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A SHARP BOUND FOR ITÔ PROCESSES

CHANGSUN CHOI

ABSTRACT. Let X and Y be Itô processes with $dX_s = \varphi_s dB_s + \psi_s ds$ and $dY_s = \zeta_s dB_s + \xi_s ds$. Burkholder obtained a sharp bound on the distribution of the maximal function of Y under the assumption that $|Y_0| \le |X_0|$, $|\zeta| \le |\varphi|$, $|\xi| \le |\psi|$, and that X is a nonnegative local submartingale. In this paper we consider a wider class of Itô processes, replace the assumption $|\xi| \le |\psi|$ by a more general one $|\xi| \le \alpha|\psi|$, where $\alpha \ge 0$ is a constant, and get a weak-type inequality between X and the maximal function of Y. This inequality, being sharp for all $\alpha \ge 0$, extends the work by Burkholder.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a right-continuous filtration $(\mathcal{F}_t)_{t\ge 0}$ such that $A \in \mathcal{F}_0$ whenever $A \in \mathcal{F}$ and $P(A) = 0$. The adapted real Brownian motion $B = (B_t)_{t\ge 0}$ starts at 0 and the process $(B_t - B_s)_{t\ge s}$ is independent of $\mathcal{F}_s$ for all $s \ge 0$.

Let φ and ψ be real predictable processes such that

$$(1.1) \qquad P\left(\int_0^t \left(|\varphi_s|^2 + |\psi_s|\right) ds < \infty \text{ for all } t > 0\right) = 1.$$

Also, let ζ and ξ be $\mathbb{R}^\nu$-valued predictable processes, where ν is a positive integer. We assume the condition (1.1) for ζ and ξ. The Itô processes X and Y are defined by

$$(1.2) \qquad \begin{cases} X_t = X_0 + \int_0^t \varphi_s \, dB_s + \int_0^t \psi_s \, ds, \\ Y_t = Y_0 + \int_0^t \zeta_s \, dB_s + \int_0^t \xi_s \, ds. \end{cases}$$

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We assume that X_0 is constant and that X and Y are continuous.

We set $Y^* = \sup_{t\geq 0} |Y_t|$ and $\|X\| = \sup \mathbb{E}\,|X_\tau|$ where the supremum is taken over all bounded stopping times τ.

The following inequality is due to Burkholder (1993). Also see Burkholder (1994) for related inequalities.

THEOREM 1.1. *If* $X \geq 0$, $\psi \geq 0$, $|Y_0| \leq |X_0|$, $|\zeta| \leq |\varphi|$ *and* $|\xi| \leq |\psi|$, *then*

$$\lambda P(Y^* \geq \lambda) \leq 3\|X\| \quad \textit{for all } \lambda > 0$$

and 3 *is best possible.*

2. A sharp probability bound

Let $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{t\geq 0}$ be as in the introduction. The adapted square integrable real martingale M starts at 0 and, for all $s \geq 0$, the process $(M_t - M_s)_{t\geq s}$ is independent of $\mathcal{F}_s$. Let $\langle M \rangle$ be the quadratic variational process of M. The adapted integrable increasing process A starts at 0. We assume that M and A are continuous. Thus $\langle M \rangle$ is also continuous.

We follow Ikeda and Watanabe (1981) for notions of stochastic processes. Thus, increasing in the above means non-decreasing. Terms like positive, negative and decreasing, will be used similarly. Also, one may see the same book for the basic facts of stochastic processes and stochastic integrals.

Consider real predictable processes φ and ψ such that for all $t > 0$

$$\int_0^t |\varphi_s|^2 d\langle M \rangle_s < \infty \text{ and } \int_0^t |\psi_s| dA_s < \infty. \tag{2.1}$$

Let $\mathbb{H}$ be a Hilbert space over $\mathbb{R}$. For $x, y \in \mathbb{H}$ we denote by $x \cdot y$ the inner product of x and y and put $|x|^2 = x \cdot x$. The $\mathbb{H}$-valued predictable processes ζ and ξ satisfy the condition (2.1). The Itô processes X and Y are defined by

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t \varphi_s\, dM_s + \int_0^t \psi_s\, dA_s, \\ Y_t = Y_0 + \displaystyle\int_0^t \zeta_s\, dM_s + \int_0^t \xi_s\, dA_s. \end{cases} \tag{2.2}$$

We assume that X_0 is constant and that X and Y are continuous.

