J. Korean Math. Soc. **35** (1998), No. 4, pp. 869–880

INTEGRAL ESTIMATES OF MAGNETOHYDRODYNAMICS EQUATIONS

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Abstract. In this paper, we show that the weak solutions of the time-dependent Magnetohydrodynamics equations in 3 dimensional periodic domain belong to $L^{\frac{2}{2r-1}}(0,T;V_r)$ following the method of Foias-Guillopé-Temam for Navier-Stokes equations.

1. Introduction

In this paper, we consider the following magnetohydrodynamics (MHD) equations in the non-dimensional form,

$$\frac{\partial}{\partial t}u+(u\cdot\nabla)u-\frac{1}{Re}\Delta u+S\nabla(\frac{1}{2}B^2)-S(B\cdot\nabla)B=f, \tag{1.1}$$

$$\frac{\partial}{\partial t}B+(u\cdot\nabla)B-(B\cdot\nabla)u+\frac{1}{Rm}\text{curl }(\text{curl}B)=0, \tag{1.2}$$

$$\text{div}u=\text{div}B=0. \tag{1.3}$$

Here we denote $u=(u_1(x,t),u_2(x,t),u_3(x,t))$ as the velocity of the particle of fluid, $B=(B_1(x,t),B_2(x,t),B_3(x,t))$ as the magnetic field, $f=f(x,t)$ as the volume density force at (x,t). The constants Re, Rm and M are the Reynolds number, the magnetic Reynolds number and the Hartan number respectively. Define $S=\frac{M^2}{ReRm}$.

The equations are important ones in the field of plasma physics. The existence of weak and strong solutions and some regularities are established by M. Sermange and R. Temam [3]. More precisely, they proved the weak solution of MHD equations belongs to $L^4(0,T;\mathbb{V})$ where $\mathbb{V}$ is

Received June 7, 1997. Revised February 24, 1998.
1991 Mathematics Subject Classification: 76W05.
Key words and phrases: Magnetohydrodynamics equations, integral estimate.

divergence-free subspece of H^1. Now we are interested in this weak solution. We prove that the weak solution belongs to $L^{\frac{2}{2r-1}}(0,T;V_r(Q))$ in 3 dimensional periodic domain for $r \geq 1$. We will follow the method of Foias-Guillopé-Temam for Navier-Stokes equations [1].

The organization of this paper is the following: In section 2, we introduce function spaces that are used in this paper. In section 3, we recall known results about existence, uniqueness and regularity of weak solutions. In section 4, we prove the main theorem, namely we show that if $f \in L^\infty(0,T;V_{r-1})$, then the weak solution belongs to $L^{\frac{2}{2r-1}}(0,T;V_r)$ for $r \geq 1$. As a corollary, we prove that the weak solution belongs to $L^1(0,T;L^\infty(Q))$ if $f \in L^\infty(0,T;H)$ following the idea of L. Tartar [1].

2. Function spaces

We supplement the system (1.1) $\sim$ (1.3) with following initial and boundary conditions

$$u(x,0) = u_0(x), B(x,0) = B_0(x) \text{ for all } x \in \mathbb{R}^3, \tag{2.1}$$

$$u(x + Le_i, t) = u(x,t), B(x + Le_i, t) = B(x,t), \tag{2.2}$$

for all $x \in \mathbb{R}^3$ and $t > 0$. Here L is the period and $\{e_i\}_{i=1}^3$ an orthonormal basis of the space. But we will regard L to be 2π for notational simplicity.

Let T > 0 and let X be a Banach space. We shall consider $L^p(0,T;X)$, $1 \leq p \leq \infty$, which is the space of functions from [0,T] into X, which are L^p for the Lebesgue measure dt. This is a Banach space for the norm

$$\left(\int_0^T \|u(t)\|_X^p dt\right)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty, \qquad \operatorname*{esssup}_{0\leq t\leq T} \|u(t)\|_X \text{ for } \quad p = \infty.$$

We denote $L^2(Q)$ as the space of $\mathbb{R}$-valued functions on Q which are square integrable for the Lebesgue measure $dx = dx_1\, dx_2\, dx_3$. This is a Hilbert space for the scalar product

$$(u,v) = \int_Q u(x) \cdot v(x)\, dx.$$

We use the same notation also for $V_0(Q)$, the space of $\mathbb{R}^3$-valued functions which are square integrable on Q.

