))$. If f has a Fourier expansion with rational coefficients, then it can be written as a homogeneous polynomial over $\mathbb{Q}$ in x and y whose degree is n.*

Proof. By Theorem 10,

$$f=\sum_{j=0}^{n} a_j x^{n-j}y^j,\quad a_j\in\mathbb{C}. \tag{4.5}$$

We must show that each a_j lies in $\mathbb{Q}$. Considering Fourier expansions of f and $x^{n-j}y^j$ gives

$$f=\sum_{i=0}^{\infty} b_i q_4^i,\quad b_i\in\mathbb{Q}$$

$$x^{n-j}y^j=\sum_{i=0}^{\infty} c_{ij} q_4^i,\quad c_{ij}\in\mathbb{Q}.$$

Plugging (4.5) into the above and comparing the coefficients of q_4-expansion, we get the following linear system:

$$(4.6) \qquad \left(c_{ij}\right)_{i\geq 0,\ 0\leq j\leq n} \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = \left(b_i\right)_{i\geq 0}.$$

Note that the j-th column of the matrix $\left(c_{ij}\right)$ corresponds to the Fourier coefficients of $x^{n-j}y^j$. Since $x^n, x^{n-1}y, \ldots, y^n$ are linearly independent over $\mathbb{C}$, the matrix $\left(c_{ij}\right)$ has rank $n+1$. This allows us to choose $n+1$ rows from $\left(c_{ij}\right)$ which are linearly independent. Without loss of generality, we may assume that the matrix $\left(c_{ij}\right)_{0\leq i,j\leq n}$ is invertible. Now, instead of (4.6), consider the following system:

$$\left(c_{ij}\right)_{0\leq i,j\leq n} \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_0 \\ \vdots \\ b_n \end{pmatrix}.$$

Multiplying through by the inverse of the matrix $\left(c_{ij}\right)_{0\leq i,j\leq n}$, we have $a_0, \ldots, a_n \in \mathbb{Q}$ as desired. □

THEOREM 18. *$\mathbb{Q}(j_4)$ coincides with the field $\mathcal{F}_4$ of all the modular functions of level 4 whose Fourier expansions with respect to q_4 have rational coefficients.*

Proof. By [14], Proposition 6.9 we know that $\mathcal{F}_4 = \mathbb{Q}(J(z), J(4z), f_{1,0}(z))$. Since x and y have rational Fourier coefficients, so also has j_4. Hence, $\mathbb{Q}(j_4)$ is contained in $\mathcal{F}_4$. For the reverse inclusion we need to show that $J(z), J(4z), f_{1,0}(z) \in \mathbb{Q}(j_4)$. From Theorem 14 (ii) we see that $J(z) \in \mathbb{Q}({j_4}^4(z))$, and so $J(z) \in \mathbb{Q}(j_4)$. Next, observe that $J(4z) \in \mathbb{Q}({j_4}^4(4z)) = \mathbb{Q}(\frac{x^4(4z)}{y^4(4z)})$. To claim $J(4z) \in \mathbb{Q}(j_4)$, it is enough to show that $x^4(4z)$ and $y^4(4z)$ are homogeneous polynomials over $\mathbb{Q}$ in x and y of degree 4. By an example in §3, we obtain $\frac{1}{2}(x+y) = \theta_3(2z)$. Simple calculation leads us to $\frac{1}{2}(x-y) = \theta_2(2z)$. Therefore, $x^4(4z) = {\theta_3}^4(2z) = \frac{1}{2}(x+y)^4$ and $y^4(4z) = {\theta_4}^4(2z) = {\theta_3}^4(2z) - {\theta_2}^4(2z) = \frac{1}{2}(x+y)^4 - \frac{1}{2}(x-y)^4$. Finally, we consider the Fricke function $f_{1,0}(z)$. Recall that $f_{1,0} = -2^7 \cdot 3^5 \frac{G_4 G_6}{\Delta} \wp_{4,\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}$. As is shown in the proof of [14], Proposition 6.9 or [11], pp. 169-170, $\pi^{-2}\wp_{4,\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}$ has rational Fourier coefficients. On the other hand $\pi^{-4}G_4, \pi^{-6}G_6$ and $\pi^{-12}\Delta$ have the same property. Furthermore, they can be viewed as modular forms of level 4. Thus, by Lemma 17, they can be written as homogeneous polynomials over $\mathbb{Q}$ in x and y. This implies that $f_{1,0}(z) \in \mathbb{Q}(j_4)$. □

