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THE $M_1, M_2/G/1/K$ RETRIAL QUEUEING SYSTEMS WITH PRIORITY

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ABSTRACT. We consider an $M_1, M_2/G/1/K$ retrial queueing system with a finite priority queue for type I calls and infinite retrial group for type II calls where blocked type I calls may join the retrial group. These models, for example, can be applied to cellular mobile communication system where handoff calls have higher priority than originating calls.

In this paper we apply the supplementary variable method where supplementary variable is the elapsed service time of the call in service. We find the joint generating function of the numbers of calls in the priority queue and the retrial group in closed form and give some performance measures of the system.

1. Introduction and model description

Retrial queueing systems are characterized by the feature that arriving calls who find the server busy join the retrial group to try again for their requests in random order and at random intervals. Retrial queues have been widely used to model many problems in telephone switching systems, computer and communication systems. For the main surveys, see Falin and Templeton [11] for retrial queue with one type of calls, and Choi and Chang [3] for retrial queue with two types of calls.

Retrial queues with two types of calls are the typical model of telephone exchange with subscriber line modules and base station in a mobile cellular radio communication system.

Choi and Park [2] investigated $M/G/1$ retrial queue with two types of calls where service times for both types of calls are independent and

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identically distributed and priority queue has infinite capacity. Khalil et al. [13] investigated the above model at Makovian level in detail. Later Falin et al. [10] extended Choi and Park's model to the case where two types of calls may have different service time distributions. B. D. Choi et al. [6] considered Choi and Park's model in which priority queue has finite capacity. They found the joint generating functions of two queue lengths by using supplementary variable method where supplementary variable is the remaining service time .

In this paper, we consider $M_1, M_2/G/1/K$ retrial queue with two type calls where blocked type I calls may allow to join the retrial group (see Figure 1). Type I calls and type II calls arrive independently of each other according to Poisson processes with rate λ_1 and λ_2, respectively. An arriving call who finds the server idle begins to be served immediately. When the server is busy, an arriving type I call joins the priority queue if there is a waiting position, but if there are no waiting positions in the priority queue, he enters the retrial group with probability α or leaves the system with probability $1-\alpha$. Type I calls have a direct access to the server and can detect the epoch of the server release and immediately enter service. If an arriving type II call finds the server busy, then he joins the retrial group in order to seek service again after random amount of time. A call in the retrial group always returns to the retrial group when he finds the server busy on his retrial attempt to the server . Note that calls in the retrial group will be served only when there are no type I calls in the priority queue. Consequently, type I calls have non-preemptive priority over type II calls.

The retrial time (the time interval between two consecutive attempts made by a call in the retrial group) is exponentially distributed with mean $1/\nu$ and is independent of all previous retrial times and all the other stochastic processes in the system. The service times of calls are independent and identically distributed with distribution function $B(x)$ and mean $1/\mu$. Let $b^*(\theta)=\int_0^\infty e^{-\theta x}dB(x)$ and $b^{*(i)}(\theta)=\frac{d^{(i)}(b^*(\theta))}{d\theta^i}$.

This paper is organized as follows. In Section 2, we find the stability condition of the system. In section 3, we obtain the joint generating functions of the number of calls in the priority queue and the retrial group in steady state. In section 4, we compute some performance measures and give some numerical examples.

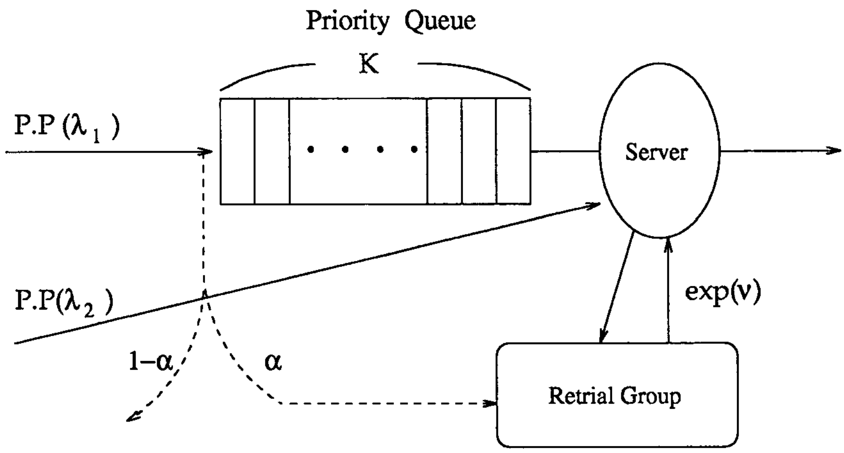

FIGURE 1. Mathematical model of the system

2. Ergodicity of the system

Define the processes $\{N_q(t)\}, \{N_r(t)\}, \{d_n\}$ as follows
$N_q(t)$= the number of calls in the priority queue at time t,
$N_r(t)$= the number of calls in the retrial group at time t,
d_n = the departure epoch of the n-th served call.
We consider the processes $\{X_n\}, \{Y_n\}$ embedded at the time d_n just after the departure of the n-th served call as follows

$$X_n = N_q(d_n+), \qquad Y_n = N_r(d_n+).$$

Then $Z_n = (X_n, Y_n)$ forms two dimensional Markov chain with state space

$$\{(i,j) : i = 0, 1, \cdots, K, j = 0, 1, \cdots\}.$$

Let

$$\begin{aligned} a_j &= P\{j \text{ arrivals of type I during a service time}\} \\ &= \int_0^\infty e^{-\lambda_1 x} \frac{(\lambda_1 x)^j}{j!} dB(x), \\ h(i) &= \text{the mean number of calls accumulated in the retrial group} \\ &\quad\ \text{during a busy period generated by } i \text{ calls in priority queue} \end{aligned}$$

and $h(0) = 0$.

