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DOMINATION IN DIGRAPHS

CHANGWOO LEE

ABSTRACT. We establish bounds for the domination number of a digraph in terms of the minimum indegree and the order, and then we find a sharp upper bound for the domination number of a weak digraph with minimum indegree one. We also determine the domination number of a random digraph.

1. Introduction

Let D be a digraph of order n. A subset S of the vertex set $V(D)$ is a *dominating set* of D if for each vertex v not in S there exists a vertex u in S such that (u,v) is an arc of D. Note that $V(D)$ itself is a dominating set of D. A dominating set of D with the smallest cardinality is called a *minimum dominating set* of D and its cardinality is the *domination number* of D. We will reserve $\alpha(D)$ or just α for the domination number of D. For subsets S and T of $V(D)$, we say that S *dominates* T if S is a dominating set of the subdigraph $D[S \cup T]$ spanned by $S \cup T$.

For each positive integer n and each number p with $0 < p < 1$, the probability space $\mathcal{D}_{n,p}$ of digraphs is defined as follows: Each point in the space is a digraph with vertex set $V = \{1, 2, \cdots, n\}$ having no loops or multiple arcs, and the probability of a given digraph D with l arcs is given by $P(D) = p^l(1-p)^{n(n-1)-l}$. In other words, each arc is present with probability p, independently of the presence or absence of other arcs. For definitions not given here see [2] or [3].

Our main object here is to establish a tight upper bound for the domination number of a digraph and to determine the domination number of a random digraph. In section 2 we show that the domination number

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$\alpha(D)$ satisfies

$$\alpha(D) \leq \left\{ 1 - \left(\frac{1}{1+\delta^-}\right)^{\frac{1}{\delta^-}} + \left(\frac{1}{1+\delta^-}\right)^{\frac{1+\delta^-}{\delta^-}} \right\} n$$

for any digraph D with order n and minimum indegree $\delta^- \geq 1$ and show that

$$\alpha(D) \leq \frac{2}{3}n$$

for any digraph with order n and minimum indegree one. In section 3 we show that a random digraph $D_n \in \mathcal{D}_{n,p}$ has domination number either

$$\lfloor k^* \rfloor + 1 \text{ or } \lfloor k^* \rfloor + 2$$

almost surely, where $k^* = \log n - 2 \log\log n + \log\log e$ and log denotes the logarithm with base $1/(1-p)$.

2. The domination number of a digraph

Let X be a random variable on a probability space Ω, and let $E[X]$ be the expectation of X. Then we know that if $E[X] \leq c$ for some constant c, there is an $s \in \Omega$ such that $X(s) \leq c$. Let $X_1, X_2, \cdots, X_n$ be random variables, and let $X = c_1X_1 + \cdots + c_nX_n$, where c_i's are constants. Linearity of expectation states that $E[X] = c_1E[X_1] + \cdots + c_nE[X_n]$.

Using these simple observations, we prove the following theorem.

THEOREM 1. *Let D be a digraph with order n and minimum indegree $\delta^- \geq 1$. Then D has a dominating set of size at most*

$$\left\{ 1 - \left(\frac{1}{1+\delta^-}\right)^{\frac{1}{\delta^-}} + \left(\frac{1}{1+\delta^-}\right)^{\frac{1+\delta^-}{\delta^-}} \right\} n.$$

Proof. The proof technique follows the same pattern used by Alon and Spencer in [1] for graphs.

Fix p with $0 < p < 1$. Let us select, randomly and independently, each vertex of $V = V(D)$ with probability p. Let S be the random set of all vertices selected, and let T be the random set of all vertices not in S that do not have any in-neighbors in S. Then the expectation $E[|S|]$ of the random variable $|S|$ is $E[|S|] = np$ since $|S|$ has a binomial distribution with parameters n and p. To find $E[|T|]$, we let $|T| = \sum_{v \in V} \chi_v$, where $\chi_v = 1$ if $v \in T$ and $\chi_v = 0$ otherwise. Note that

$$\begin{aligned} P(v \in T) &= P(v \text{ and its in-neighbors are not in } S) \\ &= (1-p)^{1+id(v)} \\ &\leq (1-p)^{1+\delta^-} \end{aligned}$$

for each $v \in V$. Thus, we have

$$\begin{aligned} E[|T|] &= E[\sum_{v \in V} \chi_v] = \sum_{v \in V} E[\chi_v] \\ &= \sum_{v \in V} P(v \in T) \leq n(1-p)^{1+\delta^-}. \end{aligned}$$