Let Y^* and $\|X\|$ be as in the introduction.

DEFINITION 2.1. For $\alpha \geq 0$ we define that Y *is α-subordinate to* X if

$$|Y_0| \leq |X_0|, \tag{2.3}$$
$$|\zeta| \leq |\varphi|, \tag{2.4}$$
$$|\xi| \leq \alpha|\psi|. \tag{2.5}$$

THEOREM 2.2. *If $X \geq 0$, $\psi \geq 0$, and Y is α-subordinate to X, then*

$$\lambda P(Y^* \geq \lambda) \leq (\alpha+2)\|X\| \quad \textit{for all } \lambda > 0 \tag{2.6}$$

and the constant $\alpha + 2$ is best possible.

Proof of the inequality. In order to make the key points of the proof clear we defer some technical details to Section 3 and use some unproved claims and lemmas in this proof.

We may assume that $\lambda = 1$ and $(\alpha+2)\|X\| < 1$.

CLAIM 2.3. *We may assume that $X > 0$ and $|Y| > 0$.*

CLAIM 2.4. *It suffices to prove*

$$P(|Y_\tau| \geq 1) \leq (\alpha+2)\|X\| \tag{2.7}$$

whenever τ is a bounded stopping time.

Let τ be a bounded stopping time. As a matter of fact, we will prove the stronger inequality

$$P(X_\tau + |Y_\tau| \geq 1) \leq (\alpha+2)\|X\|. \tag{2.8}$$

We may assume that the process $X + |Y|$ can be stopped on the surface $x + |y| = 1$.

Claim 2.5. *It suffices to prove*

$$P(X_\tau + |Y_\tau| = 1) \le (\alpha+2)\mathbb{E}\, X_\tau \tag{2.9}$$

whenever τ *is a bounded stopping time such that*

$$\mathbb{E} \int_0^\tau |\varphi_s|^2 \, d\langle M\rangle_s < \infty, \tag{2.10}$$

$$X_t + |Y_t| \le 1 \quad \text{if } 0 \le t \le \tau. \tag{2.11}$$

Put $S = \{(x,y) : x > 0 \text{ and } y \in \mathbb{H} \text{ with } |y| > 0\}$ and define functions U and V on S by

$$U(x,y) = \big(|y| - (\alpha+1)x\big)\big(x + |y|\big)^{1/(\alpha+1)} \tag{2.12}$$

and

$$V(x,y) = \begin{cases} -(\alpha+2)x & \text{if } x + |y| < 1, \\ 1 - (\alpha+2)x & \text{if } x + |y| \ge 1. \end{cases} \tag{2.13}$$

Let τ be a bounded stopping time satisfying (2.10) and (2.11). By Claim 2.3 we have $(X_\tau, Y_\tau) \in S$. And, from (2.11) and (2.13) we see that the inequality (2.9) is equivalent to the inequality

$$\mathbb{E}\, V(X_\tau, Y_\tau) \le 0. \tag{2.14}$$

Lemma 2.6. (a) *If* $x + |y| \le 1$, *then* $V(x,y) \le U(x,y)$.
(b) *If* $x \ge |y|$, *then* $U(x,y) \le 0$.

From (2.11) and (a) of Lemma 2.6 we have $\mathbb{E}\, V(X_\tau, Y_\tau) \le \mathbb{E}\, U(X_\tau, Y_\tau)$. Also, $|Y_0| \le |X_0|$ from (2.3) and X is positive, thus (b) of Lemma 2.6 implies that $\mathbb{E}\, U(X_0, Y_0) \le 0$. Hence, the inequality (2.14) follows from the inequality