Since the calls in the retrial group are served only when the priority queue is empty, intuitively $h(1)$ should be less than 1 in order for $\{Z_n\}$ to be ergodic. Indeed, $h(1) < 1$ is a necessary and sufficient condition for stability (see Theorem1). By conditioning the number of type I arrivals during a service time, We obtain

$$\begin{aligned} h(1) &= a_0\frac{\lambda_2}{\mu} + a_1(\frac{\lambda_2}{\mu}+h(1))+\cdots+a_K(\frac{\lambda_2}{\mu}+h(K)) \\ &\quad +\sum_{m=K+1}^{\infty} a_m(\frac{\lambda_2}{\mu}+h(K)+\alpha(m-K)). \end{aligned}$$

By algebraic calculation, we have

$$\begin{aligned} h(1) &= a_1h(1) + a_2h(2) + \cdots + a_{K-1}h(K-1) + \sum_{m=K}^{\infty} a_m h(K) \\ &\quad +\frac{\lambda_2}{\mu} + \alpha \sum_{m=K}^{\infty}(m-K)a_m. \end{aligned} \tag{1}$$

Similarly we obtain

$$\begin{aligned} h(i) &= a_0h(i-1) + a_1h(i) + \cdots + a_{K-i}h(K-1) + \sum_{m=K-i+1}^{\infty} a_m h(K) \\ &\quad +\frac{\lambda_2}{\mu} + \alpha \sum_{m=K-i+1}^{\infty}(m-K+i-1)a_m \quad 1 \le i \le K. \end{aligned} \tag{2}$$

Let column vector $H = (h(1), h(2), \cdots, h(K))^T, B = (b_1, b_2, \cdots, b_K)^T$, where

$$b_i = \frac{\lambda_2}{\mu} + \alpha \sum_{m=K-i+1}^{\infty}(m-K+i-1)a_m,$$

and define $K \times K$ matrix A as follows

$$
A = \begin{pmatrix}
a_1 & a_2 & a_3 & \cdots & \cdots & \cdots & a_{K-1} & \sum_{j=K}^{\infty} a_j \\
a_0 & a_1 & a_2 & \cdots & \cdots & \cdots & a_{K-2} & \sum_{j=K-1}^{\infty} a_j \\
0 & a_0 & a_1 & \cdots & \cdots & \cdots & a_{K-3} & \sum_{j=K-2}^{\infty} a_j \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \cdots & a_0 & a_1 & \sum_{j=2}^{\infty} a_j \\
0 & 0 & 0 & \cdots & \cdots & \cdots & a_0 & \sum_{j=1}^{\infty} a_j
\end{pmatrix}.
$$

Then (2) can be rewritten in matrix form as

$$H = AH + B,$$

i.e.,

$$(I - A)H = B.$$

By algebraic manipulation, we easily see that $\det(I - A) = a_0^K \neq 0$. Therefore by Cramer's Rule the above matrix equation have unique solution, in particular,

$$h(1) = \frac{\det(\hat{A})}{\det(I - A)} = a_0^{-K} \det(\hat{A}),$$

where the $K \times K$ matrix $\hat{A}$ is equal to the matrix $I - A$ except for the first column replaced by B

$$
\hat{A} = \begin{pmatrix}
b_1 & -a_2 & -a_3 & \cdots & \cdots & -a_{K-1} & -\sum_{j=K}^{\infty} a_j \\
b_2 & 1-a_1 & -a_2 & \cdots & \cdots & -a_{K-2} & -\sum_{j=K-1}^{\infty} a_j \\
b_3 & -a_0 & 1-a_1 & \cdots & \cdots & -a_{K-3} & -\sum_{j=K-2}^{\infty} a_j \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b_{K-1} & 0 & 0 & \cdots & -a_0 & 1-a_1 & -\sum_{j=2}^{\infty} a_j \\
b_K & 0 & 0 & \cdots & \cdots & -a_0 & 1-\sum_{j=1}^{\infty} a_j
\end{pmatrix}.
$$

LEMMA 1. (Mustafa criterion)[11] *For an irreducible and aperiodic Markov chain $\{Z_n\}$ with state space S, a sufficient condition for ergodicity is the existence of a non-negative function $f(s)$, $s \in S$ and $\varepsilon > 0$ such that the mean drift $x_s = E[f(Z_{n+1}) - f(Z_n) \mid Z_n = s] < \infty$ for all $s \in S$ and $x_s < -\varepsilon$ for all $s \in S$ except perhaps a finite number.*

LEMMA 2. [16] *Let $\{Z_n\}$ be a irreducible Markov chain with countable state space S. If there exists a non-constant function $f : S \to [0, \infty)$ such that*

(a) $E\{f(Z_{n+1}) - f(Z_n) \mid Z_n = i\} \geq 0$ for all $i \in S$,

(b) there is an $M > 0$ such that $E\{\mid f(Z_{n+1}) - f(Z_n) \mid\mid Z_n = i\} \leq M$ for all $i \in S$,

then $\{Z_n\}$ is not ergodic.

THEOREM 1. *The imbedded Markov chain $\{Z_n = (X_n, Y_n) \mid n = 1, 2, \cdots\}$ is ergodic if and only if $h(1) = a_0^{-K} \det(\hat{A}) < 1$.*

Proof. Suppose that $a_0^{-K}\det(\hat{A}) < 1$. To apply Lemma 1, we choose test function $f(i, j) = h(i) + h(1)j$. By conditioning the number of type