Using Fourier series expressions, $V_0(Q)$ is identified with the space of functions u satisfying

$$u = \sum_{j\in\mathbb{Z}^3} u_j(t)e^{ij\cdot x}, \quad u_j(t) \in \mathbb{C}^3,\ u_{-j} = \bar{u}_j, \quad \text{for } t \in [0,T]. \tag{2.3}$$

For $m \in \mathbb{N}$, we also introduce

$$V_m(Q) = \{u \in V_0(Q)|(2\pi)^3 \sum_{j\in\mathbb{Z}^3} |j|^{2m}|u_j|^2 < \infty,\ u_0 = 0\}$$

with inner product

$$(u,v)_{H^m} = \int_Q \sum_{|k|=m} D^k u \cdot D^k v\ dx = (2\pi)^3 \sum_{j\in\mathbb{Z}^3} |j|^{2m}\ u_j \cdot v_{-j}.$$

We will use $(\cdot,\cdot)_{H^m}$ also in scalar case if it does not make confusion. Let $V_{-m}(Q)$ be the dual space of $V_m(Q)$. Especially,

$$V = \{u \in V_1(Q)|j \cdot u_j = 0 \text{ for all } j \in \mathbb{Z}^3\},$$

$$H = \{u \in V_0(Q)|j \cdot u_j = 0 \text{ for all } j \in \mathbb{Z}^3\}$$

and V' is the dual space of V. We equip V with the scalar product

$$((u,v)) = \sum_{k=1}^{3}(\frac{\partial u}{\partial x_k}, \frac{\partial v}{\partial x_k}) = \sum_{j\in\mathbb{Z}^3}(2\pi)^3|j|^2 u_j \cdot \bar{v}_j$$

which is also scalar product on $V_1(Q)$.

$\mathbb{V}$ and $\mathbb{H}$ are defined as followings:

$$\mathbb{V} = \{(u,B)|u,B \in V\}, \quad \mathbb{H} = \{(u,B)|u,B \in H\}.$$

We equip $\mathbb{H}$ with the following scalar products

$$(\Phi,\Psi) = (u,v) + (B,C) \text{ for all } \Phi = (u,B),\ \Psi = (v,C) \in \mathbb{H}$$

with associated norm, $|\Phi| = \{(\Phi,\Phi)\}^{\frac{1}{2}}$. We also equip $\mathbb{V}$ with the scalar products

$$((\Phi,\Psi)) = ((u,v)) + ((B,C))$$

with associated norm, $\|\Phi\| = \{((\Phi,\Phi))\}^{\frac{1}{2}}$.

We define an operator $\mathcal{A} \in \mathcal{L}(V,V')$ so that

$$\langle \mathcal{A}u, v\rangle = ((u,v)) \text{ for all } u,v \in V.$$

Then we recall that $\mathcal{A}$ is unbounded operator on H, whose domain is

$$\mathcal{D}(\mathcal{A}) = \{u \in V, \mathcal{A}u \in H\} = H^2 \cap V.$$

Since we consider divergence free functions on periodic domain, $\mathcal{A}$ is actually $-\Delta$.

Now we define a trilinear form on $L^1(\mathcal{Q}) \times W^{1,1}(\mathcal{Q}) \times L^1(\mathcal{Q})$ by setting

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_{\mathcal{Q}} u_i\ D_i v_j\ w_j\ dx \ \ (\text{where } D_i = \frac{\partial}{\partial x_i}),$$

whenever the integrals make sense. We know the trilinear form b is continuous on $(H^1(\mathcal{Q}))^3$ [4]. Thus it is natural to define a continuous bilinear operator $\mathcal{B}$ from $V \times V$ into V' so that

$$\langle \mathcal{B}(u,v), w\rangle = b(u,v,w).$$

3. Known results

Let $T > 0$ be given and let us assume that (p, u, B) is a smooth solution of (1.1) $\sim$ (2.2).

