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A NEW LOOK AT THE FUNDAMENTAL THEOREM OF ASSET PRICING

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ABSTRACT. In this paper we consider a security market whose asset price process is a vector semimartingale. The market is said to be *fair* if there exists an equivalent martingale measure for the price process, deflated by a numeraire asset. It is shown that the fairness of a market is invariant under the change of numeraire. As a consequence, we show that the characterization of the fairness of a market is reduced to the case where the deflated price process is bounded. In the latter case a theorem of Kreps (1981) has already solved the problem. By using a theorem of Delbaen and Schachermayer (1994) we obtain an intrinsic characterization of the fairness of a market, which is more intuitive than Kreps' theorem. It is shown that the arbitrage pricing of replicatable contingent claims is independent of the choice of numeraire and equivalent martingale measure. A sufficient condition for the fairness of a market, modeled by an Itô process, is given.

1. Introduction

In the early 70's Black and Scholes (1973) made a breakthrough in option pricing theory by deriving the celebrated Black-Scholes formula for pricing European options via a "hedge approach". This work was further elaborated and extended by Merton (1973). Cox and Ross (1976) is the overture of a modern theory of option pricing—the risk-neutral valuation or arbitrage pricing. A key step in this direction was made

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in Harrison and Kreps (1979). They remarked that the hedge approach is not mathematically rigorous unless one excludes doubling-like strategies. Harrison and Kreps imposed some "admissibility" condition on the trading strategy and showed that the existence of an equivalent martingale measure for the deflated price processes implies the absence of arbitrage. Since then many attempts have been devoted to show the converse statement. Harrison and Pliska (1981) solved this problem in discrete-time and finite-state case. This result is referred to as the *fundamental theorem of asset pricing*. In the general state and discrete-time with finite and infinite horizon case, this problem has been solved by Dalang-Morton-Willinger (1990) and Schachermayer (1994) respectively. However, in the continuous-time case and the discrete-time with infinite horizon case the absence of arbitrage is no longer a sufficient condition for the existence of an equivalent martingale measure. A "no-free-lunch" condition, slightly stronger than no-arbitrage condition, was introduced by Kreps (1981). Under a mild but irrelevant separability assumption Kreps proved that if the deflated price process is bounded then the market is fair if and only if the market has no free-lunch. See Schachermayer (1994) for a transparent proof of this result. Without knowing this result of Kreps, the problem was attacked by Stricker (1990), who discovered that a result of Yan (1980) (or more precisely, the method of its proof) is an appropriate tool for solving the problem. The result of Stricker was re-examinated and extended by Delbaen (1992), Kusuoka (1993), Lakner (1993), Delbaen and Schachermeyer (1994), Frittelli and Lakner (1994).

In this paper we consider a semimartingale model for a market. The market is said to be *fair* if there exists an equivalent martingale measure for the deflated price process. In section 2 we show that the fairness of a market is invariant under the change of numeraire and give a characterization of self-financing strategies. In section 3, by augmenting the original market with a new asset we show that the characterization of the fairness of a market can be reduced to the case, where the deflated price process is bounded. By using a theorem of Delbaen and Schachermayer (1994) we obtain an intrinsic characterization of the fairness of a market. If the asset price process is continuous, a theorem of Delbaen (1992) implies a more elegant result. In section 4 we show that a fair market has no arbitrage with allowable strategies and the arbitrage pricing of replicatable contingent claims is independent of the choice of numeraire and equivalent martingale measure. Finally, in section 5 we give a sufficient condition for the fairness of a market modeled by an Itô process.