$$\mathbb{E}\, U(X_\tau, Y_\tau) \le \mathbb{E}\, U(X_0, Y_0). \tag{2.15}$$

Since τ is bounded and U is smooth, we may use Itô's formula to get

$$\begin{aligned}
U(X_\tau, Y_\tau) &= U(X_0, Y_0) \\
&+ \int_0^\tau \Big(U_x(X_s, Y_s)\varphi_s + U_y(X_s, Y_s)\cdot\zeta_s\Big)dM_s \\
&+ \int_0^\tau \Big(U_x(X_s, Y_s)\psi_s + U_y(X_s, Y_s)\cdot\xi_s\Big)dA_s \\
&+ \frac{1}{2}\int_0^\tau \Big(U_{xx}(X_s, Y_s)|\varphi_s|^2 + 2U_{xy}(X_s, Y_s)\cdot\varphi_s\zeta_s \\
&\qquad + U_{yy}(X_s, Y_s)\zeta_s\cdot\zeta_s\Big)d\langle M\rangle_s .
\end{aligned} \tag{2.16}$$

Here $U_{yy}(X_s, Y_s)$ can be regarded as a linear transformation from $\mathbb{H}$ to $\mathbb{H}$. For differentiation of vector functions one may see Lang (1968).

The inequality (2.15) follows if we show that the above three integrals in (2.16) have negative expectations.

LEMMA 2.7. (a) $U_x(x,y) + \alpha|U_y(x,y)| \le 0$ *for all* $(x,y) \in S$.

(b) $|U_x(x,y)| + |U_y(x,y)| \le \alpha + 2$ *if* $(x,y) \in S$ *and* $x + |y| \le 1$.

(c) $U_{xx}(x,y)|h|^2 + 2U_{xy}(x,y)\cdot hk + U_{yy}(x,y)k\cdot k \le 0$ *whenever* $(x,y) \in S$, $h \in \mathbb{R}$, $k \in \mathbb{H}$ *and* $|h| \ge |k|$.

The first integral in (2.16) has zero expectation because τ is bounded and the process

$$t \to \int_0^{\tau\wedge t} \Big(U_x(X_s, Y_s)\varphi_s + U_y(X_s, Y_s)\cdot\zeta_s\Big)dM_s$$

is a martingale starting at 0; for this observe that $|\zeta_s| \le |\varphi_s|$ from (2.4) and that (2.10), (2.11), (b) of Lemma 2.7 and the Cauchy-Schwarz inequality imply

$$\begin{aligned}
&\mathbb{E}\int_0^\tau \Big|U_x(X_s, Y_s)\varphi_s + U_y(X_s, Y_s)\cdot\zeta_s\Big|^2 d\langle M\rangle_s \\
&\le \mathbb{E}\int_0^\tau \Big(|U_x(X_s, Y_s)| + |U_y(X_s, Y_s)|\Big)^2 |\varphi_s|^2\, d\langle M\rangle_s \\
&\le (\alpha+2)^2\,\mathbb{E}\int_0^\tau |\varphi_s|^2\, d\langle M\rangle_s < \infty.
\end{aligned}$$

The rest two integrals in (2.16) have negative integrands; thus they have negative expectations because the processes A and $\langle M \rangle$ are increasing.

Since $\psi \geq 0$ and $|\xi| \leq \alpha|\psi|$ from (2.5), we use the Cauchy-Schwarz inequality and (a) of Lemma 2.7 to get

$$U_x(X_s, Y_s)\psi_s + U_y(X_s, Y_s) \cdot \xi_s \leq U_x(X_s, Y_s)\psi_s + |U_y(X_s, Y_s)|\,|\xi_s| \leq \Big(U_x(X_s, Y_s) + \alpha|U_y(X_s, Y_s)|\Big)\psi_s \leq 0.$$

Similarly, the integrand of the third integral is negative because $(X_s, Y_s) \in S$ from Claim 2.4 and $|\zeta_s| \leq |\varphi_s|$ from (2.4): put $x = X_s, y = Y_s, h = \varphi_s$, $k = \zeta_s$ and apply (c) of Lemma 2.7.

This proves the inequality in Theorem 2.2 under the assumption of Claim 2.3, Claim 2.4, Claim 2.5, Lemma 2.6 and Lemma 2.7. We will elaborate on these claims and lemmas in Section 3. In Section 4 we construct an example which shows that $\alpha + 2$ is the best constant.