We multiply (1.1) by a test function $v \in V$ and integrate over $\mathcal{Q}$. Then

$$\frac{\partial}{\partial t}(u,v) + \frac{1}{Re}((u,v)) + b(u,u,v) - Sb(B,B,v) = (f,v). \tag{3.1}$$

We multiply (1.2) by a test function $C \in V$ and integrate over $\mathcal{Q}$. Then

$$\frac{\partial}{\partial t}(B,C) + \frac{1}{Rm}((B,C)) + b(u,B,C) - b(B,u,C) = 0. \tag{3.2}$$

This suggests the following weak formulation of the problem (1.1) $\sim$ (2.2).

DEFINITION 3.1 (Weak solution). Let $N = 2$ or 3, $f \in L^2(0,T;V')$ and $\Phi_0 = (u_0, B_0)$ given in $\mathbb{H}$. Then $\Phi = (u, B)$ is called weak solution of MHD equations if it satisfies

$$\Phi \in L^2(0,T;\mathbb{V}) \cap L^\infty(0,T;\mathbb{H}),\ \Phi(0) = \Phi_0,$$

(3.1) and (3.2) for all $\Psi = (v, C) \in \mathbb{V}$.

DEFINITION 3.2 (Strong solution). Let $N = 2$ or 3, $f \in L^2(0,T;H)$ and $\Phi_0 = (u_0, B_0)$ given in $\mathbb{V}$. Then $\Phi = (u, B)$ is called strong solution of MHD equations if it satisfies

$$u, B \in L^2(0,T;\mathcal{D}(\mathcal{A})) \cap L^\infty(0,T;V),$$

(3.1) and (3.2) for all $\Psi = (v, C) \in \mathbb{V}$.

By using operators $\mathcal{A}$ and $\mathcal{B}$, equations (3.1), (3.2) are written as

$$\frac{\partial u}{\partial t} + \frac{1}{Re}\mathcal{A}u + \mathcal{B}(u,u) - S\mathcal{B}(B,B) = f, \tag{3.3}$$

$$\frac{\partial B}{\partial t} + \frac{1}{Rm}\mathcal{A}B + \mathcal{B}(u,B) - \mathcal{B}(B,u) = 0. \tag{3.4}$$

For these equations, M. Sermange and R. Temam established following results about existence and uniqueness [3].

THEOREM 3.3. *For f, u_0, B_0 given with*

$$f \in L^2(0,T;V'),\ \Phi_0 = (u_0, B_0) \in \mathbb{H}, \tag{3.5}$$

there exists a weak solution $\Phi = (u,B)$ of MHD equations satisfying

$$\Phi \in L^2(0,T;\mathbb{V}) \cap L^\infty(0,T;\mathbb{H}). \tag{3.6}$$

Furthermore,

1. *if $N = 2, \Phi$ is unique and satisfies*

$$\Phi' \in L^2(0,T;\mathbb{V}),\ \Phi \in \mathcal{C}([0,T];\mathbb{H}), \tag{3.7}$$

2. *if N=3, there is at most one weak solution of MHD equations satisfying*

$$\Phi \in L^4(0,T;\mathbb{V}). \tag{3.8}$$

THEOREM 3.4. *Let f, u_0, B_0 be given with*

$$f \in L^\infty(0,T;H),\ \Phi_0 = (u_0, B_0) \in \mathbb{V},$$

1. *if N=2, the strong solution $\Phi = (u,B)$ of MHD equations satisfies*

$$\Phi \in L^2(0,T;\mathcal{D}(\mathcal{A})) \cap L^\infty(0,T;\mathbb{V}), \tag{3.9}$$

2. *if N=3, there exists $T_* > 0$ (depending on $\Omega, f, \|\Phi\|$) and on $[0,T_*]$, there exists a unique strong solution Φ of MHD equations which satisfies (3.9) with T replaced by T_*.*

4. Integral estimate

In this section we will prove the following theorem:

THEOREM 4.1. *(N=3, space periodic) We assume that* $\Phi_0 \in H$, $f \in L^\infty(0,T;V_{m-1})$ *and that* Φ *is a weak solution of the MHD equations. Then* Φ *satisfies*

$$\int_0^T |\Phi(t)|_{V_r}^{\frac{2}{2r-1}} dt \le c_r < \infty \tag{4.1}$$

for $r = 1, \cdots, m+1$, *where the constant* $c_r = c_r(\frac{1}{Re}, \frac{1}{Rm}, Q, \Phi_0, f)$.