Therefore, we have

$$E[|S| + |T|] \leq np + n(1-p)^{1+\delta^-}. \tag{1}$$

Using elementary calculus, we minimize the right side of (1) with respect to p. Then the minimum value of it is

$$\left\{ 1 - \left(\frac{1}{1+\delta^-} \right)^{\frac{1}{\delta^-}} + \left(\frac{1}{1+\delta^-} \right)^{\frac{1+\delta^-}{\delta^-}} \right\} n,$$

which is attained when

$$p = 1 - \left(\frac{1}{1+\delta^-} \right)^{\frac{1}{\delta^-}}.$$

This means that there is at least one choice of S such that

$$|S| + |T| \leq \left\{ 1 - \left(\frac{1}{1+\delta^-} \right)^{\frac{1}{\delta^-}} + \left(\frac{1}{1+\delta^-} \right)^{\frac{1+\delta^-}{\delta^-}} \right\} n.$$

The set $S \cup T$ is clearly a dominating set of D whose cardinality is at most

$$\left\{1-\left(\frac{1}{1+\delta^-}\right)^{\frac{1}{\delta^-}}+\left(\frac{1}{1+\delta^-}\right)^{\frac{1+\delta^-}{\delta^-}}\right\}n.$$

This completes the proof. □

This theorem gives us a good upper bound for the domination number of a digraph with large minimum indegree. The coefficient of this upper bound goes to zero when the minimum indegree δ^- goes to infinity.

REMARK. Let G be an undirected graph with order n and minimum degree δ. Then, using the same argument as in Theorem 1, we can show that the domination number of G is at most

$$\left\{1-\left(\frac{1}{1+\delta}\right)^{\frac{1}{\delta}}+\left(\frac{1}{1+\delta}\right)^{\frac{1+\delta}{\delta}}\right\}n. \tag{2}$$

L. Lovász showed in [5] that the domination number of G is at most

$$\frac{1+\ln\delta}{1+\delta}n, \tag{3}$$

and N. Alon and J. Spencer found a similar upper bound

$$\frac{1+\ln(\delta+1)}{1+\delta}n \tag{4}$$

in [1]. Even though these three upper bounds for the domination number of an undirected graph are asymptotically the same, our result (2) is smaller than (3) and (4) for $\delta \geq 4$.

It is easy to see that the domination number of a digraph D is the sum of the domination numbers of all weak components of D. Therefore, we consider weak digraphs with minimum indegree at least one. Then, what is the domination number of a digraph in which every vertex has indegree one? Such a digraph is called a *contrafunctional digraph.* A vertex v of a digraph D is called a *source* of D if every vertex is reachable from v, and a *tree from a vertex* (or *arborescence*) is a digraph with a source but with no semicycles. A (*directed*) *star* S_n is a digraph on n vertices consisting of a center v and a set of arcs from v to $V(S_n) - \{v\}$.

LEMMA 2 ([4]). *A weak digraph is a tree from a vertex if and only if exactly one vertex has indegree zero and every other vertex has indegree one.*

We need the above lemma to prove the following.

THEOREM 3. *Every tree T from a vertex v has domination number*

$$1 \leq \alpha(T) \leq \lceil \frac{1}{2}|V(T)| \rceil.$$

Moreover, the bounds are sharp.

Proof. We shall state an algorithm which finds a dominating set for a tree T from a vertex v. This algorithm begins by selecting a largest star that is the farthest from the source v. Then we put the center of the star into a dominating set. Next we remove the vertices in the star from T to get a new tree from a vertex and repeat this process.

Algorithm: Let $T_1 = T$ be the given tree from the vertex v, and let $S_0 = \emptyset$. Put $i = 1$ and go to (1).