3. Proof of claims and lemmas

Proof of Claim 2.3. Let Itô processes X and Y satisfy the assumptions of Theorem 2.2. For each $\epsilon > 0$, the new processes $X + \epsilon$ and (Y, ϵ), where (Y, ϵ) is $\mathbb{H} \times \mathbb{R}$-valued, satisfy the extra assumption in Claim 2.3 as well as the assumptions in Theorem 2.2. Assuming the inequality (2.6) for these new processes with $\lambda = 1$ we have

$$P((Y, \epsilon)^* \geq 1) \leq (\alpha + 2)\|X + \epsilon\|. \tag{3.1}$$

Notice that $Y^* \leq (Y, \epsilon)^*$ and $\|X + \epsilon\| = \|X\| + \epsilon$. Thus, (3.1) yields as $\epsilon \to 0$ the inequality $P(Y^* \geq 1) \leq (\alpha + 2)\|X\|$, proving Claim 2.3. □

Proof of Claim 2.4. Define a stopping time τ by $\tau = \inf\{t > 0 : |Y_t| > 1\}$. Since $|Y_0| \leq |X_0|$ from (2.3) and $(\alpha + 2)\|X\| < 1$ we have $|Y_0| < 1$ and $\tau > 0$; recall that $|X_0|$ is constant, thus $|X_0| = \mathbb{E}\,|X_0| \leq \|X\| < 1$. We denote by n a positive integer. Observe that $\tau \wedge n$ is a bounded stopping time, $|Y_{\tau\wedge n}| \leq 1$ and that if $Y^* > 1$, then $\tau < \infty$ and $|Y_\tau| = 1$. Here we used the continuity of Y. Assuming the inequality (2.7) for $\tau \wedge n$, we get

$$P(|Y_{\tau\wedge n}| = 1) = P(|Y_{\tau\wedge n}| \geq 1) \leq (\alpha + 2)\|X\|. \tag{3.2}$$

Also, using Fatou's lemma we have

$$(3.3)\quad P(Y^* > 1) \le P(|Y_\tau| = 1 \text{ and } \tau < \infty) \le \liminf_{n\to\infty} P(|Y_{\tau\wedge n}| = 1).$$

From (3.2) and (3.3) we have

$$P(Y^* > 1) \le (\alpha+2)\|X\|,$$

from which we get $P(Y^* \ge 1) \le (\alpha+2)\|X\|$; first consider $(1+1/n)X$ and $(1+1/n)Y$, and let $n \to \infty$. This proves Claim 2.4. □

Proof of Claim 2.5. Let τ be a bounded stopping time. We define stopping times ρ and σ_n by $\rho = \inf\{t > 0 : X_t + |Y_t| \ge 1\}$ and

$$\sigma_n = \inf\left\{t > 0 : \int_0^t |\varphi_s|^2 \, d\langle M\rangle_s > n\right\}.$$

Here ρ is a stopping time because X and Y are continuous. From (2.3), the assumption that $(\alpha+2)\|X\| < 1$, and the assumption that $X_0 \ge 0$ is constant, we have $X_0 + |Y_0| \le 2X_0 = 2\mathbb{E}\, X_0 \le 2\|X\| < 1$. Thus, $X_t + |Y_t| \le 1$ if $0 \le t \le \rho$. The assumption (2.1) implies $\sigma_n \uparrow \infty$. Put $\tau_n = \tau \wedge \rho \wedge \sigma_n$. Observe that the stopping time τ_n satisfies all the conditions in Claim 2.5: here $\mathbb{E} \int_0^{\tau_n} |\varphi_s|^2 \, d\langle M\rangle_s \le \mathbb{E}\, n$. Assuming (2.9) for τ_n, we have

$$P(X_{\tau_n} + |Y_{\tau_n}| = 1) \le (\alpha+2)\mathbb{E}\, X_{\tau_n} \le (\alpha+2)\|X\|.$$