To prove the theorem, we need some lemmas. From now on, we set N=3.

LEMMA 4.2. *Let* $u, v \in H^{r+1}$ *and* $w \in H^r$. $1 \le k \le r$, *and* $1 \le l, m \le r-1$. *Then the following inequalities hold.*

$$\left|\int_Q u\ D^{r+1}v\ D^r w\ dx\right| \le c_1 |u|_{H^1}^{1-\frac{1}{2r}} |u|_{H^{r+1}}^{\frac{1}{2r}} |v|_{H^{r+1}} |w|_{H^r}, \tag{4.2}$$

$$\left|\int_Q D^k u\ D^{r-k+1}v\ \ D^r w\ dx\right| \le c_2 |u|_{H^1}^{1-\frac{k}{r}+\frac{1}{2r}} |u|_{H^{r+1}}^{\frac{k}{r}-\frac{1}{2r}} |v|_{H^1}^{\frac{k-1}{r}} |v|_{H^{r+1}}^{1-\frac{k-1}{r}} |w|_{H^r}, \tag{4.3}$$

$$\left|\int_Q u^2\ (D^r v)^2\ dx\right| \le c_3 |u|_{H^1}^{2-\frac{1}{r}} |u|_{H^{r+1}}^{\frac{1}{r}} |v|_{H^1}^{\frac{2}{r}} |v|_{H^{r+1}}^{2-\frac{2}{r}}, \tag{4.4}$$

$$\left|\int_Q u\ D^r v\ D^l u\ D^{r-l} v\ dx\right| \le c_4 |u|_{H^1}^{2-\frac{4l+1}{4r}} |u|_{H^{r+1}}^{\frac{4l+1}{4r}} |v|_{H^1}^{\frac{4l+5}{4r}} |v|_{H^{r+1}}^{2-\frac{4l+5}{4r}}, \tag{4.5}$$

$$\left|\int_Q D^m u\ D^{r-m} v\ D^l u\ D^{r-l} v\ dx\right| \le c_5 |u|_{H^1}^{2-\frac{2l+2m-1}{2r}} |u|_{H^{r+1}}^{\frac{2l+2m-1}{2r}} |v|_{H^1}^{\frac{2l+2m+1}{2r}} |v|_{H^{r+1}}^{2-\frac{2l+2m+1}{2r}}, \tag{4.6}$$

where c_1, c_2, c_3, c_4, c_5 *are constants independent of* u, v, w.

Proof. By Gagliardo-Nirenberg inequality,

$$\left|\int_Q u\ D^{r+1}v\ D^r w\ dx\right| \le |u|_{L^\infty} |v|_{H^{r+1}} |w|_{H^r} \le c|u|_{H^1}^{1-\frac{1}{2r}} |u|_{H^{r+1}}^{\frac{1}{2r}} |v|_{H^{r+1}} |w|_{H^r}.$$

By Sobolev imbedding, $H^1 \hookrightarrow L^6(Q)$, $H^{\frac{1}{2}}(Q) \hookrightarrow L^3(Q)$,

$$\begin{aligned}\left|\int_Q D^k u\ D^{r-k+1}v\ D^r w\ dx\right| &\le |D^k u|_{L^3} |D^{r-k+1} v|_{L^6} |D^r w|_{L^2} \\ &\le c|u|_{H^{k+\frac{1}{2}}} |v|_{r-k+2} |w|_{H^r}.\end{aligned}$$

Thus by interpolation inequalities, we infer that

$$\left|\int_Q D^k u\ D^{r-k+1} v\ D^r w\ dx\right| \leq c_1 |u|_{H^1}^{1-\frac{k}{r}+\frac{1}{2r}} |u|_{H^{r-1}}^{\frac{k}{r}-\frac{1}{2r}} |v|_{H^1}^{\frac{k-1}{r}} |v|_{H^{r+1}}^{1-\frac{k-1}{r}} |w|_{H^r}.$$