2. The characterization of self-financing strategies and fair market

We fix a finite time-horizon $[0, T]$ and consider a security market which consists of $m+1$ assets whose price processes $(S_t^i), i = 0, \cdots, m$ are assumed to be strictly positive semimartingales, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ satisfying the usual conditions. Moreover, we assume that $\mathcal{F}_0$ is the trivial σ-algebra. For notational convenience, we take asset 0 as the numeraire asset. We set $\gamma_t \hat{=} (S_t^0)^{-1}$ and call γ_t the *deflator* at time t. We set $S_t = (S_t^1, \cdots, S_t^m)$ and $\widetilde{S}_t = (\widetilde{S}_t^1, \cdots, \widetilde{S}_t^m)$, where $\widetilde{S}_t^i = \gamma_t S_t^i, 1 \leq i \leq m$. We call $(\widetilde{S}_t)$ the *deflated* price process of the assets. Note that the deflated price process of asset 0 is the constant 1.

The continuous trading is modeled by a stochastic integral. In order to be able to define a trading strategy we need the notion of integration w.r.t. a vector-valued semimartingale (see Jacod (1980)). Such integral is defined globally and not componentwise. A basic fact is that a vector-valued predictable process H is integrable w.r.t. a vector-valued semimartingale X if and only if the sequence $(I_{[|H| \leq n]} H).X$ converges in the semimartingale topology. In this case the limit gives the integral $H.X$. Consequently, if $H = (H^0, \cdots, H^m)$ is integrable w.r.t. a semimartingale $(X^0, \cdots, X^m)$ and H^0 is integrable w.r.t. X^0, then we have

$$H.(X^0, \cdots, X^m) = H^0.X^0 + (H^1, \cdots, H^m).(X^1, \cdots, X^m). \tag{2.1}$$

A *trading strategy* is a $\mathbf{R}^{m+1}$-valued $\mathcal{F}_t$-predictable process $\phi = \{\theta^0, \theta\}$, where

$$\theta(t) = (\theta^1(t), \cdots, \theta^m(t)),$$

such that ϕ is integrable w.r.t semimartingale (S^0, S) with $S = (S^1, \cdots, S^m)$. $\theta^i(t)$ represents the numbers of units of asset i held at time t. This notion of trading strategy is not very realistic. However it is convenient for mathematical studies. The wealth $V_t(\phi)$ at time t of a trading strategy $\phi = \{\theta^0, \theta\}$ is

$$V_t(\phi) = \theta^0(t) S_t^0 + \theta(t) \cdot S_t, \tag{2.2}$$

where $\theta(t)\cdot S_t=\sum_{i=1}^m \theta^i(t)S_t^i$. The deflated wealth at time t is $\widetilde{V}_t(\phi)=V_t(\phi)\gamma_t$. A trading strategy $\{\theta^0,\theta\}$ is said to be *self-financing*, if

$$V_t(\phi)=V_0(\phi)+\int_0^t \phi(u)d(S_u^0,S_u). \tag{2.3}$$

In this paper we always use notation $\int_0^t H_u dX_u$ or $(H.X)_t$ to stand for the integral of H w.r.t. X over the interval $(0,t]$. In particular, we have $(H.X)_0=0$.

It is easy to see that for any given $\mathbf{R}^m$-valued predictable process θ which is integrable w.r.t (S_t) and a real number x there exists a real-valued predictable process (θ_t^0) such that $\{\theta^0,\theta\}$ is a self-financing strategy with initial wealth x.

A process $\{\theta^0,\theta\}$ is said to be *elementary*, if there exist a finite partition of $[0,T]$: $0=t_0\le t_1<\cdots<t_n=T$ and a sequence of $\mathbf{R}^{m+1}$-valued random variables $(\xi_1,\cdots,\xi_n)$, with each ξ_i being $\mathcal{F}_{t_{i-1}}$-measurable, such that

$$\theta^i(t)=\sum_{k=1}^n \xi_k^i I_{(t_{k-1},t_k]}(t),\quad t\in[0,T],\ 0\le i\le m.$$

If we take stopping times $t_k's$ instead of deterministic times, the corresponding process is said to be *simple.* If furthermore $(\xi_1,\cdots,\xi_n)$ are elementary random variables (i.e. taken only a finite number of values), the corresponding process is said to be *very simple.*

DEFINITION 2.1. A security market is said to be *fair* if there exists a probability measure $\mathbf{Q}$ equivalent to the "objective" probability measure $\mathbf{P}$ such that the deflated price processes $(\widetilde{S}_t)$ is a (vector-valued) $\mathbf{Q}$-martingale. We call such a $\mathbf{Q}$ an *equivalent martingale measure* for the market.