Observe that if $X_\tau + |Y_\tau| \ge 1$, then $\rho \le \tau$ and $\rho = \tau \wedge \rho$. Thus, Fatou's lemma gives

$$\begin{aligned} P(X_\tau + |Y_\tau| \ge 1) &\le P(X_{\tau\wedge\rho} + |Y_{\tau\wedge\rho}| = 1) \\ &\le \liminf_{n\to\infty} P(X_{\tau_n} + |Y_{\tau_n}| = 1) \le (\alpha+2)\|X\|, \end{aligned}$$

which yields (2.8) and completes the proof of the Claim 2.5. □

Proof of Lemma 2.6. Let $(x,y) \in S$.
Proof of (a). We may assume $x + |y| < 1$ because $U(x,y) = V(x,y)$ if $x + |y| = 1$. Write $x + |y| = r^{\alpha+1}$. Since $0 < r < 1$, we have

$$V(x,y) - U(x,y) = -r^{\alpha+2} - (1-r)(\alpha+2)x < 0.$$

Proof of (b). If $|y| \leq x$, then $|y| - (\alpha+1)x \leq |y| - x \leq 0$, hence $U(x,y) \leq 0$. □

Proof of Lemma 2.7. Differentiating U in (2.12), we have

$$\text{(3.4)} \qquad \begin{cases} U_x(x,y) = -\dfrac{(\alpha+1)(\alpha+2)x + \alpha(\alpha+2)|y|}{(\alpha+1)\big(x+|y|\big)^{1/(\alpha+1)}} \\ U_y(x,y) = \dfrac{(\alpha+2)y}{(\alpha+1)\big(x+|y|\big)^{1/(\alpha+1)}}. \end{cases}$$

Now (a) and (b) of Lemma 2.7 are clear from (3.4).
Proof of (c). Let $(x,y) \in S$, $h \in \mathbb{R}$, $k \in \mathbb{H}$ and $|h| \geq |k|$. We define a function G on the set $I = \{t \in \mathbb{R} : x + th > 0 \text{ and } |y + tk| > 0\}$ by

$$G(t) = U(x+th, y+tk).$$

Observe that I is an open set, $0 \in I$ and that $G(t)$ is smooth on I. By the chain rule one has

$$G''(0) = U_{xx}(x,y)|h|^2 + 2U_{xy}(x,y) \cdot hk + U_{yy}(x,y)k \cdot k\,.$$

Hence the proof is complete if we can check $G''(0) \leq 0$. If no confusion arises, we will not write the argument $t \in I$. On I define more functions K, Q and R by $K = K(t) = x + th$, $Q = |y + tk|$ and $R = K + Q$. Writing

$$G = R^{(\alpha+2)/(\alpha+1)} - (\alpha+2)KR^{1/(\alpha+1)},$$

we get

$$G' = \frac{\alpha+2}{\alpha+1}R'R^{1/(\alpha+1)} - (\alpha+2)hR^{1/(\alpha+1)} - \frac{\alpha+2}{\alpha+1}KR'R^{-\alpha/(\alpha+1)},$$

and

$$\eta G'' = R''R^2 + \frac{1}{\alpha+1}(R')^2R - 2hR'R - KR''R + \frac{\alpha}{\alpha+1}K(R')^2,$$

where

$$\eta = \frac{\alpha+1}{\alpha+2}R^{(2\alpha+1)/(\alpha+1)}.$$

Rearranging terms and inserting $(R')^2R - R(R')^2$, we have

$$\begin{aligned}\eta G'' &= \Big(R''R - KR'' - 2hR' + (R')^2\Big)R \\ &\quad + \Big(-R + \frac{1}{\alpha+1}R + \frac{\alpha}{\alpha+1}K\Big)(R')^2 \\ &= (|k|^2 - |h|^2)R - \frac{\alpha}{\alpha+1}Q(R')^2 \le 0.\end{aligned}$$

Here we used the observation that $K' = h$, $Q' = R' - h$, $QQ' = k \cdot (y + tk)$ and $QR'' = QQ'' = |k|^2 - (Q')^2$. Putting $t = 0$ we get $G''(0) \le 0$ and this proves (b) of Lemma 2.7. □

4. About the Best Constant

Let $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)_{t\ge0}$ and $B = (B_t)_{t\ge0}$ be as in the introduction. Let $\alpha \ge 0$ and $0 < \beta < \alpha + 2$. We will construct real Itô processes X and Y satisfying all the conditions of Theorem 2.2 for which we have $\lambda P(Y^* \ge \lambda) > \beta\|X\|$ for some $\lambda > 0$.