Inequalities (4.4) $\sim$ (4.6) come from interpolation inequality and the following inequalities:

$$\left|\int_Q u^2\ (D^r v)^2\ dx\right| \leq c|u|_{L^\infty}^2 |v|_{H^r}^2,$$

$$\left|\int_Q u\ D^r v\ D^l u\ D^{r-l} v\ dx\right| \leq c|u|_{L^\infty} |v|_{H^r} |D^l u|_{L^4} |D^{r-l} v|_{L^4},$$

$$\left|\int_Q D^m u\ D^{r-m} v\ D^l u\ D^{r-l} v\ dx\right| \leq c|D^m u|_{L^4} |D^{r-m} v|_{L^4} |D^l u|_{L^4} |D^{r-l} v|_{L^4}.$$

□

LEMMA 4.3. *If Φ is a smooth solution of MHD equations, then for each $t > 0$ and $r \geq 1$*

$$\frac{d}{dt}|\Phi(t)|_{H^r}^2 + \frac{R}{2}|\Phi(t)|_{H^{r+1}}^2 \leq L_r(1 + |\Phi(t)|_{H^1}^2 |\Phi(t)|_{H^r}^{\frac{4r}{2r-1}}), \tag{4.7}$$

where $R = \min(\frac{1}{Re}, \frac{1}{Rm})$, $L_r = L_r(Re, Rm, \mathcal{Q}, N_{r-1}(f))$ and $N_{r-1}(f) = |f|_{L^\infty(0,T;V_{r-1})}$. Moreover, we have

$$\frac{d}{dt}|\Phi(t)|_{H^r}^2 + \frac{R}{2}|\Phi(t)|_{H^{r-1}}^2 \leq L_r'(1 + |\Phi(t)|_{H^1}^2)^{2r+1} \tag{4.8}$$

where $L_r' = L_r'(Re, Rm, \mathcal{Q}, N_{r-1}(f))$.

Proof. We take the scalar product of H in (3.3) and (3.4) with $\mathcal{A}^r u$ and $\mathcal{A}^r B$ respectively. Then we obtain that

$$\frac{1}{2}\frac{d}{dt}|u|_{H^r}^2 + \frac{1}{Re}|u|_{H^{r+1}}^2 = (f, u)_{H^r} - (\mathcal{B}(u, u), \mathcal{A}^r u) + S(\mathcal{B}(B, B), \mathcal{A}^r u), \tag{4.9}$$

$$\frac{1}{2}\frac{d}{dt}|B|_{H^r}^2 + \frac{1}{Rm}|B|_{H^{r-1}}^2 = (\mathcal{B}(B, u), \mathcal{A}^r B) - (\mathcal{B}(u, B), \mathcal{A}^r B). \tag{4.10}$$

The first term in the right-hand side of (4.9) is majorized by

$$|f(t)|_{H^{r-1}} |u(t)|_{H^{r+1}} \leq \frac{1}{4R}|u(t)|_{H^{r+1}}^2 + R\{N_{r-1}(f)\}^2. \tag{4.11}$$

Then by (4.2) and (4.3), other terms in right-hand side of (4.9) and (4.10) are governed by $|\Phi|_{H^1}^{1-\frac{1}{2r}}|\Phi|_{H^{r-1}}^{1+\frac{1}{2r}}|\Phi|_{H^r}$. Thus we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u|_{H^r}^2 &+ \frac{1}{Re}|u|_{H^{r-1}}^2 \\ (4.12) \qquad &\leq \frac{R}{4}|u|_{H^{r-1}}^2 + \frac{1}{R}\{N_{r-1}(f)\}^2 + c_4'|\Phi|_{H^1}^{1-\frac{1}{2r}}|\Phi|_{H^{r-1}}^{1+\frac{1}{2r}}|\Phi|_{H^r} \\ &\leq \frac{R}{4}|u|_{H^{r+1}}^2 + \frac{1}{R}\{N_{r-1}(f)\}^2 + \frac{R}{4}|\Phi|_{H^{r+1}}^2 + c_5'|\Phi|_{H^1}^2|\Phi|_{H^r}^{\frac{4r}{2r-1}},\end{aligned}$$

$$(4.13) \qquad \frac{1}{2}\frac{d}{dt}|B|_{H^r}^2 + \frac{1}{Rm}|B|_{H^{r+1}}^2 \leq \frac{R}{4}|\Phi|_{H^{r+1}}^2 + c_6'|\Phi|_{H^1}^2|\Phi|_{H^r}^{\frac{4r}{2r-1}}.$$