I arrivals during one service time, we have for $i \geq 1, j \geq 0$,

$$
\begin{aligned}
&E[h(X_{n+1}) \mid (X_n, Y_n) = (i,j)] \\
&\qquad = a_0 h(i-1) + a_1 h(i) + \cdots + a_{K-i} h(K-1) + \sum_{m=K-i+1}^{\infty} a_m h(K) \\
&\qquad = h(i) - \frac{\lambda_2}{\mu} - \alpha \sum_{m=K-i+1}^{\infty} (m-K+i-1) a_m, \\
&E[Y_{n+1} \mid (X_n, Y_n) = (i,j)] \\
&\qquad = j + \frac{\lambda_2}{\mu} + \alpha \sum_{m=K-i+1}^{\infty} (m-K+i-1) a_m.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&E[f(Z_{n+1}) - f(Z_n) \mid Z_n = (i,j)] \\
&\quad = E[h(X_{n+1}) + h(1) Y_{n+1} \mid (X_n, Y_n) = (i,j)] - h(i) - h(1) j \\
&\quad = (h(1)-1)\frac{\lambda_2}{\mu} + (h(1)-1)\alpha \sum_{m=K-i+1}^{\infty} (m-K+i-1) a_m \\
&\quad \leq (h(1)-1)\frac{\lambda_2}{\mu}.
\end{aligned}
$$

Similarly, for $i = 0,\ j \geq 1$

$$
\begin{aligned}
&E[h(X_{n+1}) \mid (X_n, Y_n) = (0,j)] \\
&\qquad = a_1 h(1) + \cdots + a_{K-1} h(K-1) + \sum_{m=K}^{\infty} a_m h(K) \\
&\qquad = h(1) - \frac{\lambda_2}{\mu} - \alpha \sum_{m=K}^{\infty} (m-K) a_m,
\end{aligned}
$$

$$
\begin{aligned}
&E[Y_{n+1} \mid (X_n, Y_n) = (0,j)] \\
&\qquad = \frac{\lambda}{\lambda + j\nu} \left\{ j + \frac{\lambda_2}{\mu} + \alpha sum_{m=K}^{\infty} (m-K) a_m \right\} \\
&\qquad + \frac{j\nu}{\lambda + j\nu} \left\{ j - 1 + \frac{\lambda_2}{\mu} + \alpha sum_{m=K}^{\infty} (m-K) a_m \right\} \\
&\qquad = j + \frac{\lambda_2}{\mu} - \frac{j\nu}{\lambda + j\nu} + \alpha \sum_{m=K}^{\infty} (m-K) a_m.
\end{aligned}
$$

Therefore

$$\begin{aligned}
&E[f(Z_{n+1}) - f(Z_n) \mid Z_n = (0,j)] \\
&\quad = E[h(X_{n+1}) + h(1)Y_{n+1} \mid Z_n = (0,j)] - h(1)j \\
&\quad = (h(1)-1)(\frac{\lambda_2}{\mu} + \alpha \sum_{m=K+1}^{\infty} (m-K)a_m) + h(1)\frac{\lambda}{\lambda + j\nu} \\
&\quad < (h(1)-1)\frac{\lambda_2}{\mu} + h(1)\frac{\lambda}{\lambda + j\nu} \to (h(1)-1)\frac{\lambda_2}{\mu} \quad as\ j \to \infty.
\end{aligned}$$

Since $h(1) < 1$, this implies that there exists an N such that for $j \geq N$, $E[f(Z_{n+1}) - f(Z_n) \mid Z_n = (i,j)] < -\varepsilon = -\frac{1-h(1)}{2}\frac{\lambda_2}{\mu}$. By Mustafa criterion the Markov chain is ergodic.

Suppose $h(1) \geq 1$. We easily obtain, for all $i \geq 0$,

$$E\{f(Z_{n+1}) - f(Z_n) \mid Z_n = (i,j)\} \geq (h(1)-1)\frac{\lambda_2}{\mu} \geq 0,$$

$$E\{\mid f(Z_{n+1}) - f(Z_n) \mid\mid Z_n = (i,j)\} \leq (h(1)+1)\frac{\lambda}{\mu} < \infty.$$

By Lemma 2, $\{Z_n\}$ is not ergodic. □

3. Queue size distribution in steady state

If the distribution of service time is not exponential, then the stochastic process $\{(N_q(t), N_r(t)); t \geq 0\}$ is not Markov process. We introduce random variables $X(t)$ and $I(t)$, where $X(t)$ is the elapsed service time of the call in service at time t and $I(t)$ is the server state, $I(t) = 0$ if the server is idle at time t and $I(t) = 1$ otherwise. Then

$$\{(N_q(t), N_r(t), X(t), I(t)); t \geq 0\}$$

is Markov process with state space

$$\{(i,j,x,l); i = 0,1,\cdots,K, j = 0,1,\cdots, 0 \leq x < \infty, l = 0,1\}.$$

We define the probabilities,

$$\begin{aligned}
q_j(t) &= P\{N_q(t) = 0, N_r(t) = j, I(t) = 0\} \\
p_{i,j}(t,x)dx &= P\{N_q(t) = i, N_r(t) = j, x < X(t) \leq x + dx, I(t) = 1\} \\
& \qquad\qquad i = 0,1,\cdots,K.
\end{aligned}$$

We assume that the system is ergodic. Let us define steady state probabilities

$$q_j = \lim_{t\to\infty} q_j(t), \qquad p_{i,j}(x)dx = \lim_{t\to\infty} p_{i,j}(t,x)dx$$

and define $b(x) = \frac{B'(x)}{1-B(x)}$.

By the procedure of supplementary variable method, we easily obtain the following system equations in the steady state:

$$\begin{aligned}
&(3) \quad (\lambda + j\nu)q_j &=& \int_0^\infty p_{0,j}(x)b(x)dx,\\
&(4) \quad p'_{i,j}(x) &=& -(\lambda + b(x))p_{i,j}(x) + \lambda_1 p_{i-1,j}(x) + \lambda_2 p_{i,j-1}(x)\\
& & & 0 \le i \le K-1,\\
&(5) \quad p'_{K,j}(x) &=& -(\lambda_2 + \alpha\lambda_1 + b(x))p_{K,j}(x) + \lambda_1 p_{K-1,j}(x)\\
& & & +(\lambda_2 + \alpha\lambda_1)p_{K,j-1}(x),\\
&(6) \quad p_{0,j}(0) &=& \lambda q_j + (j+1)\nu q_{j+1} + \int_0^\infty p_{1,j}(x)b(x)dx,\\
&(7) \quad p_{i,j}(0) &=& \int_0^\infty p_{i+1,j}(x)b(x)dx \qquad \text{for } 1 \le i \le K-1\\
& \quad p_{K,j}(0) &=& 0,
\end{aligned}$$

and the normalization condition

$$\sum_{j=0}^{\infty}\sum_{i=0}^{K}\int_0^\infty p_{i,j}(x)dx + \sum_{j=0}^{\infty} q_j = 1,$$

where $\lambda = \lambda_1 + \lambda_2$ and $p_{i,-1}(x) = p_{-1,j}(x) = 0$ for any fixed x.