We denote by $\mathcal{M}^j$ the set of all equivalent martingale measures for the market, if asset j is taken as the numeraire asset.

The following theorem shows that the definition of fair market does not depend on the choice of numeraire.

THEOREM 2.2. *The fairness of a market is invariant under the change of numeraire.*

Proof. Assume that $\mathcal{M}^0 \neq \emptyset$. For a $\mathbf{P}^* \in \mathcal{M}^0$ we define a probability measure $\mathbf{Q}$ by

$$\frac{d\mathbf{Q}}{d\mathbf{P}^*} = \frac{S_0^0}{S_0^j}(S_T^0)^{-1}S_T^j. \tag{2.4}$$

We denote $\mathbf{Q}$ by $h_j(\mathbf{P}^*)$. We are going to show that h_j is a bijection from $\mathcal{M}^0$ onto $\mathcal{M}^j$. Let $\gamma_t' = (S_t^j)^{-1}$ and put

$$\widehat{S}_t^i = \gamma_t' S_t^i, \quad 0 \leq i \leq m.$$

Since $(S_t^0)^{-1}S_t^j = \widetilde{S}_t^j$ is a $\mathbf{P}^*$-martingale, we must have

$$M_t := \mathbf{E}^*\left[\frac{d\mathbf{Q}}{d\mathbf{P}^*}\Big|\, \mathcal{F}_t\right] = \frac{S_0^0}{S_0^j}(S_t^0)^{-1}S_t^j, \quad 0 \leq t \leq T. \tag{2.5}$$

From the fact that

$$M_t\widehat{S}_t^i = M_t(S_t^j)^{-1}S_t^i = \frac{S_0^j}{S_0^0}\widetilde{S}_t^i, \quad 0 \leq i \leq m$$

we know that $\mathbf{Q} \in \mathcal{M}^j$. The theorem is proved. □

A strategy is said to be *admissible*, if its wealth process is non-negative. A strategy is said to be *tame*, if its deflated wealth process is bounded from below by some real constant. The weakness of the notion of tame strategy is that it is not invariant under the change of numeraire. Moreover, all bounded elementary or simple strategies are not tame. We propose below to extend the notion of tame strategy to a notion of "allowable strategy".

DEFINITION 2.3. A strategy $\phi = \{\theta^0, \theta\}$ is said to be *allowable*, if there exists a positive constant c such that the wealth $(V_t(\phi))$ at any time t is bounded from below by $-c\sum_{i=0}^m S_t^i$.

It is easy to see that all bounded elementary or simple strategies are allowable, and the notion of allowable strategy does not involve the numeraire.

DEFINITION 2.4. A market is said to have no arbitrage with allowable strategies if there exists no allowable self-financing strategy with initial wealth zero and a non-negative terminal wealth V_T such that $\mathbf{P}(V_T > 0) > 0$.

A key point of arbitrage pricing of contingent claims is the following characterization of the self-financing strategy.

THEOREM 2.5. *A strategy $\phi = \{\theta^0, \theta\}$ is self-financing if and only if its wealth process (V_t) satisfies*

$$d\widetilde{V}_t = \theta(t) d\widetilde{S}_t, \tag{2.6}$$

where $\widetilde{V}_t = V_t \gamma_t$. In particular, the deflated wealth process of an allowable self-financing strategy is a local $\mathbf{Q}$-martingale and a $\mathbf{Q}$-supermartingale for any $\mathbf{Q} \in \mathcal{M}^0$.