By adding (4.12) and (4.13),

$$\frac{1}{2}\frac{d}{dt}|\Phi|_{H^r}^2 + \frac{1}{4}R|\Phi|_{H^{r+1}}^2 \leq c_7|\Phi|_{H^1}^2|\Phi|_{H^r}^{\frac{4r}{2r-1}} + \{N_{r-1}(f)\}^2.$$

Thus we obtain (4.7). By interpolation inequality,

$$\begin{aligned}1 + |\Phi(t)|_{H^1}^2|\Phi(t)|_{H^r}^{\frac{4r}{2r-1}} &\leq 1 + c|\Phi(t)|_{H^1}^{\frac{4r+2}{2r-1}}|\Phi(t)|_{H^{r+1}}^{\frac{4(r-1)}{2r-1}} \\ &\leq 1 + \epsilon|\Phi(t)|_{H^{r+1}}^2 + C_\epsilon|\Phi(t)|_{H^1}^{4r+2}\end{aligned}$$

for all $\epsilon > 0$. This yields to (4.8). □

An immediate consequence of (4.8) is that Φ remains in V_r as long as $\|\Phi\|$ remains bounded.

LEMMA 4.4. *If $\Phi_0 \in V_r$, $f \in L^\infty(0,T;V_{r-1})$ and $r \geq 1$, then the strong solution Φ of MHD equations given by* Theorem 3.4 *belongs to $\mathcal{C}([0,T_*];V_r)$. If $\Phi_0 \in V$, $f \in L^\infty(0,T;V_{r-1})$ and $r \geq 1$, then $\Phi \in \mathcal{C}((0,T_*];V_r)$.*

Proof. We first consider the case $\Phi_0 \in V_r$, and we first show that Φ belongs to $L^\infty(0,T_*;V_r)$. For that it suffices to prove that the Galerkin approximation u_m of u remains bounded in $L^\infty(0,T_*;V_r)$ as $m \to \infty$.

(4.14)
$$\frac{du_m}{dt}(t) + \frac{1}{Re}\mathcal{A}u_m(t) + P_m\mathcal{B}(u_m(t),u_m(t)) - SP_m\mathcal{B}(B_m(t),B_m(t)) = P_mf,$$

(4.15)
$$\frac{dB_m}{dt}(t) + \frac{1}{Rm}\mathcal{A}B_m(t) + P_m\mathcal{B}(B_m(t),u_m(t)) - P_m\mathcal{B}(u_m(t),B_m(t)) = 0,$$

$$u_m(0) = P_m(u_0), \quad B_m(0) = P_m(B_0). \tag{4.16}$$

We take the scalar product in H of (4.14) with $\mathcal{A}^r u_m = (-1)^r \Delta^r u_m$, and of (4.15) with $\mathcal{A}^r B_m = (-1)^r \Delta^r B_m$. Since P_m is self-adjoint in H and $P_m \mathcal{A}^r u_m = \mathcal{A}^r u_m$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_m|^2_{H^r} &+ \frac{1}{Re}|u_m|^2_{H^{r+1}} \\ &= (f, u_m)_{H^r} - (-1)^r b(u_m, u_m, \Delta^r u_m) + S(-1)^r b(B_m, B_m, \Delta^r u_m),\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|B_m|^2_{H^r} &+ \frac{1}{Rm}|B_m|^2_{H^{r+1}} \\ &= (-1)^r b(B_m, u_m, \Delta^r B_m) - (-1)^r b(u_m, B_m \Delta^r B_m).\end{aligned}$$

This is similar to (4.9) and (4.10). Thus, exactly as in Lemma 4.3, we get the analogue of (4.8),

$$\frac{d}{dt}|\Phi_m|^2_{H^r} + \frac{R}{2}|\Phi_m|^2_{H^{r+1}} \le L'_r(1 + |\Phi_m|^2_{H^1})^{2r+1}.$$

Since $\Phi_m \in L^\infty(0, T_*; V)$, for $0 \le t \le T_*$

$$|\Phi_m(t)|^2_{H^r} \le c'_1 T_* + |\Phi_m(0)|^2_{H^r},$$

$$\frac{R}{2}\int_0^{T_*} |\Phi_m(t)|^2_{H^{r+1}} dt \le c'_1 T_* + |\Phi_m(0)|^2_{H^r}.$$

$|P_m \Phi_0|_{H^r} \le |\Phi_0|_{H^r}$ implies that Φ_m remains bounded in $L^\infty(0, T_*; V_r)$ and $L^2(\,0, T_*; V_{r+1})$.