To solve equations (3)$\sim$(7), we define the partial generating functions for a complex number z with $|\, z \,| \le 1$,

$$\begin{aligned}
Q(z) &= \sum_{j=0}^{\infty} q_j z^j\\
P_i(z,x) &= \sum_{j=0}^{\infty} p_{i,j}(x) z^j\\
P_i(z) &= \sum_{j=0}^{\infty}\left\{\int_0^\infty p_{i,j}(x)dx\right\} z^j \qquad 0 \le i \le K
\end{aligned}$$

and let

$$P(z,x) = \sum_{i=0}^{K} P_i(z,x), \qquad P(z) = \sum_{i=0}^{K} P_i(z).$$

Note that $Q(z)$ and $P_i(z)$ are the partial generating function of the number of calls in the retrial group in steady state when the server is idle and when the server is busy with i calls in the priority queue, respectively.

Multiplying both side of (3)$\sim$(7) by z^j respectively and summing over j, we obtain the following basic equations after some algebraic calculations:

$$
\begin{aligned}
&(8) & 0 &= -(\lambda Q(z) + \nu z Q'(z)) + \int_0^\infty P_0(z,x)b(x)dx,\\
&(9) & \frac{\partial P_i(z,x)}{\partial x} &= -(\lambda - \lambda_2 z + b(x))P_i(z,x) + \lambda_1 P_{i-1}(z,x),\\
& & & \quad 0 \le i \le K-1,\\
&(10) & \frac{\partial P_K(z,x)}{\partial x} &= -((\lambda_2 + \alpha\lambda_1)(1-z) + b(x))P_K(z,x)\\
& & & \quad +\lambda_1 P_{K-1}(z,x),\\
&(11) & P_0(z,0) &= \lambda Q(z) + \nu Q'(z) + \int_0^\infty P_1(z,x)b(x)dx,\\
&(12) & P_i(z,0) &= \int_0^\infty P_{i+1}(z,x)b(x)dx \qquad 1 \le i \le K-1,\\
& & P_K(z,0) &= 0,
\end{aligned}
$$

and normalization condition

$$(13) \qquad \sum_{i=0}^{K} P_i(z)\mid_{z=1} + Q(z)\mid_{z=1} = 1.$$

By recursively solving the equations (9) and (10), we obtain

$$
\begin{aligned}
(14)\, P_i(z,x) &= \left\{ \sum_{j=0}^{i} \frac{(\lambda_1 x)^j}{j!} P_{i-j}(z,0) \right\} (1-B(x)) e^{-(\lambda-\lambda_2 z)x}, \\
&\quad 0 \le i \le K{-}1, \\
P_K(z,x) &= \left\{ \sum_{i=0}^{K-1} (1-\alpha+\alpha z)^{-i-K} P_{K-1-i}(z,0) \right. \\
&\quad \times \left(1 - \sum_{j=0}^{i} e^{-\lambda_1(1-\alpha+\alpha z)x} \frac{(\lambda_1(1-\alpha+\alpha z)x)^{-(i-j)}}{(i-j)!} \right) \\
(15) \qquad &\quad \left. + P_K(z,0) \right\} (1-B(x)) e^{-(\lambda_2+\lambda_1\alpha)(1-z)x}.
\end{aligned}
$$

For simplicity of notation, let

$$
\begin{aligned}
\beta_j(z) &= \int_0^\infty \frac{(\lambda_1 x)^j}{j!} (1-B(x)) e^{-(\lambda-\lambda_2 z)x} dx, \qquad 0 \le i \le K-1, \\
\gamma_0(z) &= \int_0^\infty e^{-(\lambda_2+\lambda_1\alpha)(1-z)x} (1-B(x)) dx = \frac{1-b^*(\lambda_2+\lambda_1\alpha)(1-z))}{(\lambda_2+\lambda_1\alpha)(1-z))}.
\end{aligned}
$$

Integrating eqs.(14) and (15) from $x=0$ to $x=\infty$, we obtain

$$
\begin{aligned}
(16) \qquad P_i(z) &= \sum_{j=0}^{i} \beta_j(z) P_{i-j}(z,0) \qquad 0 \le i \le K-1, \\
(17) \qquad P_K(z) &= \sum_{i=0}^{K-1} (1-\alpha+\alpha z)^{i-K} P_{K-1-i}(z,0) \\
&\quad \times \left\{ \gamma_0(z) - \sum_{j=0}^{i} (1-\alpha+\alpha z)^j \beta_j(z) \right\}.
\end{aligned}
$$

Now we need to find $P_i(z,0)$. Substituting eqs.(14) and (15) into eqs.(8), (11) and (12), we obtain

$$
\begin{aligned}
(18)\qquad P_0(z,0) &= \frac{\lambda Q(z)+\nu z Q'(z)}{b^*(\lambda-\lambda_2 z)},\\
(19)\qquad P_1(z,0) &= c_1(z)P_0(z,0)-\frac{\lambda Q(z)+\nu Q'(z)}{b^*(\lambda-\lambda_2 z)},\\
(20)\qquad P_i(z,0) &= \sum_{j=0}^{i-1} c_{i-j}P_j(z,0) \qquad 2\le i\le K-1,\\
P_K(z,0) &= 0,
\end{aligned}
$$

where

$$
c_i(z)=\begin{cases} \dfrac{1+\lambda_1 b^{*'}(\lambda-\lambda_2 z)}{b^*(\lambda-\lambda_2 z)} & \text{if } i=1,\\ \dfrac{-(\lambda_1)^i b^{*(i)}(\lambda-\lambda_2 z)}{i!\,b^*(\lambda-\lambda_2 z)} & \text{if } i\ge 2.\end{cases}
$$