5. Application to quadratic forms

LEMMA 19. (i) *Let $f \in M_{2k}(\Gamma(1))$. Then f is a symmetric homogeneous polynomial over $\mathbb{C}$ in $x^4(z)$ and $y^4(z)$ whose degree is k.*

(ii) *Let $g \in M_{2k}(\Gamma(2))$. Then g is a symmetric homogeneous polynomial in $x^2(z)$ and $y^2(z)$ whose degree is $2k$.*

Proof. By Theorem 9 and 10,

$$f(z) = p_1(X(z), Y(z)) = p_2(x(z), y(z)) \tag{5.1}$$

where p_1 and p_2 are homogeneous polynomials in two variables with deg $p_1 = k$ and deg $p_2 = 4k$. We claim that p_1 and p_2 are symmetric. In fact,

$$\begin{aligned} p_1(X, Y) &= f = f|_{[STS]_{2k}} \quad \text{since } f \in M_{2k}(\Gamma(1)) \\ &= p_1(X|_{[STS]_2}, Y|_{[STS]_2}) = p_1(Y, X) \quad \text{by Theorem 4}. \end{aligned}$$

Also,

$$\begin{aligned} p_2(x, y) &= f = f|_{[T^2]_{2k}} \text{ since } f \in M_{2k}(\Gamma(1)) \\ &= p_2(x(z+2), y(z+2)) = p_2(y, x) \text{ by Corollary 5}. \end{aligned}$$

Recall from Theorem 12 that $X = \frac{1}{4}(x^2 - y^2)^2$ and $Y = \frac{1}{4}(x^2 + y^2)^2$. Substituting x for x and $-y$ for y we see that X and Y are unchanged. This implies by (5.1) that $p_2(x, -y) = p_2(x, y)$, that is, p_2 involves terms whose degree of y is even. Also, substituting x for x and iy for y, X and Y interchange with each other. Since p_1 is symmetric, by (5.1) we have $p_2(x, iy) = p_2(x, y)$, i.e., p_2 has terms whose degree of y is a multiple of 4. In the case of (ii), p_2 is symmetric and the equality $p_2(x, -y) = p_2(x, y)$ still holds. The assertion follows from these facts. □

For $p(x) \in \mathbb{C}[x]$, we call $p(x)$ *symmetric* if $p(x) = x^k p(\frac{1}{x})$ with $k =$ deg $p(x)$.

COROLLARY 20. (i) *Let $f_1, f_2 \in M_{2k}(\Gamma(1))$. Then,*

$$\frac{f_1(z)}{f_2(z)} = \frac{p({j_4}^4(z))}{q({j_4}^4(z))}$$

where p and q are symmetric polynomials in one variable whose degrees are less than or equal to k.

(ii) *Let* $g_1, g_2 \in M_{2k}(\Gamma(2))$. *Then,*

$$\frac{g_1(z)}{g_2(z)} = \frac{p({j_4}^2(z))}{q({j_4}^2(z))}$$

where p and q are symmetric polynomials of degree less than or equal to $2k$.

Proof. Obvious. □

Now, we will consider the theta series associated to quardratic forms. Let $Q(n,1)$ be the set of even unimodular positive definite integral quadratic forms in n-variables. Then $n \equiv 0 \bmod 8$ ([12], ch.V). For $A[X]$ in $Q(n,1)$, the theta series defined by

$$\theta_A(z) = \sum_{X \in \mathbb{Z}^n} e^{\pi i z A[X]} \quad (z \in \mathfrak{H})$$

is a modular form of weight $\frac{n}{2}$ and level 1. In cases $n = 8$ and 16, the quotients $\frac{\theta_A}{\theta_B}$ are 1 for $A[X], B[X] \in Q(n,1)$. If $n \geq 24$, then we have the following theorem.