We will need to consider sequences $(a_n)_{n\ge1}$, $(b_n)_{n\ge1}$ and $(c_n)_{n\ge1}$. They satisfy $a_1 = b_1 = 1/2$, $c_1 = 0$, and if $n > 1$, then $a_n + b_n + c_n = 1$ and

$$a_n = \frac{n-1}{n}a_{n-1} + \frac{a_{n-1}}{(\alpha+1)n(2n-1)} + \frac{b_{n-1}+c_{n-1}}{(\alpha+1)n}. \tag{4.1}$$

PROPOSITION 4.1. $\lim_{n\to\infty} a_n = 1/(\alpha+2)$.

Proof. Noting $b_{n-1}+c_{n-1}=1-a_{n-1}$, we compute

$$a_n = a_{n-1}\left(1-\frac{(\alpha+1)(2n-1)+2(n-1)}{(\alpha+1)n(2n-1)}\right)+\frac{1}{(\alpha+1)n}.$$

With $t_n = a_n - 1/(\alpha+2)$ one has

$$\begin{aligned} t_n &= t_{n-1}\left(1-\frac{(\alpha+1)(2n-1)+2(n-1)}{(\alpha+1)n(2n-1)}\right) \\ &\quad + \frac{1}{(\alpha+1)(\alpha+2)n(2n-1)} \\ &< t_{n-1}\Big(1-\frac{1}{n}\Big)+\frac{1}{n^2} \end{aligned}$$

for $n > 1$. Also, $0 \le t_1 = \alpha/\big(2(\alpha+2)\big) < 1$ and $t_n > 0$ for all $n > 1$. Thus, if $1 < K < N$, then iteration gives

$$\begin{aligned} 0 < t_N &< t_1 \prod_{n=2}^{N}\Big(1-\frac{1}{n}\Big)+\frac{1}{N^2}+\sum_{n=2}^{N-1}\frac{1}{n^2}\prod_{k=n+1}^{N}\Big(1-\frac{1}{k}\Big) \\ &< \prod_{n=2}^{N}\Big(1-\frac{1}{n}\Big)+\sum_{n=K}^{N}\frac{1}{n^2}+\left(\sum_{n=2}^{K-1}\frac{1}{n^2}\right)\left(\prod_{n=K}^{N}\Big(1-\frac{1}{n}\Big)\right) \\ &< \exp\left(-\sum_{n=2}^{N}\frac{1}{n}\right)+\sum_{n=K}^{N}\frac{1}{n^2}+\left(\sum_{n=1}^{K}\frac{1}{n^2}\right)\exp\left(-\sum_{n=K}^{N}\frac{1}{n}\right). \end{aligned}$$

Now, one can see, as $N\to\infty$, that $t_N \to 0$, hence $a_N \to 1/(\alpha+2)$.

Since $1/\beta > 1/(\alpha+2)$ we may choose N so that $1/\beta > a_N$, or $1 > \beta a_N$.

Define a sequence of stopping times $(\sigma_n : 3 \le n \le 4N+1)$. Put $\sigma_3 = 3$. For $1 \le n \le 4N$, we let

(4.2) $$\sigma_{4n} = \inf\{s > \sigma_{4n-1} : B_s - B_{\sigma_{4n-1}} \notin (-2n+1, 1)\}.$$

And for $1 < n \le 4N$, put

(4.3)

$$\begin{aligned} \sigma_{4n-3} &= \inf\{s > \sigma_{4n-4} : B_s - B_{\sigma_{4n-4}} \notin (-2n+2, 1)\}, \\ \sigma_{4n-2} &= \sigma_{4n-3} + \frac{2}{\alpha+1}, \\ \sigma_{4n-1} &= \inf\left\{s > \sigma_{4n-2} : B_s - B_{\sigma_{4n-2}} \notin \Big(-\frac{2}{\alpha+1}, 2n-\frac{2}{\alpha+1}\Big)\right\}. \end{aligned}$$

Finally, put $\sigma_{4N+1} = 1 + \sigma_{4N}$.

