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FORCED OSCILLATIONS OF SOLUTIONS OF IMPULSIVE NONLINEAR PARABOLIC DIFFERENTIAL–DIFFERENCE EQUATIONS

DRUMI BAINOV AND EMIL MINCHEV

ABSTRACT. Sufficient conditions for forced oscillations of the solutions of impulsive nonlinear parabolic differential-difference equations are obtained.

1. Introduction

The impulsive differential equations are adequate apparatus for mathematical simulation in the science and technology. These equations provide natural mathematical description of processes which are subject to short-time perturbations during their evolution. In contrast to the big number of results for impulsive ordinary differential equations collected in seven monographs during the last eight years [2], [7]–[11], [15], the first results for impulsive partial differential equations were obtained in the recent years [1], [3]–[6], [12]–[14].

The present paper is concerned with the forced oscillations of solutions of impulsive nonlinear parabolic differential-difference equations subject to certain boundary conditions. The oscillation properties of the solutions are investigated via averaging technique.

2. Preliminary notes

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$ and $\overline{\Omega} = \Omega \cup \partial\Omega$. Suppose that $0 = t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ are given

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numbers and $t_{k+l} = t_k + \sigma$, $k = 0, 1, \dots$, where $\sigma = \text{const} > 0$ and l is a fixed natural number.

Define $J_{imp} = \{t_k\}_{k=1}^{\infty}$, $\mathbb{R}_+ = [0, +\infty)$, $E^0 = [-\sigma, 0] \times \overline{\Omega}$, $E = (0, +\infty) \times \Omega$, $E^* = \mathbb{R}_+ \times \overline{\Omega}$, $E_{imp} = \{(t,x) \in E \colon t \in J_{imp}\}$, $E^*_{imp} = \{(t,x) \in E^* \colon t \in J_{imp}\}$.

Let $C_{imp}[E^0 \cup E^*, \mathbb{R}]$ be the class of all functions $u \colon E^0 \cup E^* \to \mathbb{R}$ such that:

(i) The restriction of u to the set $E^0 \cup E^* \setminus E^*_{imp}$ is a continuous function.

(ii) For each $(t,x) \in E^*_{imp}$ there exist the limits

$$\lim_{\substack{(q,s)\to(t,x)\\ q<t}} u(q,s) = u(t^-, x), \qquad \lim_{\substack{(q,s)\to(t,x)\\ q>t}} u(q,s) = u(t^+, x)$$

and $u(t,x) = u(t^+, x)$ for $(t,x) \in E^*_{imp}$.

The class of functions $C_{imp}[E^*, \mathbb{R}]$ is defined analogously as E^* is written instead of $E^0 \cup E^*$ in the above definition.

Consider the nonlinear parabolic differential-difference equation

$$\begin{aligned} u_t(t,x) &- a(t)\Delta u(t,x) + p(t,x) f(u(t-\sigma, x)) \\ &= H(t,x), \ (t,x) \in E \setminus E_{imp}, \end{aligned} \tag{1}$$

subject to the impulsive condition

$$u(t,x) - u(t^-, x) = g(t, x, u(t^-, x)), \quad (t,x) \in E^*_{imp}, \tag{2}$$

and the boundary conditions

$$\frac{\partial u}{\partial n}(t,x) + \gamma(t,x) u(t,x) = 0, \quad (t,x) \in (\mathbb{R}_+ \setminus J_{imp}) \times \partial\Omega, \tag{3}$$

or,

$$u(t,x) = 0, \quad (t,x) \in (\mathbb{R}_+ \setminus J_{imp}) \times \partial\Omega. \tag{4}$$

The functions $a \colon \mathbb{R}_+ \to \mathbb{R}$, $p \colon E^* \to \mathbb{R}$, $f \colon \mathbb{R} \to \mathbb{R}$, $H \colon E^* \to \mathbb{R}$, $g \colon E^*_{imp} \times \mathbb{R} \to \mathbb{R}$, $\gamma \colon \mathbb{R}_+ \times \partial\Omega \to \mathbb{R}$ are given.

DEFINITION 1. The function $u \colon E^0 \cup E^* \to \mathbb{R}$ is called a solution of the problem (1)-(3) ((1), (2), (4)) if:

(i) $u \in C_{imp}[E^0 \cup E^*, \mathbb{R}]$, there exist the derivatives $u_t(t,x)$, $u_{x_i x_i}(t,x)$, $i = 1, \dots, n$ for $(t,x) \in E \setminus E_{imp}$ and u satisfies (1) on $E \setminus E_{imp}$.