THEOREM 21. *For any two quadratic forms* $A[X], B[X] \in Q(n,1)$,

$$\frac{\theta_A(z)}{\theta_B(z)} = \frac{p(j_4(z))}{q(j_4(z))}$$

where p and q are symmetric polynomials over $\mathbb{Q}$ *in* j_4 *of degree* n.

Proof. From Lemma 17 and Lemma 19 we see that θ_A and θ_B are symmetric homogeneous polynomials over $\mathbb{Q}$ in $x(z)$ and $y(z)$ whose degree is n. In both cases the coefficients of the term x^n do not vanish because $\theta_A(0) = \theta_B(0) = 1$, $x(0) \neq 0$ and $y(0) = 0$ by Appendix A. Now the result follows. □

6. Examples

In case $n = 24$, we are able to completely determine the polynomials discussed in Theorem 21.

LEMMA 22. *Let E_6 be the Eisenstein series of weight 6 of level 1 with $E_6(\infty) = 1$ and $F = (2\pi)^{-12}\Delta$ where Δ is the modular discriminant. Then we have*

$$E_6 = -\frac{1}{64}x^{12} + \frac{33}{64}x^8y^4 + \frac{33}{64}x^4y^8 - \frac{1}{64}y^{12}$$
$$F = \frac{1}{2^{16}}x^4y^4(x^4 - y^4)^4.$$

Proof. By Lemma 17 and Lemma 19, E_6 can be written as

$$E_6 = ax^{12} + bx^8y^4 + bx^4y^8 + ay^{12}$$

for some $a, b \in \mathbb{Q}$. Evaluating both sides at some cusps of $\Gamma(4)$, we will determine a and b. First, at $s = 0$, $1 = E_6(0) = a \cdot x(0)^{12} = a \cdot (\sqrt{-2i})^{12} = a \cdot (-64)$; hence $a = -\frac{1}{64}$. Next, at $s = \infty$, $1 = E_6(\infty) = a + b + b + a = 2 \cdot (-\frac{1}{64}) + 2b$ and hence $b = \frac{33}{64}$. Now, consider the case of F. As is well known ([10], p. 222), we have the following equality:

$$\begin{aligned} F &= \frac{1}{2^8}{\theta_2}^8{\theta_3}^8{\theta_4}^8 \\ &= \frac{1}{2^8}X^2Y^2(Y-X)^2 \quad \text{by the relation } {\theta_3}^4 = {\theta_2}^4 + {\theta_4}^4 \text{ and Fact 2} \\ &= \frac{1}{2^8}\frac{1}{4^4}(x^4 - y^4)^4(x^2y^2)^2 \quad \text{by Theorem 12} \\ &= \frac{1}{2^{16}}x^4y^4(x^4 - y^4)^4. \end{aligned}$$

This completes the lemma. □

PROPOSITION 23. *For $A \in Q(24, 1)$,*

$$\begin{aligned} \theta_A(z) &= a^2x^{24} + (2ab + \frac{g_A}{2^{16}})x^{20}y^4 + (b^2 + 2ab - \frac{g_A}{2^{14}})x^{16}y^8 \\ &+ (2a^2 + 2b^2 + \frac{3g_A}{2^{15}})x^{12}y^{12} \\ &+ (b^2 + 2ab - \frac{g_A}{2^{14}})x^8y^{16} + (2ab + \frac{g_A}{2^{16}})x^4y^{20} + a^2y^{24} \end{aligned}$$

where $a = -\frac{1}{64}, b = \frac{33}{64}$ and $g_A = c_A + \frac{762048}{691} = r_A(1) + 1008$ $(\in \mathbb{Z})$ depending on Niemeier's classification ([8]).