Observe, from the strong Markov property of the Brownian motion and the basic facts of exit times of the Brownian motion, that σ_n is finite almost surely, and that

(4.4)
$$\begin{aligned}
&P(B_{\sigma_{4n}} - B_{\sigma_{4n-1}} = -2n+1)\\
&\quad = 1 - P(B_{\sigma_{4n}} - B_{\sigma_{4n-1}} = 1) = 1/(2n),\\
&P(B_{\sigma_{4n-3}} - B_{\sigma_{4n-4}} = -2n+2)\\
&\quad = 1 - P(B_{\sigma_{4n-3}} - B_{\sigma_{4n-4}} = 1) = 1/(2n-1),\\
&P\Big(B_{\sigma_{4n-1}} - B_{\sigma_{4n-2}} = -\frac{2}{\alpha+1}\Big)\\
&\quad = 1 - P\Big(B_{\sigma_{4n-1}} - B_{\sigma_{4n-2}} = 2n - \frac{2}{\alpha+1}\Big) = \Big(2n - \frac{2}{\alpha+1}\Big)\frac{1}{2n}.
\end{aligned}$$

We write $\operatorname{sgn} x$=1 if $x > 0$ and $\operatorname{sgn} x = -1$ if $x < 0$. Also, write $(x, y)c$ for the scalar multiplication $c(x, y)$ and 1_A for the indicator random variable on the set A.

Itô processes X and Y are defined by $X_0 = Y_0 = 1$ and the formula (1.2). Here we define φ, ψ, ζ and ξ as follows:

(4.5)

if $0 < s \le 3 = \sigma_3$, let $\varphi_s = \zeta_s = \psi_s = \xi_s = 0$;

if $1 \le n \le N$ and $\sigma_{4n-1} < s \le \sigma_{4n}$, let $\psi_s = \xi_s = 0$ and
$$(\varphi_s, \zeta_s) = (1, -\mathrm{sgn} Y_{\sigma_{4n-1}}) 1_{\{|Y_{\sigma_{4n-1}}|=1\}};$$

if $1 < n \le N$ and $\sigma_{4n-4} < s \le \sigma_{4n-3}$, let $\psi_s = \xi_s = 0$ and
$$(\varphi_s, \zeta_s) = \Big(1, \mathrm{sgn}(B_{1/(n-1)} - B_{1/n})\Big) 1_{\{Y_{\sigma_{4n-4}}=0\}};$$

if $1 < n \le N$ and $\sigma_{4n-3} < s \le \sigma_{4n-2}$, let $\varphi_s = \zeta_s = 0$ and
$$(\psi_s, \xi_s) = (1, \alpha\, \mathrm{sgn} Y_{\sigma_{4n-3}}) 1_{\{|Y_{\sigma_{4n-3}}|=2n-2\}};$$

if $1 < n \le N$ and $\sigma_{4n-2} < s \le \sigma_{4n-1}$, let $\psi_s = \xi_s = 0$ and
$$(\varphi_s, \zeta_s) = (1, -\mathrm{sgn} Y_{\sigma_{4n-2}}) 1_{\{|Y_{\sigma_{4n-2}}|=2n-2/(\alpha+1)\}};$$

if $\sigma_{4N} < s \le \sigma_{4N+1}$, let $\psi_s = \xi_s = 0$ and
$$(\varphi_s, \zeta_s) = (1, \mathrm{sgn}(B_{1/N} - B_{1/(N+1)})) 1_{\{|Y_{\sigma_{4N}}|=0\}};$$

if $s > \sigma_{4N+1}$, let $\varphi_s = \zeta_s = \psi_s = \xi_s = 0$.