LEMMA 3. *Let $y_i(z)$ be the coefficient of η^i in Taylor series expansion of*

$$
\frac{1}{b^*(\lambda-\lambda_2 z-\lambda_1\eta)-\eta},\quad |\, z \,|\le 1,\quad i=0,1,\cdots .
$$

Then we have that for $i=0,1,\cdots,K-1$,

$$
(21)\; P_i(z,0) = (y_i(z)-y_{i-1}(z))\lambda Q(z)+(zy_i(z)-y_{i-1}(z))\nu Q'(z).
$$

Proof. It is known (Choi and Park [2]) that for each z, $|\, z\,|\le 1$, $b^*(\lambda-\lambda_2 z-\lambda_1\eta)-\eta=\eta$ has no solution in a neighborhood of $\eta=0$. So $y_i(z), i=0,1,\cdots$ are well defined for $|\, z\,|\le 1$.

$y_i(z)$'s satisfy the following recurrence relation:

$$
y_{-1}(z)=0,\;\; y_0(z)=\frac{1}{b^*(\lambda-\lambda_2 z)},\;\; y_i(z)=\sum_{j=0}^{i-1} c_{i-j}(z)y_j(z),\;\; i=1,2,\cdots .
$$

Using this relation and eqs.(18)~(20), we easily obtain for $i=0,1,\cdots$, $K-1$,

$$
P_i(z,0)=(y_i(z)-y_{i-1}(z))\lambda Q(z)+(zy_i(z)-y_{i-1}(z))\nu Q'(z). \qquad \square
$$

Substituting (21) into eqs.(16) and (17) yields the following equations:

$$
\begin{aligned}
P_i(z) &= \sum_{j=0}^{i} \beta_{i-j}(z) P_j(z,0) \\
&= \left\{ \sum_{j=0}^{i} \beta_{i-j}(z)(y_j(z) - y_{j-1}(z)) \right\} \lambda Q(z) \\
&\quad + \left\{ \sum_{j=0}^{i} \beta_{i-j}(z)(zy_j(z) - y_{j-1}(z)) \right\} \nu Q'(z), \\
&\quad 0 \le i \le K-1
\end{aligned} \tag{22}
$$

$$
P_K(z) = D_K(z)\lambda Q(z) + \hat{D}_K(z)\nu Q'(z), \tag{23}
$$

where

$$
\begin{aligned}
D_K(z) &= \sum_{i=0}^{K-1} (1-\alpha+\alpha z)^{-i-1}(y_i(z) - y_{i-1}(z)) \\
&\quad \times \left\{ \gamma_0(z) - \sum_{j=0}^{K-1-i} (1-\alpha+\alpha z)^j \beta_j(z) \right\},
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
\hat{D}_K(z) &= \sum_{i=0}^{K-1} (1-\alpha+\alpha z)^{-i-1}(zy_i(z) - y_{i-1}(z)) \\
&\quad \times \left\{ \gamma_0(z) - \sum_{j=0}^{K-1-i} (1-\alpha+\alpha z)^j \beta_j(z) \right\}.
\end{aligned} \tag{25}
$$

It remains to find $Q(z)$. Integrating eqs.(9) and (10) from $x = 0$ to $x = \infty$, we obtain

$$
\int_0^\infty P_i(z,x)b(x)dx = P_i(z,0) - (\lambda - \lambda_2 z)P_i(z) + \lambda_1 P_{i-1}(z), \quad 0 \le i \le K-1, \tag{26}
$$

$$
\int_0^\infty P_K(z,x)b(x)dx = -(\lambda_2 + \alpha\lambda_1)(1-z)P_K(z) + \lambda_1 P_{K-1}(z). \tag{27}
$$

Substituting eqs.(26), (27) into (8), (11) and (12), we obtain

$$
\begin{aligned}
(28)\qquad 0 &= -(\lambda Q(z)+\nu z Q'(z))+P_0(z,0)-(\lambda-\lambda_2 z)P_0(z),\\
(29)\qquad P_0(z,0) &= \lambda Q(z)+\nu z Q'(z)+P_1(z,0)\\
&\quad -(\lambda-\lambda_2 z)P_1(z)+\lambda_1 P_0(z),\\
(30)\qquad P_i(z,0) &= P_{i+1}(z,0)-(\lambda-\lambda_2 z)P_{i+1}(z)+\lambda_1 P_i(z)\\
&\qquad 1\le i\le K-2,\\
(31)P_{K-1}(z,0) &= P_K(z,0)-(\lambda_2+\alpha\lambda_1)(1-z)P_K(z)+\lambda_1 P_{K-1}(z).
\end{aligned}
$$

Adding up (28) $\sim$(31), we obtain

$$\nu Q'(z)(1-z)=\lambda_2(1-z)P(z)+\alpha\lambda_1 P_K(z)(1-z),$$

$$\nu Q'(z)=\lambda_2 P(z)+\alpha\lambda_1 P_K(z), \qquad\qquad |\, z \,|< 1.$$

By the continuity of analytic functions, we have

$$(32)\qquad \nu Q'(z)=\lambda_2 P(z)+\alpha\lambda_1 P_K(z), \qquad\qquad |\, z \,|\le 1.$$

Substituting eqs.(22) and (23) into (32), we obtain

$$
\begin{aligned}
&\left\{1-\sum_{i=0}^{K-1}\sum_{j=0}^{i}\lambda_2\beta_{i-j}(z)(zy_j(z)-y_{j-1}(z))-(\lambda_2+\alpha\lambda_1)\hat{D}_K(z)\right\}\nu Q'(z)\\
&=\left\{\sum_{i=0}^{K-1}\sum_{j=0}^{i}\lambda_2\beta_{i-j}(z)(y_j(z)-y_{j-1}(z))+(\lambda_2+\alpha\lambda_1)D_K(z)\right\}\lambda Q(z).
\end{aligned}
$$