Proof. Since E_{12} and F span $M_{12}(\Gamma(1))$, we can express

$$\theta_A = E_{12} + c_AF = {E_6}^2 + g_AF. \tag{6.1}$$

By comparing q-expansion we get $g_A = c_A + \frac{762048}{691}$. Now, plugging the results in Lemma 22 into (6.1), we obtain the assertion. □

Appendix A

For 6 cusps of $\Gamma(4)$, we have the following table:

	∞	0	1	-1	-2	$\frac{1}{2}$
X	0	-1	1	1	-1	0
Y	1	-1	0	0	-1	1
λ	0	1	∞	∞	1	0
x	1	$\sqrt{-2i}$	$-i$	$-i$	0	$-i$
y	1	0	-1	1	$\sqrt{-2i}$	i
j_4	1	∞	i	$-i$	0	-1

Appendix B

From Proposition 23, the formula (9) in [4] and following Niemeier's notation,

$$\theta_{3\times E_8}(z) = \theta_{E_8\oplus D_{16}}(z) =$$
$$\tfrac{1}{4096}x^{24} + \tfrac{21}{2048}x^{20}y^4 + \tfrac{591}{4096}x^{16}y^8 + \tfrac{707}{1024}x^{12}y^{12} + \tfrac{591}{4096}x^8y^{16} + \tfrac{21}{2048}x^4y^{20} + \tfrac{1}{4096}y^{24}$$

$$\theta_{E_7\oplus E_7\oplus D_{10}}(z) = \theta_{E_7\oplus A_{17}}(z) =$$
$$\tfrac{1}{4096}x^{24} + \tfrac{3}{512}x^{20}y^4 + \tfrac{663}{4096}x^{16}y^8 + \tfrac{85}{128}x^{12}y^{12} + \tfrac{663}{4096}x^8y^{16} + \tfrac{3}{512}x^4y^{20} + \tfrac{1}{4096}y^{24}$$

$$\theta_{D_{24}}(z) =$$
$$\tfrac{1}{4096}x^{24} + \tfrac{33}{2048}x^{20}y^4 + \tfrac{495}{4096}x^{16}y^8 + \tfrac{743}{1024}x^{12}y^{12} + \tfrac{495}{4096}x^8y^{16} + \tfrac{33}{2048}x^4y^{20} + \tfrac{1}{4096}y^{24}$$

$$\theta_{D_{12}\oplus D_{12}}(z) =$$
$$\tfrac{1}{4096}x^{24} + \tfrac{15}{2048}x^{20}y^4 + \tfrac{639}{4096}x^{16}y^8 + \tfrac{689}{1024}x^{12}y^{12} + \tfrac{639}{4096}x^8y^{16} + \tfrac{15}{2048}x^4y^{20} + \tfrac{1}{4096}y^{24}$$

$$\theta_{3\times D_8}(z) =$$
$$\tfrac{1}{4096}x^{24} + \tfrac{9}{2048}x^{20}y^4 + \tfrac{687}{4096}x^{16}y^8 + \tfrac{671}{1024}x^{12}y^{12} + \tfrac{687}{4096}x^8y^{16} + \tfrac{9}{2048}x^4y^{20} + \tfrac{1}{4096}y^{24}$$

$$\theta_{D_9\oplus A_{15}}(z) =$$
$$\tfrac{1}{4096}x^{24} + \tfrac{21}{4096}x^{20}y^4 + \tfrac{675}{4096}x^{16}y^8 + \tfrac{1351}{2048}x^{12}y^{12} + \tfrac{675}{4096}x^8y^{16} + \tfrac{21}{4096}x^4y^{20} + \tfrac{1}{4096}y^{24}$$

$$\theta_{4\times E_6}(z) = \theta_{E_6\oplus D_7\oplus A_{11}}(z) =$$
$$\tfrac{1}{4096}x^{24} + \tfrac{15}{4096}x^{20}y^4 + \tfrac{699}{4096}x^{16}y^8 + \tfrac{1333}{2048}x^{12}y^{12} + \tfrac{699}{4096}x^8y^{16} + \tfrac{15}{4096}x^4y^{20} + \tfrac{1}{4096}y^{24}$$

$\theta_{4\times D_6}(z) = \theta_{D_6 \bigoplus A_9 \bigoplus A_9}(z) =$
$\frac{1}{4096}x^{24} + \frac{3}{1024}x^{20}y^4 + \frac{711}{4096}x^{16}y^8 + \frac{331}{512}x^{12}y^{12} + \frac{711}{4096}x^8y^{16} + \frac{3}{1024}x^4y^{20} + \frac{1}{4096}y^{24}$