Since $|u|_{H^1}, |B|_{H^1}$ are uniformly bounded for $0 \le t \le T_*$, $\mathcal{B}(u,u)$, $\mathcal{B}(u,B)$, $\mathcal{B}(B,u)$ and $\mathcal{B}(B,B)$ are in $L^2(0, T_*; V_{r-1})$ by Lemma 4.2. Furthermore, $f \in L^2(0, T_*; V_{r-1})$ and $\mathcal{A}u, \mathcal{A}B \in L^2(0, T_*; V_{r-1})$. Thus it follows that $\Phi' \in L^2(0, T_*; V_{r-1})$. Therefore $\Phi \in \mathcal{C}([0, T_*]; V_r)$ [5] (Chap III. §1.4).

For $\Phi_0 \in V$, we observe that the strong solution of MHD equations belongs to $L^2(0, T_*; \mathcal{D}(\mathcal{A}))$. Thus $\Phi(t) \in \mathcal{D}(\mathcal{A}) = V_2$ almost everywhere on $(0, T_*)$, and we can find t_1 that is arbitrarily small so that $\Phi(t_1) \in V_2$. The first part of the proof shows that $\Phi \in \mathcal{C}([t_1, T_*]; V_2) \cap L^2(t_1, T_*; V_3)$. Hence $\Phi(t_2) \in V_3$ for some $t_2 \in [t_1, T_*]$ arbitrarily close to t_1, and $\Phi \in \mathcal{C}([t_2, T_*]; V_3) \cap L^2(t_2, T_*; V_4)$. By induction we arrive at $\Phi \in \mathcal{C}([t_{r-1}, T_*]; V_r) \cap L^2(t_{r-1}, T_*; V_{r+1})$. Since t_{r-1} is arbitrarily close to 0, the lemma is proved. □

Let $m \geq 1$. We say that a solution Φ of MHD equations is V_m-regular on (t_1, t_2) $(0 \leq t_1 < t_2)$ if $\Phi \in \mathcal{C}((t_1, t_2); V_m(\mathcal{Q}))$. We say that V_m-regularity interval (t_1, t_2) is maximal if there does not exist an interval of V_m-regularity greater than (t_1, t_2).
The local existence of an interval of V_m-regular solution is given by above lemma : if $\Phi \in V_m$ and $f \in L^\infty(0, T; V_{m-1})$, then there exists an V_m-regular solution of the MHD equations defined on some interval $(0, t_0)$. Also, it follows easily by above lemma that if (t_1, t_2) is a maximal interval of V_m-regularity of a solution Φ, then

$$\limsup_{t \to t_2 - 0} |u(t)|_{V_m} = \infty. \tag{4.17}$$

Now we prove the main theorem.

Proof of the theorem Let (α_i, β_i) be maximal interval of V_m-regularity of Φ for $i \in \mathbb{N}$. On each interval (α_i, β_i), the inequality (4.7) is satisfied for $r = 1, \cdots, m$. We write them in the slightly weaker form

$$\frac{d}{dt}|\Phi|^2_{V_r} + \frac{R}{2}|\Phi(t)|^2_{V_{r+1}} \leq L_r(1 + |\Phi(t)|^2_{V_1})(1 + |\Phi(t)|^2_{V_r})^{\frac{2r}{2r-1}}. \tag{4.18}$$

Then we deduce

$$\frac{\frac{d}{dt}(1 + |\Phi|^2_{V_r})}{(1 + |\Phi(t)|^2_{V_r})^{\frac{2r}{2r-1}}} \leq L_r(1 + |\Phi(t)|^2_{V_1}).$$