(1) Take a vertex v_i with maximum distance from v in T_i.
(2) If $v_i = v$, then let $S = S_{i-1} \cup \{v\}$ and stop. If $v_i \neq v$ (i.e., $id_{T_i}(v_i) = 1$), let u_i be the vertex of T_i that is adjacent to v_i and go to (3).
(3) If $od_{T_i}(u_i) = 1$ and $u_i = v$, then let $S = S_{i-1} \cup \{u_i\}$ and stop. If $od_{T_i}(u_i) = 1$ and $u_i \neq v$, then let $S_i = S_{i-1} \cup \{u_i\}$ and $T_{i+1} = T_i - \{u_i, v_i\}$ and next return to (1) putting $i = i - 1$. If $od_{T_i}(u_i) \geq 2$, go to (4).
(4) If $u_i = v$, then let $S = S_{i-1} \cup \{v\}$ and stop. If $u_i \neq v$, then let $S_i = S_{i-1} \cup \{u_i\}$ and $T_{i+1} = T_i - N^+[u_i]$, and next return to (1) putting $i = i + 1$.

From this algorithm, it is easily seen that S is a dominating set for T and that $|S| \leq \lceil \frac{1}{2}|V(T)| \rceil$ since in each step except (possibly) the last, we take at least two vertices and put only one vertex into S that dominates the rest of them.

Extremal digraphs are a star S_n on n vertices and a path P_n on n vertices. □

Here we note that the complexity of this algorithm is $\mathcal{O}(n^2)$, where $n = |V(T)|$.

LEMMA 4 ([4]). *The following statements are equivalent for a weak digraph D.*

(1) *D is contrafunctional.*

(2) *D has exactly one cycle C and the removal of any one arc of C results in a tree from a vertex.*

The removal of any arc in a given digraph never decreases its domination number. Therefore, combining Theorem 3 and Lemma 4, we have the following corollary.

COROLLARY 5. *Every weak contrafunctional digraph D has domination number*

$$1 \leq \alpha(D) \leq \lceil \frac{1}{2}|V(D)| \rceil.$$

Moreover, the bounds are sharp.

Proof. To see the latter, we construct a digraph D as follows. We add one new vertex u to a star S_{n-1} and add two new arcs between u and the center of S_{n-1}. Then D is an extremal digraph, and a cycle C_n will do for the other extreme. □

If a digraph D has a spanning subdigraph H of D such that H is a disjoint union of stars, then H is called a *vertex disjoint star cover* (*vds-cover*) of D.

THEOREM 6. *Let D be a digraph with order n and minimum indegree $\delta^- \geq 1$. Then, we have*

$$1 \leq \alpha(D) \leq \frac{\delta^- + 1}{2\delta^- + 1} n.$$

Proof. It is easy to see that D has a vds-cover H, namely, take H as the empty digraph on $V(D)$. Among all such vds-covers of D, let H^* be one with minimum number of copies of S_1. For each $k = 1, 2, \cdots$, let H_k^* be the subdigraph of H^* consisting of weak components that are isomorphic to S_k and let h_k denote the number of weak components in H_k^*.

First, the subdigraph of D induced by $V(H_1^*)$ has no arcs at all since otherwise, H^* violates the minimality. Next, there are no arcs of D

from vertices in $\bigcup_{k\geq 3} H_k^*$ to vertices in H_1^* because if not, H^* violates the minimality also. However, each vertex in H_1^* is the terminal vertex of at least δ^- arcs. Hence these arcs must be incident from vertices in H_2^*. Let uv be a star in H_2^* with center u. Then, because of the minimality of H^*, u is not adjacent to any vertex in H_1^* and v is adjacent to at most one vertex in H_1^*. Since each vertex in H_1^* has indegree at least δ^-, we have $h_2 \geq \delta^- h_1$.

Now let S be the set of all centers of the stars in H^*. Then S is a dominating set of D and $|S| = \sum_{i\geq 1} h_i$. Note that

$$\frac{\delta^- + 1}{2\delta^- + 1} \geq \frac{1}{i}$$

for $i = 3, 4, \cdots$ and that

$$\frac{\delta^- + 1}{2\delta^- + 1}(h_1 + 2h_2) - (h_1 + h_2) = \frac{h_2 - \delta^- h_1}{2\delta^- + 1} \geq 0.$$

Since

$$|V(D)| = n = \sum_{i\geq 1} ih_i,$$

we have

$$\begin{aligned}\frac{\delta^- + 1}{2\delta^- + 1}n &= \frac{\delta^- + 1}{2\delta^- + 1}(h_1 + 2h_2) + \sum_{i\geq 3} \frac{\delta^- + 1}{2\delta^- + 1} ih_i \\ &\geq (h_1 + h_2) + \sum_{i\geq 3} h_i = |S|.\end{aligned}$$

This completes the proof. □

This theorem gives a better upper bound for the domination number of a digraph with $\delta^- = 1$ or 2 than that of Theorem 1.