(ii) u satisfies (2), (3) ((2), (4)).

DEFINITION 2. The nonzero solution $u(t,x)$ of equation (1) is said to be nonoscillating if there exists a number $\mu \geq 0$ such that $u(t,x)$ has a constant sign for $(t,x) \in [\mu, +\infty) \times \Omega$. Otherwise, the solution is said to oscillate.

For the function sign we have adopted the following definition

$$\operatorname{sign} x = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Introduce the following assumptions:
H1. $a \in C_{imp}[\mathbb{R}_+, \mathbb{R}_+]$.
H2. $p \in C_{imp}[E^*, \mathbb{R}_+]$.
H3. $g \in C(E^*_{imp} \times \mathbb{R}, \mathbb{R})$.
H4. $\gamma \in C_{imp}[\mathbb{R}_+ \times \partial\Omega, \mathbb{R}_+]$.
H5. $f \in C(\mathbb{R}, \mathbb{R})$, $f(u) = -f(-u)$ for $u \geq 0$, f is a positive and convex function in the interval $(0, +\infty)$.
H6. $H \in C_{imp}[E^*, \mathbb{R}]$.
In the sequel the following notations will be used:

$$P(t) = \min\{p(t,x) \colon\ x \in \overline{\Omega}\},$$

$$V(t) = \int_\Omega u(t,x)dx \left(\int_\Omega dx\right)^{-1},$$

$$H_0(t) = \int_\Omega H(t,x)dx \left(\int_\Omega dx\right)^{-1}.$$

3. Main results

We give sufficient conditions for oscillation of the solutions of problem (1)-(3).

LEMMA 1. *Let the following conditions hold:*
1. *Assumptions* H1–H6 *are fulfilled.*
2. $u \in C^2(E \backslash E_{imp}) \cap C^1(E^* \backslash E^*_{imp})$ *is a positive solution of the problem* (1)-(3) *in the domain* E.
3. $g(t_k, x, \xi) \leq L_k \xi$, $k = 1, 2, \ldots$, $x \in \overline{\Omega}$, $\xi \in \mathbb{R}_+$, $L_k \geq 0$ *are constants.*

Then the function $V(t)$ satisfies for $t \geq \sigma$ the impulsive differential inequality

$$V'(t) + P(t)f(V(t-\sigma)) \leq H_0(t), \quad t \neq t_k, \tag{5}$$

$$V(t_k) \leq (1 + L_k)V(t_k^-). \tag{6}$$

Proof. Let $t \geq \sigma$. Integrating the equation (1) with respect to x over the domain Ω, we obtain

$$\frac{d}{dt}\int\limits_\Omega u(t,x)dx - a(t)\int\limits_\Omega \Delta u(t,x)dx +$$

$$+\int\limits_\Omega p(t,x)f(u(t-\sigma,x))dx = \int\limits_\Omega H(t,x)dx, \quad t \neq t_k. \tag{7}$$

From the Green formula and H4 it follows that

$$\int\limits_\Omega \Delta u(t,x)dx = \int\limits_{\partial\Omega} \frac{\partial u}{\partial n}dS = -\int\limits_{\partial\Omega} \gamma(t,x)u(t,x)dS \leq 0, \quad t \neq t_k. \tag{8}$$

Moreover, for $t \neq t_k$, the Jensen inequality enables us to get

$$\int\limits_\Omega p(t,x)f(u(t-\sigma,x))dx \geq P(t)\int\limits_\Omega f(u(t-\sigma,x))dx \geq$$

$$\geq P(t)f\left(\int\limits_\Omega u(t-\sigma,x)dx\left(\int\limits_\Omega dx\right)^{-1}\right)\int\limits_\Omega dx = P(t)f(V(t-\sigma))\int\limits_\Omega dx. \tag{9}$$