Proof. Assume that $\phi = \{\theta^0, \theta\}$ is a self-financing strategy. First of all, by Itô's formula, we have

(2.7)

$$d(1, \widetilde{S}_t) = d(\gamma_t S_t^0, \gamma_t S_t) = \gamma_t d(S_t^0, S_t) + (S_t^0, S_t) d\gamma_t + d([S^0, \gamma]_t, [S, \gamma]_t).$$

Secondly, by (2.1) we have

$$\phi(t) d(1, \widetilde{S}_t) = \theta(t) d\widetilde{S}_t. \tag{2.8}$$

Thirdly, by (2.3) we have

$$\Delta V_t = \theta^0(t) \Delta S_t^0 + \theta(t) \cdot \Delta S_t,$$

which together with (2.2) implies

$$V_{t-} = \theta^0(t) S_{t-}^0 + \theta(t) \cdot S_{t-}. \tag{2.9}$$

Finally, applying Itô's formula to the product $V_t \gamma_t$ we get from (2.3) and (2.7)-(2.9)

$$\begin{aligned} d\widetilde{V}_t &= V_{t-}(\phi) d\gamma_t + \gamma_{t-} dV_t + d[V, \gamma]_t \\ &= (\theta^0(t) S_{t-}^0 + \theta(t) \cdot S_{t-}) d\gamma_t + \gamma_{t-}(\theta^0(t), \theta(t)) d(S_t^0, S_t) \\ &\quad + \theta^0(t) d[S^0, \gamma]_t + \theta(t) \cdot d[S, \gamma]_t) \\ &= \theta(t) d\widetilde{S}_t. \end{aligned}$$

Similarly, we can prove the "if" part. □

Now assume that $\phi = \{\theta^0, \theta\}$ is an allowable self-financing strategy. By definition there exists a positive constant c such that $V_t(\phi) \geq -c \sum_{i=0}^m S_t^i$, $t \in [0, T]$. Put

$$\theta_1^i = \theta^i + c, \quad 0 \leq i \leq m, \quad \phi_1 = \{\theta_1^0, \theta_1\}.$$

Then by (2.6) we have $d\widetilde{V}_t(\phi_1) = \theta_1(t) d\widetilde{S}_t$ and

$$\widetilde{V}_t(\phi_1) = \widetilde{V}_t(\phi) + c \sum_{i=0}^m \widetilde{S}_t^i \geq 0.$$

By a theorem of Ansel and Stricker (1994), $(\widetilde{V}_t(\phi_1))$ is a local $\mathbf{Q}$-martingale and $\mathbf{Q}$-supermartingale. Thus so is $(\widetilde{V}_t)$ because $(\sum_{i=0}^{m} \widetilde{S}_t^i)$ is a $\mathbf{Q}$-martingale.

As a corollary we obtain:

THEOREM 2.6. *A fair market has no arbitrage with allowable strategies.*

Proof. Let $\mathbf{Q} \in \mathcal{M}^0$. Let $\{\theta, \theta^0\}$ be an allowable self-financing strategy with initial wealth zero. By Theorem 2.5 the deflated wealth process of ϕ is a $\mathbf{Q}$-supermartingale. Therefore, we must have $\mathbf{E}_{\mathbf{Q}}[\widetilde{V}_T] \leq 0$. So the market has no arbitrage with allowable strategies. □

3. The fundamental theorem of asset pricing

The fairness of a security market is the basis of the so-called "pricing by arbitrage". By the *fundamental theorem of asset pricing* we mean a characterization of the fairness of a market. Roughly speaking, such a characterization states that the market is fair if and only if the market has no "free-lunch". In the literature several notions of "free-lunch" have been introduced in different circumstances. A common feature of these notions is that they involve an appropriate topological closure of the set $V - L_+^\infty$, where V is the set of all achievable gains by a certain bounded elementary (or simple) strategy. If the deflated price process is a bounded (vector-valued) semimartingale, several characterizations of the fairness are available.

Now we introduce a new asset, indexed by $m+1$, whose price process is:

$$S_t^{m+1} = \sum_{i=0}^{m} S_t^i. \tag{3.1}$$

We augment the market with this new asset. It is readily seen that the new market is fair if and only if the old one is fair. According to Theorem 2.2 in order to characterize the fairness of the new market one can choose asset $m+1$ as the numeraire asset. In doing so the deflated price process becomes bounded. This trick not only reduces the problem to the easy case but also leads to an *intrinsic* characterization of the fairness of a market in the sense that no numeraire asset is involved.