$\theta_{D_5 \bigoplus D_5 \bigoplus A_7 \bigoplus A_7}(z) =$
$\frac{1}{4096}x^{24} + \frac{9}{4096}x^{20}y^4 + \frac{723}{4096}x^{16}y^8 + \frac{1315}{2048}x^{12}y^{12} + \frac{723}{4096}x^8y^{16} + \frac{9}{4096}x^4y^{20} + \frac{1}{4096}y^{24}$

$\theta_{3\times A_8}(z) =$
$\frac{1}{4096}x^{24} + \frac{21}{8192}x^{20}y^4 + \frac{717}{4096}x^{16}y^8 + \frac{2639}{4096}x^{12}y^{12} + \frac{717}{4096}x^8y^{16} + \frac{21}{8192}x^4y^{20} + \frac{1}{4096}y^{24}$

$\theta_{A_{24}}(z) =$
$\frac{1}{4096}x^{24} + \frac{69}{8192}x^{20}y^4 + \frac{621}{4096}x^{16}y^8 + \frac{2783}{4096}x^{12}y^{12} + \frac{621}{4096}x^8y^{16} + \frac{69}{8192}x^4y^{20} + \frac{1}{4096}y^{24}$

$\theta_{A_{12} \bigoplus A_{12}}(z) =$
$\frac{1}{4096}x^{24} + \frac{33}{8192}x^{20}y^4 + \frac{693}{4096}x^{16}y^8 + \frac{2675}{4096}x^{12}y^{12} + \frac{693}{4096}x^8y^{16} + \frac{33}{8192}x^4y^{20} + \frac{1}{4096}y^{24}$

$\theta_{6\times D_4}(z) = \theta_{D_4 \bigoplus (4\times A_5)}(z) =$
$\frac{1}{4096}x^{24} + \frac{3}{2048}x^{20}y^4 + \frac{735}{4096}x^{16}y^8 + \frac{653}{1024}x^{12}y^{12} + \frac{735}{4096}x^8y^{16} + \frac{3}{2048}x^4y^{20} + \frac{1}{4096}y^{24}$

$\theta_{4\times A_6}(z) =$
$\frac{1}{4096}x^{24} + \frac{15}{8192}x^{20}y^4 + \frac{729}{4096}x^{16}y^8 + \frac{2621}{4096}x^{12}y^{12} + \frac{729}{4096}x^8y^{16} + \frac{15}{8192}x^4y^{20} + \frac{1}{4096}y^{24}$

$\theta_{6\times A_4}(z) =$
$\frac{1}{4096}x^{24} + \frac{9}{8192}x^{20}y^4 + \frac{741}{4096}x^{16}y^8 + \frac{2603}{4096}x^{12}y^{12} + \frac{741}{4096}x^8y^{16} + \frac{9}{8192}x^4y^{20} + \frac{1}{4096}y^{24}$

$\theta_{8\times A_3}(z) =$
$\frac{1}{4096}x^{24} + \frac{3}{4096}x^{20}y^4 + \frac{747}{4096}x^{16}y^8 + \frac{1297}{4096}x^{12}y^{12} + \frac{747}{4096}x^8y^{16} + \frac{3}{4096}x^4y^{20} + \frac{1}{4096}y^{24}$

$\theta_{12\times A_2}(z) =$
$\frac{1}{4096}x^{24} + \frac{3}{8192}x^{20}y^4 + \frac{753}{4096}x^{16}y^8 + \frac{2585}{4096}x^{12}y^{12} + \frac{753}{4096}x^8y^{16} + \frac{3}{8192}x^4y^{20} + \frac{1}{4096}y^{24}$

$\theta_{24\times A_1}(z) =$
$\frac{1}{4096}x^{24} + \frac{759}{4096}x^{16}y^8 + \frac{121}{256}x^{12}y^{12} + \frac{759}{4096}x^8y^{16} + \frac{1}{4096}y^{24}$

$\theta_{G_0}(z) =$
$\frac{1}{4096}x^{24} - \frac{3}{4096}x^{20}y^4 + \frac{771}{4096}x^{16}y^8 + \frac{1279}{2048}x^{12}y^{12} + \frac{771}{4096}x^8y^{16} - \frac{3}{4096}x^4y^{20} + \frac{1}{4096}y^{24}$

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