COROLLARY 7. *Let D be a weak contrafunctional digraph. Then we have the following:*

(1) $\alpha(D) = \frac{2}{3}|V|$ *if and only if* $D = C_3$.

(2) $\alpha(D) < \frac{2}{3}|V|$ *if and only if* $D \neq C_3$.

Here, C_3 denotes a directed 3-cycle.

Proof. (1) The sufficiency is trivial. For the necessity, first note that for integer $n \geq 2$, $\frac{2}{3}n \leq \lceil \frac{n}{2} \rceil$ iff $n = 3$. Suppose that $\alpha(D) = \frac{2}{3}|V|$. Then $\frac{2}{3}|V| = \alpha(D) \leq \lceil \frac{1}{2}|V| \rceil$ by Corollary 5 and so $|V| = 3$ by the note. Moreover, C_3 is the only digraph on 3 vertices whose domination number is 2. This completes the proof of the first part.

(2) Since a weak contrafunctional digraph D has $\delta^- = 1$, we have $\alpha(D) \leq \frac{2}{3}|V|$ by Theorem 6, and so the second part follows. □

THEOREM 8. *Let D be a contrafunctional digraph. Then we have the following:*

(1) $\alpha(D) = \frac{2}{3}|V|$ *if and only if D is a disjoint union of 3-cycles.*
(2) $\alpha(D) < \frac{2}{3}|V|$ *if and only if D is not a disjoint union of 3-cycles.*

Proof. (1) The sufficiency is trivial. To prove the necessity, let $\alpha(D) = \frac{2}{3}|V|$ and let $\{H_1, H_2, \cdots, H_l\}$ be the set of weak components of D. Suppose that there exists a component that is not a 3-cycle. Then by Corollary 7, we have

$$\frac{2}{3}|V| = \alpha(D) = \sum_{i=1}^{l} \alpha(H_i) < \sum_{i=1}^{l} \frac{2}{3}|V(H_i)| = \frac{2}{3}|V|,$$

which is a contradiction. Thus every weak component of D is a 3-cycle and hence D is a disjoint union of 3-cycles.

(2) Suppose that D is not a disjoint union of 3-cycles and let $\{H_1, H_2, \cdots, H_l\}$ be the set of weak components of D. Then all H_i's are weak contrafunctional digraphs, and $H_i \neq C_3$ for some i. Hence we have

$$\alpha(D) = \sum_{j=1}^{l} \alpha(H_j) < \sum_{j=1}^{l} \frac{2}{3}|V(H_j)| = \frac{2}{3}|V|$$

and so the sufficiency has been established.

To prove the necessity, we let $\alpha(D) < \frac{2}{3}|V|$ and assume D is a disjoint union of 3-cycle $Z_i's$. Then we have

$$\alpha(D) = \sum_{i \geq 1} \alpha(Z_i) = \sum_{i \geq 1} \frac{2}{3}|V(Z_i)| = \frac{2}{3}|V|,$$

which contradicts $\alpha(D) < \frac{2}{3}|V|$. Therefore D is not a disjoint union of 3-cycles. □

The bound in Theorem 6 can be sharpened for weak digraphs with $3k$ vertices as follows.

THEOREM 9. *Let D be a weak digraph with minimum indegree $\delta^- = 1$ and let $|V(D)| = n$. Then we have the following:*
(1) *If $n \equiv 0 \pmod 3$ and $n \geq 6$, then $1 \leq \alpha(D) \leq \frac{2}{3}n - 1$.*
(2) *If $n \equiv 1 \pmod 3$ and $n \geq 4$, then $1 \leq \alpha(D) \leq \lfloor \frac{2}{3}n \rfloor$.*
(3) *If $n \equiv 2 \pmod 3$ and $n \geq 2$, then $1 \leq \alpha(D) \leq \lfloor \frac{2}{3}n \rfloor$.*
Moreover, all bounds are sharp.