In virtue of (8) and (9) we obtain from (7) that

$$V'(t) + P(t)f(V(t-\sigma)) \leq H_0(t), \qquad t \neq t_k.$$

For $t = t_k$ we have that

$$V(t_k) - V(t_k^-) \leq L_k\left(\int\limits_\Omega dx\right)^{-1}\int\limits_\Omega u(t_k^-,x)dx = L_kV(t_k^-),$$

that is,

$$V(t_k) \leq (1 + L_k)V(t_k^-).$$

□

DEFINITION 3. The solution $V \in C_{imp}[[-\sigma, 0] \cup \mathbb{R}_+, \mathbb{R}] \cap C^1(\cup_{k=0}^{\infty}(t_k, t_{k+1}), \mathbb{R})$ of the differential inequality (5), (6) is called eventually positive (negative), if there exists a number $t^* \geq 0$ such that $V(t) > 0$ ($V(t) < 0$) for $t \geq t^*$.

THEOREM 1. *Let the following conditions hold:*

1. *Assumptions* H1–H6 *are fulfilled.*

2. $g(t_k, x, \xi) \leq L_k \xi$, $k = 1, 2, \dots$, $x \in \overline{\Omega}$, $\xi \in \mathbb{R}_+$, $L_k \geq 0$ *are constants and* $g(t_k, x, \xi) = -g(t_k, x, -\xi)$.

3. *The differential inequality* (5), (6) *and the differential inequality*

$$V'(t) + P(t)f(V(t-\sigma)) \leq -H_0(t), \quad t \neq t_k, \tag{10}$$

$$V(t_k) \leq (1 + L_k)V(t_k^-), \tag{11}$$

have no eventually positive solutions.

Then each nonzero solution $u \in C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$ *of problem* (1)-(3) *oscillates in the domain* E.

Proof. Suppose the conclusion of the theorem is not true, i.e., $u(t,x)$ is a nonzero solution of the problem (1)-(3) which is of the class $C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$, and it has a constant sign in the domain $E_\mu = [\mu, +\infty) \times \Omega$, $\mu \geq 0$. If $u(t,x) > 0$ for $(t,x) \in E_\mu$, then it follows from Lemma 1 that $V(t)$ is a positive solution of the differential inequality (5), (6) for $t \geq \mu + \sigma$, which contradicts condition 3 of the theorem. If $u(t,x) < 0$ for $(t,x) \in E_\mu$, then the function $-u(t,x)$ is a solution of the problem

$$\begin{aligned} u_t(t,x) - a(t)\Delta u(t,x) + p(t,x)f(u(t-\sigma,x)) \\ = -H(t,x), \ (t,x) \in E \setminus E_{imp}, \end{aligned}$$

$$u(t,x) - u(t^-,x) = g(t,x,u(t^-,x)), \quad (t,x) \in E^*_{imp},$$

$$\frac{\partial u}{\partial n}(t,x) + \gamma(t,x)u(t,x) = 0, \quad (t,x) \in (\mathbb{R}_+ \setminus J_{imp}) \times \partial\Omega,$$

which is positive in E_μ. From Lemma 1 it follows that

$$\int_\Omega [-u(t,x)]dx \left(\int_\Omega dx \right)^{-1}$$

is a positive solution of the differential inequality (10), (11) for $t \geq \mu + \sigma$ which also contradicts condition 3 of the theorem. □

THEOREM 2. *Let the following conditions hold:*

1. $P \in C_{imp}[\mathbb{R}_+, \mathbb{R}_+]$, $H_0 \in C_{imp}[\mathbb{R}_+, \mathbb{R}]$.
2. $f(u) \geq 0$ *for* $u \geq 0$.
3. $\sum_{k=1}^{\infty} L_k < +\infty$, $L_k \geq 0$, $k = 1, 2, \dots$, *are constants.*
4. *For any number* $\tilde{t}_0 \geq \sigma$ *we have*

$$\liminf_{t\to\infty} \int_{\tilde{t}_0}^{t} \prod_{s<t_k\leq t} (1 + L_k) H_0(s) ds = -\infty.$$

Then the differential inequality (5), (6) *has no eventually positive solutions.*

Proof. Suppose the conclusion of the theorem is not true and let $V(t)$ be a positive solution of the differential inequality (5), (6) in the interval $[t^*, +\infty)$, $t^* \geq 0$. Then it follows from conditions 1 and 2 of the theorem that

$$V'(t) \leq H_0(t), \qquad t \geq t^* + \sigma, \quad t \neq t_k.$$

Integrating over the interval $[\tilde{t}_1, t]$, $t^* + \sigma \leq \tilde{t}_1 < t$, we obtain

$$V(t) \leq \prod_{\tilde{t}_1<t_k\leq t} (1 + L_k) V(\tilde{t}_1) + \int_{\tilde{t}_1}^{t} \prod_{s<t_k\leq t} (1 + L_k) H_0(s) ds. \tag{12}$$