In the following we consider the augmented market and choose the new asset as the numeraire asset. We denote by (X_t^i) the deflated price process of asset i (i.e. $X_t^i = (S_t^{m+1})^{-1} S_t^i$) and set $X_t = (X_t^0, \cdots, X_t^m)$.

A theorem of Lakner (1993, Theorem 2.1) implies immediately the following characterization of the fairness of a market.

THEOREM 3.1. *Let process (X_t) be defined as above. Put*

$$V = \{(H.X)_T : H \text{ is a very simple process}\}. \tag{3.2}$$

Then the (original) market is fair if and only if

$$\overline{V - L_+^\infty} \cap L_+^\infty = \{0\}, \tag{3.3}$$

where $\overline{V - L_+^\infty}$ is the closure of $V - L_+^\infty$ in the $\sigma(L^\infty, L^1(\mathbf{P}))$-topology.

REMARK. Condition (3.3) can be interpreted as "no-free-lunch" in a certain sense. In fact, if condition (3.3) is violated, then there is an $f_0 \in L_+^\infty \setminus \{0\}$ and a net $(\phi_\alpha)_{\alpha \in J}$ of very simple self-financing strategies with initial wealth 0 such that at the terminal time the agent "throws away" the amount of money $h_\alpha S_T^{m+1}$ with $h_\alpha \in L_+^\infty$ the random variable $(S_T^{m+1})^{-1} V_T(\phi_\alpha) - h_\alpha$ becomes close to f_α w.r.t $\sigma(L^\infty, L^1(\mathbf{P}))$-topology. On the other hand, according to Schachermayer (1994) the Kreps' "no-free-lunch" condition can be stated as follows:

$$\overline{(V_0 - L_+^0) \cap L^\infty} \cap L_+^\infty = \{0\}, \tag{3.4}$$

where

$$V_0 = \{(H.X)_T : H \text{ is an elementary process}\}. \tag{3.5}$$

So the economic meaning of Lakner's no-free-lunch condition (3.3) is more convincing than the Kreps' one. We refer the reader to Kusuoka (1993) for another "no-free-lunch" condition which is similar to condition (3.3). In view of the economic meaning of no-free-lunch, an equivalent martingale measure is also called a *risk-neutral probability measure.*

As pointed out in Delbaen and Schachermayer (1994) the drawback of a variant of Kreps' theorem is twofold. First it is stated in terms of nets or topological closure, a highly non intuitive concept. Second it involves the use of very risky positions. The main theorem of Delbaen and Schachermayer (1994) remedies this drawback. By using this theorem we obtain the following intrinsic characterization of the fairness of a market.

THEOREM 3.2. *The market is fair if and only if there is no sequence* (ϕ_n) *of allowable self-financing strategies with initial wealth 0 such that* $V_T(\phi_n) \geq -\frac{1}{n}\sum_{i=0}^{m} S_T^i$, *a.s., for all* $n \geq 1$ *and such that* $V_T(\phi_n)$, *a.s., tends to a non-negative random variable* ξ *satisfying* $\mathbf{P}(\xi > 0) > 0$.

Proof. Consider the market augmented with asset $m+1$ and choose asset $m+1$ as the numeraire asset. Let $\phi = \{\phi^0, \cdots, \phi^m\}$ be an "admissible" integrand for the vector semimartingale $X = (X^0, \cdots, X^m)$, in the sense of Delbaen and Schachermayer (1994) that there is a positive constant c such that $(\phi.X)_T \geq -c$. We can introduce a predictable process ϕ^{m+1} such that ϕ together with ϕ^{m+1} constitutes a self-financing strategy with initial wealth 0 for the augmented market. By Theorem 2.5 we have

$$(S_T^{m+1})^{-1}V_T(\phi, \phi^{m+1}) = \int_0^T \phi(t)dX_t. \tag{3.6}$$

On the other hand, we have

$$V_t(\phi, \phi^{m+1}) = V_t(\phi) + \phi_t^{m+1}S_t^{m+1}, \quad 0 \leq t \leq T.$$