Proof. Since (2) and (3) are the same as Theorem 6, it suffices to prove (1). For each vertex in D, color one incoming arc green and the others red and next choose only green arcs. Then we have a spanning contrafunctional subdigraph H of D. First, consider the case that H is not a disjoint union of 3-cycles. Clearly, $\alpha(D) \leq \alpha(H) < \frac{2}{3}n$ by Theorem 8 and hence $\alpha(D) \leq \frac{2}{3}n - 1$. Next, consider the case that H is a disjoint union of 3-cycles. Since D is weak but H is not, the arc set $E(D)$ of D consists of $E(H)$ and some arcs not in H. In addition, if we add some arcs in $E(D) - E(H)$ to H, then the resulting digraph has a strictly smaller domination number than that of H. Therefore, $\alpha(D) < \alpha(H) = \frac{2}{3}n$ and hence $\alpha(D) \leq \frac{2}{3}n - 1$. This completes the proof of (1).

For the sharpness of the lower bound in all cases, we take a digraph D as follows:

$$V(D) = \{v_1, v_2, \cdots, v_n\},$$
$$E(D) = \{v_2v_1, v_1v_2, v_1v_3, \cdots, v_1v_n\}.$$

For an extremal digraph of the case (1), we define a digraph D as follows: Take a disjoint union of k 3-cycles $Z_1, Z_2, \cdots, Z_k$, and let v_i be a vertex in Z_i for each i. Add $k-1$ new arcs v_iv_1 for $i = 2, 3, \cdots, k$, and let D be the resulting digraph. Next, for an extremal digraph of the case (2), we define a digraph as follows: Take a disjoint union of k 3-cycles $Z_1, Z_2, \cdots, Z_k$ and a new vertex u. Let v_i be a vertex in Z_i for each i. Add k new arcs v_iu and let D be the resulting digraph.

Finally, for an extremal digraph of the case (3), we define a digraph D as follows: Take a disjoint union of k 3-cycles and a 2-cycle C_2. Let u be a vertex in C_2 and v_i in Z_i. Add k new arcs v_iu and let D be the resulting digraph. □

OPEN PROBLEM. We have shown in Theorem 9 that the upper bound

$$\lfloor \frac{\delta^- + 1}{2\delta^- + 1} n \rfloor$$

in Theorem 6 is sharp for infinitely many n when $\delta^- = 1$. For $\delta^- = 2$, can we either sharpen this upper bound or construct a digraph with order n and $\delta^- = 2$ whose domination number is $\lfloor \frac{\delta^- + 1}{2\delta^- + 1} n \rfloor$?

3. The domination number of a random digraph

Let $\mathcal{Q}$ be a property of digraphs. If $\mathcal{A}$ is the set of digraphs of order n with property $\mathcal{Q}$ and the probability $P(\mathcal{A})$ of $\mathcal{A}$ has limit 1 as $n \to \infty$, then we say *almost all digraphs have property $\mathcal{Q}$* or *a random digraph has property $\mathcal{Q}$ almost surely.*

K. Weber determined the domination number for almost all graphs [6]. Using the same techniques as in [6] for analyzing the first and the second moments, we establish a similar result for digraphs.

THEOREM 10. *For p fixed, $o < p < 1$, a random digraph $D_n \in \mathcal{D}_{n,p}$ has domination number either*

$$\lfloor k^* \rfloor + 1 \text{ or } \lfloor k^* \rfloor + 2$$

almost surely, where $k^* = \log n - 2 \log \log n + \log \log e$ and $\log$ denotes the logarithm with base $1/(1-p)$.

Proof. Let X be a nonnegative random variable such that $X(D_n)$ is the number of dominating k-sets in D_n for each $D_n \in \mathcal{D}_{n,p}$. Since P(a fixed vertex v does not dominate another fixed vertex u) $= 1-p := q$, we have P(a fixed k-set $K \subseteq V$ does not dominate a fixed vertex in $V - K$) $= q^k$ and hence P(a fixed k-set of vertices is a dominating set) $= (1-q^k)^{n-k}$. Therefore the expected value of X is

$$E[X] = \binom{n}{k}(1-q^k)^{n-k}.$$

Now the result comes from [6]. □

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Department of Mathematics
University of Seoul
Seoul 130–743, Korea
E-mail: chlee@uoscc.uos.ac.kr