Define a_n, b_n and c_n for $1 \le n \le N$ by

$$\begin{aligned} a_n &= P((X_{\sigma_{4n}}, Y_{\sigma_{4n}}) = (2n, 0)), \\ b_n &= P((X_{\sigma_{4n}}, Y_{\sigma_{4n}}) = (0, 2n)), \\ c_n &= P((X_{\sigma_{4n}}, Y_{\sigma_{4n}}) = (0, -2n)). \end{aligned}$$

Then we have $a_1 = b_1 = 1/2$, $c_1 = 0$ and for $1 < n \le N$ inductively we can check (4.1) and $a_n + b_n + c_n = 1$: we do not check this fully. To see how the induction can be carried out let's just consider the change of (X, Y) from the time σ_{4n-4} to the time σ_{4n-3}, where $1 < n \le N$. Observe that $\{Y_{\sigma_{4n-4}} = 0\}$ depends only on $\{B_s : 3 \le s \le \sigma_{4n-4}\}$. From the definition (4.5) we get

$$\begin{aligned} &Y_{\sigma_{4n-3}} - Y_{\sigma_{4n-4}} \\ &= \int_{\sigma_{4n-4}}^{\sigma_{4n-3}} \operatorname{sgn}(B_{1/(n-1)} - B_{1/n}) 1_{\{Y_{\sigma_{4n-4}}=0\}} dB_s \\ &= \left(\operatorname{sgn}(B_{1/(n-1)} - B_{1/n}) 1_{\{Y_{\sigma_{4n-4}}=0\}}\right) \left(B_{\sigma_{4n-3}} - B_{\sigma_{4n-4}}\right) \end{aligned}$$

because the integrand is $\mathcal{F}_{\sigma_{4n-4}}$ measurable. The random variables $1_{\{Y_{\sigma_{4n-4}}=0\}}$, $\operatorname{sgn}(B_{1/(n-1)} - B_{1/n})$ and $B_{\sigma_{4n-3}} - B_{\sigma_{4n-4}}$ are independent because of the strong Markov property of the Brownian motion . Observe that $\operatorname{sgn}(B_{1/(n-1)} - B_{1/n}) = 1$, or -1, each with probability $1/2$. Hence writing

$$Z = \left(X_{\sigma_{4n-3}}, Y_{\sigma_{4n-3}}\right) - \left(X_{\sigma_{4n-4}}, Y_{\sigma_{4n-4}}\right),$$

and using (4.4) we have

$$\begin{aligned} &P\big(Z = (1,1)\big) = P\big(Z = (1,-1)\big) = \frac{1}{2}\,\frac{2n-2}{2n-1} a_{n-1}, \\ &P\big(Z = (-2n+2, -2n+2)\big) \\ &\qquad = P\big(Z = (-2n+2, 2n-2)\big) = \frac{1}{2}\,\frac{1}{2n-1} a_{n-1}, \\ &P\big(Z = (0,0)\big) = 1 - a_{n-1}. \end{aligned}$$

Similarly, from (4.2)-(4.5) we can check that the Itô processes X and Y satisfy all the conditions of Theorem 2.2. The random vector $(X_{\sigma_{4N+1}}, Y_{\sigma_{4N+1}})$ is placed at four positions $(0, 2N)$, $(0, -2N)$,

$(4N, 2N)$ and $(4N, -2N)$ with probability $b_N + a_N/4$, $c_N + a_N/4$, $a_N/4$ and $a_N/4$, respectively. Thus

$$P(Y^* \geq 2N) = P(Y_{\sigma_{4N+1}} = 2N) = 1.$$

Since X is stopped at σ_{4N+1} we have $X_t \to X_{\sigma_{4N+1}}$ as $t \to \infty$. Also, $|\varphi| \leq 1$ and $0 \leq \psi \leq 1$, hence X is a submartingale: we also have $|X| \leq 4N$. Thus, for any bounded stopping time τ, Doob's optional sampling theorem gives

$$\mathbb{E}\, X_\tau \leq \mathbb{E}\, X_{\sigma_{4N+1}} = 4N\frac{a_N}{4} + 4N\frac{a_N}{4}.$$

Hence $\|X\| \leq 2Na_N$. Since $1 > \beta a_N$, with $\lambda = 2N$ we have

$$\lambda P(Y^* \geq \lambda) = 2N > 2N\beta a_N \geq \beta\|X\|.$$

This proves that $\alpha + 2$ is the best constant. □

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Department of Mathematics
KAIST
Taejon 305-701, Korea
E-mail: cschoi@math.kaist.ac.kr