Conditions 3 and 4 of the theorem imply that the right-hand side of (12) is not bounded from below and hence $V(t)$ cannot be eventually positive solution. □

COROLLARY 1. *Let the following conditions hold:*

1. *Assumptions* H1–H6 *are fulfilled.*
2. $g(t_k, x, \xi) \leq L_k \xi$, $k = 1, 2, \dots$, $x \in \overline{\Omega}$, $\xi \in \mathbb{R}_+$, $L_k \geq 0$ *are constants such that* $\sum_{k=1}^{\infty} L_k < +\infty$ *and* $g(t_k, x, \xi) = -g(t_k, x, -\xi)$.
3. *For any number* $\tilde{t}_0 \geq \sigma$ *we have*

$$\liminf_{t\to\infty} \int_{\tilde{t}_0}^{t} \prod_{s<t_k\leq t} (1 + L_k) H_0(s) ds = -\infty$$

and

$$\limsup_{t\to\infty} \int_{\tilde{t}_0}^{t} \prod_{s<t_k\le t} (1+L_k)H_0(s)ds = +\infty.$$

Then each nonzero solution $u \in C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$ *of the problem* (1)-(3) *oscillates in the domain* E.

Corollary 1 follows from Theorem 1 and Theorem 2.

Now we give sufficient conditions for oscillation of the solutions of problem (1), (2), (4). Consider the following Dirichlet problem

$$\begin{aligned} &\Delta\varphi + \alpha\varphi = 0 \quad \text{in} \quad \Omega, \\ &\varphi_{|\partial\Omega} = 0, \end{aligned} \tag{13}$$

where $\alpha =$ const. It is known that the smallest eigenvalue α_0 of the problem (13) is positive and the corresponding eigenfunction $\varphi_0(x) > 0$ for $x \in \Omega$. Without loss of generality we may assume that φ_0 is normalized, i.e., $\int_\Omega \varphi_0(x)dx = 1$.

Introduce the notations:

$$W(t) = \int_\Omega u(t,x)\varphi_0(x)dx,$$

$$H_1(t) = \int_\Omega H(t,x)\varphi_0(x)dx.$$

LEMMA 2. *Let the following conditions hold:*

1. *Assumptions* H1–H3, H5, *and* H6 *are fulfilled.*

2. $u \in C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$ *is a positive solution of the problem* (1), (2), (4) *in the domain* E.

3. $g(t_k, x, \xi) \le L_k\xi$, $k = 1, 2, \dots$, $x \in \overline{\Omega}$, $\xi \in \mathbb{R}_+$, $L_k \ge 0$ *are constants.*

Then the function $W(t)$ *satisfies for* $t \ge \sigma$ *the impulsive differential inequality*

$$W'(t) + \alpha_0 a(t)W(t) + P(t)f(W(t-\sigma)) \le H_1(t), \quad t \ne t_k, \tag{14}$$

$$W(t_k) \le (1 + L_k)W(t_k^-). \tag{15}$$

Proof. Let $t \geq \sigma$. We multiply the both sides of equation (1) by the eigenfunction $\varphi_0(x)$ and integrating with respect to x over Ω, we obtain

$$\frac{d}{dt}\int_\Omega u(t,x)\varphi_0(x)dx - a(t)\int_\Omega \Delta u(t,x)\varphi_0(x)dx +$$

$$+\int_\Omega p(t,x)f(u(t-\sigma,x))\varphi_0(x)dx = \int_\Omega H(t,x)\varphi_0(x)dx,\ t \neq t_k. \tag{16}$$

From the Green formula it follows that

$$\int_\Omega \Delta u(t,x)\varphi_0(x)dx = \int_\Omega u(t,x)\Delta\varphi_0(x)dx =$$

$$= -\alpha_0\int_\Omega u(t,x)\varphi_0(x)dx = -\alpha_0 W(t), \quad t \neq t_k, \tag{17}$$

where $\alpha_0 > 0$ is the smallest eigenvalue of the problem (13).