The general solution of this differential equation is

$$(33)\,Q(z)=\hat{C}\exp\left\{-\frac{\lambda}{\nu}\int_z^1\frac{E_K(s)+(\lambda_2+\alpha\lambda_1)D_K(s)}{1-\hat{E}_K(s)-(\lambda_2+\alpha\lambda_1)\hat{D}_K(s)}ds\right\},$$

where

$$
\begin{aligned}
(34)\qquad E_K(z) &= \sum_{i=0}^{K-1}\sum_{j=0}^{i}\lambda_2\beta_{i-j}(z)(y_i(z)-y_{j-1}(z)),\\
(35)\qquad \hat{E}_K(z) &= \sum_{i=0}^{K-1}\sum_{j=0}^{i}\lambda_2\beta_{i-j}(z)(zy_j(z)-y_{j-1}(z)).
\end{aligned}
$$

To find C, summing eqs.(9) and (10) over i, and letting $z = 1$, we can solve $P(1, x)$ as

$$P(1,x) = P(1,0)(1 - B(x)).$$

Integrating above equation from $x = 0$ to $x = \infty$, we obtain

$$P(1) \;=\; P(1,0)\mu^{-1}.$$

Since $P(1) = P\{\text{the server is busy}\} = 1 - P\{\text{idle}\} = 1 - Q(1)$, we have

$$P(1,0) \;=\; (1 - Q(1))\mu. \tag{36}$$

Set $z = 1$ in eqs. (28)~(31) and summing them, we have

$$P_0(1,0) \;=\; \lambda_1 P_0(1) + \lambda Q(1) + \nu Q'(1), \tag{37}$$

$$P_i(1,0) \;=\; \lambda_1 P_i(1) \qquad 1 \le i \le K-1, \tag{38}$$

$$P(1,0) \;=\; \lambda_1 P(1) + \lambda Q(1) + \nu Q'(1) - \lambda_1 P_K(1). \tag{39}$$

Similarly letting $z = 1$ in (21) and summing them over i, we obtain

$$P(1,0) \;=\; y_{K-1}(1)(\lambda Q(1) + \nu Q'(1)). \tag{40}$$

Also letting $z = 1$ in (32), we obtain

$$\nu Q'(1) \;=\; \lambda_2 P(1) + \alpha\lambda_1 P_K(1). \tag{41}$$

Using eqs.(36), (39), (40) and (41), we can calculate C as

$$C = Q(1) = \frac{1 - \alpha + (\alpha - \alpha\rho_1 - \rho_2) y_{K-1}(1)}{1 - \alpha + (\alpha + (1-\alpha)\rho_1) y_{K-1}(1)}, \tag{42}$$

where $\rho_1 = \frac{\lambda_1}{\mu}, \rho_2 = \frac{\lambda_2}{\mu}$.

Thus we have obtained the following results.

THEOREM 2. (a) *In steady state, when the server is idle, the partial generating function $Q(z)$ of the number of calls in the retrial group is given by*

$$\begin{aligned} Q(z) &= E[z^{N_r}; I = 0] \\ &= \frac{1 - \alpha + (\alpha - \alpha\rho_1 - \rho_2) y_{K-1}(1)}{1 - \alpha + (\alpha + (1-\alpha)\rho_1) y_{K-1}(1)} \\ &\quad \times \exp\left\{ -\frac{\lambda}{\nu} \int_z^1 \frac{E_K(s) + (\lambda_2 + \alpha\lambda_1) D_K(s)}{1 - \hat{E}_K(s) - (\lambda_2 + \alpha\lambda_1)\hat{D}_K(s)} ds \right\}. \end{aligned} \tag{43}$$

(b) The partial generating function $P_i(z)$ of the number of calls in the retrial group when the server is busy with i calls in the priority queue in steady state is given by

$$
\begin{aligned}
(1)\qquad P_i(z) &= E[z^{N_r}; N_q = i, I = 1] \\
&= \lambda Q(z) \sum_{j=0}^{i} \beta_{i-j}(z) \Bigg\{ (y_j(z) - y_{j-1}(z)) \\
&\quad + \frac{(zy_j(z) - y_{j-1}(z))(E_K(z) + (\lambda_2 + \alpha\lambda_1)D_K(z)}{1 - \hat{E}_K(z) - (\lambda_2 + \alpha\lambda_1)\hat{D}_K(z)} \Bigg\}, \qquad (44) \\
&\quad 0 \le i \le K - 1,
\end{aligned}
$$

$$
\begin{aligned}
(2) P_K(z) &= E[z^{N_r}; N_q = K, I = 1] \\
&= \lambda Q(z) \left\{ \frac{\hat{D}_K(z)(E_K(z) + (\lambda_2 + \alpha\lambda_1)D_K(z))}{1 - \hat{E}_K(z) - (\lambda_2 + \alpha\lambda_1)\hat{D}_K(z)} + D_K(z) \right\}, \qquad (45)
\end{aligned}
$$

where $D_K(z), \hat{D}_K(z), E_K(z)$ and $\hat{E}_K(z)$ are given by (24), (25), (34) *and* (35) *respectively.*

REMARK 1. As we expected, we will show that the probability that the server is idle converges to $1-\rho$ as $K \to \infty$. Note that the probability that the server is idle is equal to $1 - \rho$ when $K = \infty$ (Choi and Park[2]). The function $\sum_{i=0}^{\infty} y_i(1)\eta^i = \frac{1}{(b^*(\lambda_1 - \lambda_1\eta) - \eta)}$ is analytic in $|\eta| < 1$, by the Abelian theorem, we have

$$\lim_{K\to\infty} y_{K-1}(1) = \lim_{\eta\to 1} \frac{1-\eta}{b^*(\lambda_1 - \lambda_1\eta) - \eta} = (1-\rho_1)^{-1}.$$