Thus, if we put

$$\phi'^i_t = \phi^i_t + \phi^{m+1}_t, \quad 0 \leq i \leq m,$$

then we have $V_t(\phi') = V_t(\phi, \phi^{m+1})$. Consequently, by (3.6) ϕ' is an allowable strategy for the original market. It is easy to see that ϕ' is self-financing and its initial wealth is 0. Conversely, for any allowable strategy ϕ for the original market, $\{\phi, 0\}$ is a self-financing strategy for the augmented market and we have

$$(S_t^{m+1})^{-1}V_t(\phi) = (S_t^{m+1})^{-1}V_t(\phi, 0) = \int_0^t \phi(t)dX_t.$$

Thus, by the vector versions of Theorem 1.1 and Corollary 3.7 of Delbaen and Schachermayer (1994) we can conclude the theorem. □

REMARK. According to Delbaen and Schachermayer (1994) the condition in Theorem 3.2 is called the condition of *no free lunch with vanishing risk.*

If the asset price process is continuous, a theorem of Delbaen (1992) (Theorem 5.1) gives us a more elegant characterization of the fairness of a market.

THEOREM 3.3. *Assume that the asset price process is continuous. Then the market is fair if and only if the following condition is satisfied:*

If (ϕ_n) *is a sequence of very simple self-financing strategies such that* $V_0(\phi_n) = 0, \quad |V(\phi_n)| \le S^{m+1}, \forall n \ge 1$ *and* $V_T(\phi_n)^- \to 0$ *in probability, then* $V_T(\phi_n)^+ \to 0$ *in probability.*

4. Arbitrage pricing of contingent claims in a fair market

In this section we will study the problem of the pricing of European contingent claims in a fair market. By a *(European) contingent claim* we mean a non-negative $\mathcal{F}_T$-measurable random variable. Let ξ be a contingent claim. One raises naturally a question: what is a "fair" price process of ξ? Assume that $\gamma_T\xi$ is $\mathbf{P}^*$-integrable for some $\mathbf{P}^* \in \mathcal{M}^0$. We put

$$V_t = \gamma_t^{-1}\mathbf{E}^*[\gamma_T\xi \,|\, \mathcal{F}_t]. \tag{4.1}$$

If we consider (V_t) as the price process of an asset, then the market augmented with this asset is still fair, because the deflated price process of this asset is a $\mathbf{P}^*$-martingale. So it seems that (V_t) can be considered as a candidate for a"fair" price process of ξ. However this definition of "fair" price depends on the choice of equivalent martingale measure. We will show that for replicatable contingent claims (see Definition 4.1) this definition is reasonable.

DEFINITION 4.1. Let $\mathbf{P}^* \in \mathcal{M}^0$. A European contingent claim ξ is said to be $\mathbf{P}^*$-*replicatable* (or *attainable*) if $\gamma_T\xi$ is $\mathbf{P}^*$-integrable and there exists an admissible self-financing strategy ϕ such that its terminal wealth is equal to ξ and its deflated wealth process is a $\mathbf{P}^*$-martingale (i.e. $\mathbf{E}^*[\gamma_T\xi] = \gamma_0 V_0(\phi)$). Such a strategy is called a $\mathbf{P}^*$-*hedging strategy* for ξ.

The following theorem shows that the "fair" price process of a replicatable contingent claim is uniquely determined.