Moreover, from the Jensen inequality

$$\int_\Omega p(t,x)f(u(t-\sigma,x))\varphi_0(x)dx \geq P(t)\int_\Omega f(u(t-\sigma,x))\varphi_0(x)dx \geq$$

$$\geq P(t)f\left(\int_\Omega u(t-\sigma,x)\varphi_0(x)dx\right) = P(t)f(W(t-\sigma)),\ t \neq t_k. \tag{18}$$

Making use of (17) and (18), we obtain from (16) that

$$W'(t) + \alpha_0 a(t)W(t) + P(t)f(W(t-\sigma)) \leq H_1(t), \qquad t \neq t_k.$$

For $t = t_k$ we have that

$$W(t_k) - W(t_k^-) \leq L_k\int_\Omega u(t_k^-,x)\varphi_0(x)dx = L_k W(t_k^-),$$

that is,

$$W(t_k) \leq (1 + L_k)W(t_k^-). \qquad \square$$

Analogously to Theorem 1, we can prove the following theorem.

THEOREM 3. *Let the following conditions hold:*

1. *Assumptions* H1–H3, H5, *and* H6 *are fulfilled.*

2. $g(t_k,x,\xi) \leq L_k\xi$, $k = 1,2,\ldots$, $x \in \overline{\Omega}$, $\xi \in \mathbb{R}_+$, $L_k \geq 0$ *are constants and* $g(t_k,x,\xi) = -g(t_k,x,-\xi)$.

3. *The differential inequality* (14), (15) *and the differential inequality*

$$W'(t) + \alpha_0 a(t) W(t) + P(t) f(W(t-\sigma)) \le -H_1(t), \quad t \ne t_k,$$

$$W(t_k) \le (1 + L_k) W(t_k^-),$$

have no eventually positive solutions.

Then each nonzero solution $u \in C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$ *of problem* (1), (2), (4) *oscillates in the domain* E.

THEOREM 4. *Let the following conditions hold:*

1. $a,\ P \in C_{imp}[\mathbb{R}_+, \mathbb{R}_+]$, $H_1 \in C_{imp}[\mathbb{R}_+, \mathbb{R}]$.
2. $f(u) \ge 0$ *for* $u \ge 0$.
3. $\sum_{k=1}^{\infty} L_k < +\infty$, $L_k \ge 0$, $k = 1, 2, \dots$, *are constants.*
4. *For any number* $\tilde{t}_0 \ge \sigma$ *we have*

$$\liminf_{t\to\infty} \int_{\tilde{t}_0}^{t} \prod_{s<t_k\le t} (1 + L_k) e^{\alpha_0 \int_{t_0}^{s} a(\tau) d\tau} H_1(s) ds = -\infty.$$

Then the differential inequality (14), (15) *has no eventually positive solutions.*

The proof of Theorem 4 is analogous to the proof of Theorem 2. It is omitted here.

COROLLARY 2. *Let the following conditions hold:*

1. *Assumptions* H1–H3, H5, *and* H6 *are fulfilled.*
2. $g(t_k, x, \xi) \le L_k \xi$, $k = 1, 2, \dots$, $x \in \bar{\Omega}$, $\xi \in \mathbb{R}_+$, $L_k \ge 0$ *are constants such that* $\sum_{k=1}^{\infty} L_k < +\infty$ *and* $g(t_k, x, \xi) = -g(t_k, x, -\xi)$.
3. *For any number* $\tilde{t}_0 \ge \sigma$ *we have*

$$\liminf_{t\to\infty} \int_{\tilde{t}_0}^{t} \prod_{s<t_k\le t} (1 + L_k) e^{\alpha_0 \int_{t_0}^{s} a(\tau) d\tau} H_1(s) ds = -\infty$$

and

$$\limsup_{t\to\infty} \int_{\tilde{t}_0}^{t} \prod_{s<t_k\le t} (1 + L_k) e^{\alpha_0 \int_{t_0}^{s} a(\tau) d\tau} H_1(s) ds = +\infty.$$

Then each nonzero solution $u \in C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$ *of problem* (1), (2), (4) *oscillates in the domain* E.

Corollary 2 follows from Theorem 3 and Theorem 4.

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Higher Medical Institute
P. O. Box 45, Sofia-1504
Bulgaria