Therefore we conclude that

$$
\begin{aligned}
&P\{\text{server is idle}\} \\
&= Q(1) = \frac{1 - \alpha + (\alpha - \alpha\rho_1 - \rho_2)y_{K-1}(1)}{1 - \alpha + (\alpha + (1-\alpha)\rho_1)y_{K-1}(1)} \\
&\to \frac{1 - \alpha + (\alpha - \alpha\rho_1 - \rho_2)(1-\rho_1)^{-1}}{1 - \alpha + (\alpha + (1-\alpha)\rho_1)(1-\rho_1)^{-1}} = 1 - \rho, \qquad \text{as } K \to \infty.
\end{aligned}
$$

REMARK 2. For a special case of $\lambda_1 = 0$, our model is reduced to the classical retrial queueing system (Falin[11]).

In this case

$$\begin{aligned}
\beta_0(z) &= \int_0^\infty (1-B(x))e^{-\lambda_2(1-z)x}dx \\
&= \frac{1-b^*(\lambda_2-\lambda_2 z)}{\lambda_2(1-z)} = \gamma_0(z), \\
\beta_i(z) &= 0, \qquad\qquad i \geq 1, \\
y_i(z) &= (b^*(\lambda_2-\lambda_2 z))^{-(i+1)} \qquad 1 \leq i \leq K-1.
\end{aligned}$$

So we have

$$\begin{aligned}
D_K(z) &= \hat{D}_K(z) = 0, \\
E_K(z) &= \lambda_2\beta_0(z)\sum_{i=0}^{K-1}(y_i(z)-y_{i-1}(z)) = \frac{1-b^*(\lambda_2-\lambda_2 z)}{(1-z)b^{*K}(\lambda_2-\lambda_2 z)}, \\
\hat{E}_K(z) &= \lambda_2\beta_0(z)\sum_{i=0}^{K-1}(zy_i(z)-y_{i-1}(z)) = 1-\frac{b^*(\lambda_2-\lambda_2 z)-z}{(1-z)b^{*K}(\lambda_2-\lambda_2 z)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
Q(z) &= E[z^{N_r}; I=0] \\
&= (1-\rho_2)exp\{-\frac{\lambda}{\nu}\int_z^1 \frac{1-b^*(\lambda_2-\lambda_2 s)}{b^*(\lambda_2-\lambda_2 s)-s}ds\}, \\
P(z) &= E[z^{N_r}; I=1]\frac{1-b^*(\lambda_2-\lambda_2 z)}{b^*(\lambda_2-\lambda_2 z)-z}Q(z).
\end{aligned}$$

These agree with equations (1.23) and (1.25) in Falin [11].

REMARK 3. When $\alpha = 0$, our model is identical to the $M/G/1$ retrial queueing system with two types of calls and finite capacity[6].

In this case

$$\begin{aligned}
\nu Q'(z) &= \lambda_2 P(z), \\
\beta_0(z) &= \frac{1-b^*(\lambda-\lambda_2 z)}{\lambda-\lambda_2 z}, \\
E_k(z)+\lambda_2 D_K(z) &= \frac{y_{K-1}(z)(1-b^*(\lambda_2-\lambda_2 z))}{1-z}, \\
\hat{E}_k(z)+\lambda_2\hat{D}_K(z) &= \frac{(1-b^*(\lambda_2-\lambda_2 z))}{1-z}\left\{y_{K-1}(z)-(1-z)\sum_{i=0}^{K-1}y_i(z)\right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
Q(z) &= E[z^{N_r}; I=0] \\
&= \frac{1-\rho_2 y_{K-1}}{1+\rho_1 y_{K-1}} \exp\left\{-\frac{\lambda}{\nu}\int_z^1 \frac{y_{K-1}(s)(1-b^*(\lambda_2-\lambda_2 s))}{D(s)} ds\right\}, \\
P_0(z) &= E[z^{N_r}; N_q=0, I=1] \\
&= \frac{\lambda(1-z)(1-b^*(\lambda-\lambda_2 z))B(z)Q(z)}{(\lambda-\lambda_2 z)b^*(\lambda-\lambda_2 z))D(z)},
\end{aligned}$$

where

$$\begin{aligned}
D(z) &= (1-z)-(1-b^*(\lambda_2-\lambda_2 z))[y_{K-1}(z)-(1-z)\sum_{i=0}^{K-1} y_i(z)], \\
B(z) &= 1+(1-b^*(\lambda_2-\lambda_2 z))(\sum_{i=0}^{K-1} y_i(z)-y_{K-1}(z)).
\end{aligned}$$

These agree with equations (3.14), (3.16a) in Choi and K. B. Choi [6].

4. System performance measures and numerical examples

(1) The probability P_l that type I calls are blocked in priority queue and leave the system:

$$\begin{aligned}
P_l &= (1-\alpha)P_K(1) \\
&= (1-\alpha)\frac{\lambda}{\lambda_1}\frac{1-\rho_2 y_{K-1}(1)}{1+(\rho_1+\frac{\alpha}{1-\alpha})y_{K-1}(1)}.
\end{aligned}$$

(2) The system throughput P_u:

$$P_u = 1-Q(1) = \frac{\rho y_{K-1}(1)}{1-\alpha+(\alpha+(1-\alpha)\rho_1)y_{K-1}(1)}.$$