THEOREM 4.2. *Let* $\mathbf{P}^*, \mathbf{P}' \in \mathcal{M}^0$ *and* ξ *be* $\mathbf{P}^*$- *and* $\mathbf{P}'$-*replicatable. Let* (V_t) *(resp.* (U_t)*) be the wealth process of a* $\mathbf{P}^*$- *(resp.* $\mathbf{P}'$-*)hedging strategy for* ξ*. Then* (V_t) *and* (U_t) *are the same. Moreover,* V_t *is given by* (4.1) *and we have*

$$V_t = \operatorname{essinf}_{\mathbf{Q}\in\mathcal{M}^0}\gamma_t^{-1}\mathbf{E}_{\mathbf{Q}}[\gamma_T\xi|\mathcal{F}_t]. \tag{4.2}$$

Proof. Put $\widetilde{V}_t = \gamma_t V_t$, $\widetilde{U}_t = \gamma_t U_t$. Then $(\widetilde{V}_t)$ is a $\mathbf{P}^*$-martingale and a $\mathbf{P}'$-supermartingale and $(\widetilde{U}_t)$ is a $\mathbf{P}'$-martingale and a $\mathbf{P}^*$-supermartingale. Note that $U_T = V_T = \xi$ and we have

$$\mathbf{E}^*[\widetilde{V}_T | \mathcal{F}_t] = \widetilde{V}_t \geq \mathbf{E}'[\widetilde{V}_T | \mathcal{F}_t] = \mathbf{E}'[\widetilde{U}_T | \mathcal{F}_t] = \widetilde{U}_t.$$

Thus we have $V_t \geq U_t$, a.s. Similarly, we have $U_t \geq V_t$, a.s. Hence $V = U$. The last assertion of the theorem is obvious. □

REMARK. According to Theorem 4.2, for a $\mathbf{P}^*$-replicatable contingent claim ξ it is natural to define its *"fair" price* at time t by (4.1). We call this method of pricing the *arbitrage pricing* (or *pricing by arbitrage*, or *risk-neutral valuation*).

The following theorem shows that the arbitrage pricing of replicatable contingent claims is independent of the choice of numeraire.

THEOREM 4.3. *Let $\mathbf{P}^* \in \mathcal{M}^0$ and ξ be a $\mathbf{P}^*$-replicatable contingent claim and ϕ be a fair hedging strategy for ξ. Then for any $0 \leq j \leq m$ ξ is an $h_j(\mathbf{P}^*)$-replicatable contingent claim, and its "fair" price process remains the same.*

Proof. We keep the notations in the proof of Theorem 2.2. We have by (4.1)

$$\mathbf{E}_{\mathbf{Q}}[\gamma'_T \xi] = \mathbf{E}^*[M_T \gamma'_T \xi] = \frac{S_0^0}{S_0^j} \mathbf{E}^*[\gamma_T \xi] = \gamma'_0 V_0.$$

This implies that a $\mathbf{P}^*$-hedging strategy for ξ is also a $\mathbf{Q}$-hedging strategy for ξ. So ξ is a $\mathbf{Q}$-replicatable contingent claim. Moreover, by the Bayes rule we have

$$\begin{aligned}(\gamma'_t)^{-1}\mathbf{E}_{\mathbf{Q}}[\gamma'_T \xi | \mathcal{F}_t] &= (\gamma'_t)^{-1} M_t^{-1} \mathbf{E}^*[M_T \gamma'_T \xi | \mathcal{F}_t] \\ &= \gamma_t^{-1} \mathbf{E}^*[\gamma_T \xi | \mathcal{F}_t].\end{aligned}$$

This proves that the "fair" price process of ξ is invariant under the change of numeraire.

Let $\mathbf{P}^* \in \mathcal{M}^0$. The market is said to be $\mathbf{P}^*$-*complete*, if every contingent claim ξ with $\gamma_T \xi$ being $\mathbf{P}^*$-integrable is $\mathbf{P}^*$-replicatable. If there exists a unique martingale measure for the market then by a general result of Jacod-Yor (1977), the market is complete. In a complete market, the fair price process of a replicatable contingent claim ξ is given by (4.1). □

5. The Itô process model

We fix a finite time-horizon T. Let $B = (B^1, \cdots, B^d)$ be a Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We denote by $(\mathcal{F}_t)$ the natural filtration of (B_t) and by $\mathcal{L}$ the set of all measurable $(\mathcal{F}_t)$-adapted processes.