By integration in t from α_i to β_i, we get

$$-\frac{2r-1}{(1 + |\Phi(\beta_i - 0)|^2_{V_r})^{\frac{1}{2r-1}}} + \frac{2r-1}{(1 + |\Phi(\alpha_i + 0)|^2_{V_r})^{\frac{1}{2r-1}}}$$
$$+R\int_{\alpha_i}^{\beta_i} \frac{|\Phi(t)|^2_{V_{r+1}}}{(1 + |\Phi(t)|^2_{V_r})^{\frac{2r}{2r-1}}} dt \leq L_r \int_{\alpha_i}^{\beta_i} (1 + |\Phi(t)|^2_{V_1}) dt.$$

From (4.17), the first term in the left-hand side of the inequality vanishes along a sequence converging to β_i, since (α_i, β_i) is a maximal interval of V_m-regularity. Thus

$$\int_{\alpha_i}^{\beta_i} \frac{|\Phi(t)|^2_{V_{r+1}}}{(1 + |\Phi(t)|^2_{V_r})^{\frac{2r}{2r-1}}} dt \leq \frac{L_r}{R} \int_{\alpha_i}^{\beta_i} (1 + |\Phi(t)|^2_{V_1}) dt.$$

By summation of these relations for $i \in N$, we find, since $\Phi \in L^2(0, T; V)$,

$$\int_0^T \frac{|\Phi(t)|^2_{V_{r+1}}}{(1 + |\Phi(t)|^2_{V_r})^{\frac{2r}{2r-1}}} dt \leq c_r \text{ for } r = 1, \cdots, m. \tag{4.19}$$

The proof of (4.1) is now made by induction. The result is true for $r = 1$. We assume that it is true for $1, \cdots, r$, and prove it for $r+1$ $(r \leq m)$. We have

$$\begin{aligned}
&\int_0^T |\Phi(t)|_{V_{r-1}}^{\frac{2}{2r-1}} dt \\
&= \int_0^T \left[\frac{|\Phi(t)|_{V_{r-1}}^2}{(1+|\Phi(t)|_{V_r}^2)^{\frac{2r}{2r-1}}} \right]^{\frac{1}{2r-1}} \left[(1+|\Phi(t)|_{V_r}^2)^{\frac{2r}{2r-1}} \right]^{\frac{1}{2r-1}} dt \\
&\leq \left[\int_0^T \frac{|\Phi(t)|_{V_{r-1}}^2}{(1+|\Phi(t)|_{V_r}^2)^{\frac{2r}{2r-1}}} dt \right]^{\frac{1}{2r-1}} \left[\int_0^T (1+|\Phi(t)|_{V_r}^2)^{\frac{1}{2r-1}} dt \right]^{\frac{2r}{2r-1}}.
\end{aligned}$$

Therefore (4.1) holds for $r+1$ due to (4.19) and the induction assumption.

Now we obtain the following corollary using Tartar's idea [1].

COROLLARY 4.5. *We assume $N = 3$, periodic. We assume $\Phi_0 \in H$ and $f \in L^\infty(0,T;H)$. Then any weak solution Φ of MHD equations belongs to $L^1(0,T;L^\infty(Q))$.*

Proof. By Gagliardo-Nirenberg inequality

$$|u(t)|_{L^\infty(Q)} \leq c\|u\|^{\frac{1}{2}} |\mathcal{A}u(t)|^{\frac{1}{2}},$$

$$|B(t)|_{L^\infty(Q)} \leq c\|B\|^{\frac{1}{2}} |\mathcal{A}B(t)|^{\frac{1}{2}}.$$

Thus we obtain that

$$|\Phi(t)|_{L^\infty(Q)} \leq c'\|\Phi\|^{\frac{1}{2}} |\mathcal{A}\Phi|^{\frac{1}{2}}.$$

By Hölder's inequality it follows that

$$\int_0^T |\Phi(t)|_{L^\infty(Q)} \leq c \left(\int_0^T |\mathcal{A}\Phi(t)|^{\frac{2}{3}} dt \right)^{\frac{3}{4}} \left(\int_0^T \|\Phi(t)\|^2 dt \right)^{\frac{1}{4}}.$$

Thus the right-hand side is finite thanks to (4.1). □

ACKNOWLEDGMENTS. The author thanks Professor Dongho Chae for suggesting the problem and for many helpful discussions.

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