(3) Mean number $E[N_q]$ of calls in the priority queue :

$$\begin{aligned}
\sum_{i=1}^{K-1} iP_i(1) &= \frac{1}{\lambda_1}\sum_{i=1}^{K-1} iP_i(1,0)\\
&= \frac{1}{\lambda_1}\sum_{i=1}^{K-1} i(y_i(1)-y_{i-1}(1))(\lambda Q(1)+\nu Q'(1))\\
&= \frac{1}{\lambda_1}\frac{1-Q(1)}{\mu^{-1}y_{K-1}}\sum_{i=1}^{K-1} i(y_i(1)-y_{i-1}(1))\\
&= \frac{\lambda}{\lambda_1}\frac{\sum_{i=1}^{K-1} i(y_i(1)-y_{i-1}(1))}{1-\alpha+(\alpha+(1-\alpha)\rho_1)y_{K-1}(1)},\\
E[N_q] &= \sum_{i=1}^{K} iP_i(1)\\
&= \sum_{i=1}^{K-1} iP_i(1)+KP_K(1)\\
&= \frac{\lambda}{\lambda_1}\frac{(K+K\rho_1 y_{K-1}(1)-\sum_{i=0}^{K-1} y_i(1)}{1-\alpha+(\alpha+(1-\alpha)\rho_1)y_{K-1}(1)}.
\end{aligned}$$

(4) Mean waiting time of type I calls in the priority queue W_p:
By Little's theorem

$$W_p = \frac{E[N_q]}{\lambda_1(1-P_K(1))}.$$

Now we give numerical examples on the some performance measures. Assume that the mean service time is 1 and the retrial rate $\nu = 0.3$. In Figure 2, the service time distribution was taken as hyper-exponential with parameter (1/3,2/3), in Figure 3 and Figure 4, we consider the service time distribution as exponential $exp(1)$.

Figure 2 displays the loss probability of type I calls for two cases ($\alpha = 0$ and $\alpha = 0.3$) versus the capacity K and arrival rate of type I calls under a fixed $\lambda_2 = 0.1$. For each case, the loss probability decreases as the capacity K increases and the arrival rate of type I calls decreases.

Figure 3 displays the loss probability of type I calls as functions of the arrival rate λ_2 under the parameters: $K = 8$ and $\lambda_1 = 0.4\lambda_2$. This figure shows that the loss probability increases as the arrival rate of type

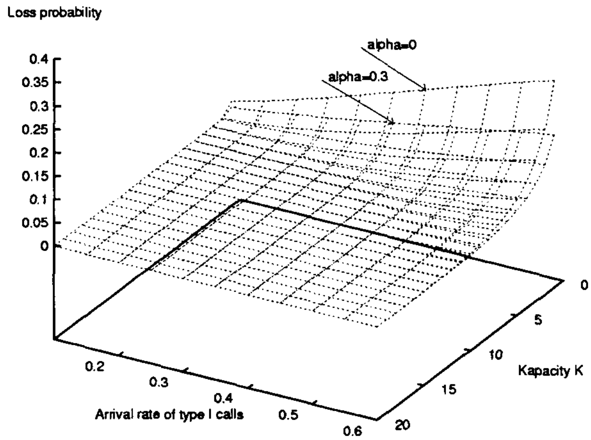

FIGURE 2. Loss probability of type I calls: $Hexp(1/3, 2/3)$ service time, $\nu = 0.3, \lambda_2 = 0.1$

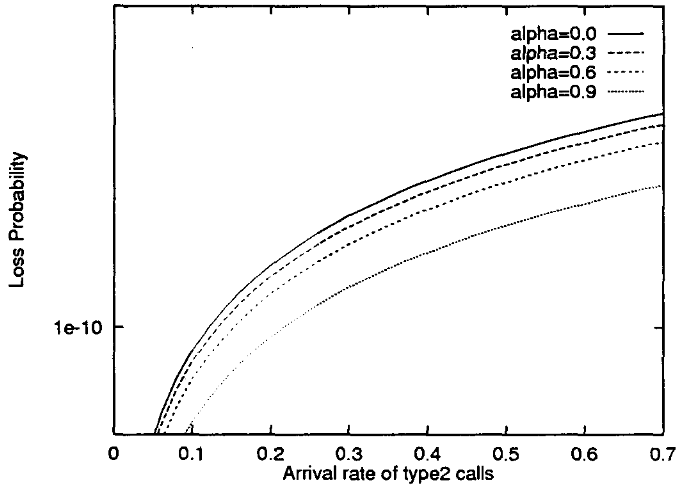

FIGURE 3. Loss probability of type I calls: $exp(1)$ service time, $K = 8$, $\lambda_1 = 0.4\lambda_2$

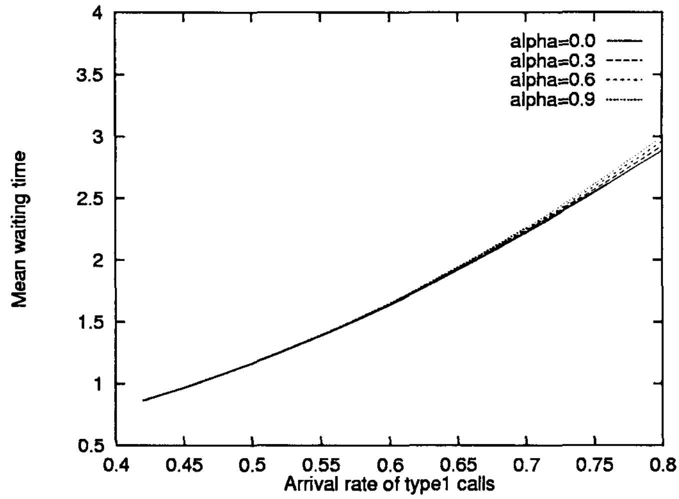

FIGURE 4. Mean waiting time of type I calls in priority queue: $exp(1)$ service time, $K = 8$, $\lambda_2 = 0.2\lambda_1$

II calls increases, and decreases as the probability α of entering to retrial group increases.

Figure 4 displays the mean waiting time of type I calls in priority queue as functions of the arrival rate λ_1 under the parameters: $K = 8$ and $\lambda_2 = 0.2\lambda_1$. This figure, shows that the mean waiting time of type I calls increases as the arrival rate of type I calls increases, but has no a great difference according to the probability α.

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