We consider a financial market which consists of $m+1$ assets. The price process (S_t^i) of each asset i is assumed to be a strictly positive Itô process. Since its logarithm is also an Itô process, we can represent (S_t^i) as

$$dS_t^i = S_t^i\Big[\sigma^i(t)dB_t + \mu^i(t)dt\Big],\ S_0^i = p_i,\quad 0 \le i \le m. \tag{5.1}$$

We call $\mu = (\mu^0, \cdots, \mu^m)$ the *vector of expected rate of return* and σ the *volatility matrix.*

We specify asset 0 as the numeraire asset and set $\gamma_t := (S_t^0)^{-1}$. By Itô's formula we have

$$d\gamma_t = -\gamma_t\Big[\sigma^0(t)dB_t + (\mu^0(t) - |\sigma^0(t)|^2)dt\Big], \tag{5.2}$$

$$d\widetilde{S}_t^i = \widetilde{S}_t^i\Big[a^i(t)dB_t + b^i(t)dt\Big],\quad 1 \le i \le m, \tag{5.3}$$

where

$$a^i(t) = \sigma^i(t) - \sigma^0(t);\quad b^i(t) = \mu^i(t) - \mu^0(t) + |\sigma^0(t)|^2 - \sigma^i(t)\cdot\sigma^0(t).$$

In particular, If asset 0 is a bank account with interest rate process $(r(t))$, then

$$a^i(t) = \sigma^i(t),\quad b^i(t) = \mu^i(t) - r(t).$$

One raises naturally a question: what conditions we should impose on coefficients a and b of the Itô process $(\widetilde{S}_t)$ such that the market is fair? The following theorem gives a partial answer to this question.

THEOREM 5.1. *If the market is fair, the linear equation*

$$a(t)\psi(t) = b(t),\ dt\times d\mathbf{P}\text{-a.e., a.s., on } [0,T]\times\Omega \tag{5.4}$$

has a solution $\psi \in (\mathcal{L}^2)^d$, *where* $\mathcal{L}^2$ *stands for the set of all adapted process* ϕ *with* $\int_0^T \phi^2(u)du < \infty$. *Conversely, if*

$$\mathbf{E}\left[\exp\left\{\frac{1}{2}\int_0^T |a^i(t)|^2 dt\right\}\right] < \infty,\quad 1 \le i \le m, \tag{5.5}$$

and equation (5.4) *has a solution* $\psi \in (\mathcal{L}^2)^d$ *satisfying*

$$\mathbf{E}\left[\exp\left\{\frac{1}{2}\int_0^T |\psi(t)|^2 dt\right\}\right] < \infty, \tag{5.6}$$

then the probability measure $\mathbf{Q}$ *with Radon-Nikodym derivative* $\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathcal{E}(-\psi.B)_T$ *is an equivalent martingale measure.*

Proof. Let $\mathbf{Q} \in \mathcal{M}^0$. Put

$$M_t = \mathbf{E}\big(\frac{d\mathbf{Q}}{d\mathbf{P}}|\mathcal{F}_t\big).$$

Then (M_t) is a $\mathbf{P}$-martingale. By the martingale representation theorem for Brownian motion there exists $\phi \in (\mathcal{L}^2)^d$ such that $dM_t = \phi(t)dB_t$. Set $\psi(t) = -\phi(t)/M_t$. Then $M = \mathcal{E}(-\psi.B)$ and by Girsanov's theorem $B^*_t = B_t + \int_0^t \psi(s)ds$ is a Brownian motion under $\mathbf{Q}$. Moreover, by a Theorem of Fujisaki, M., G. Kallianpur and H. Kunita (1972), (B^*_t) has also the martingale representation property w.r.t. $(\mathcal{F}_t)$ under $\mathbf{Q}$. Thus there exists some $\sigma^* \in (\mathcal{L}^2)^{m\times d}$ such that

$$d\widetilde{S}_t = \sigma^*(t)dB^*_t = \sigma^*(t)(dB_t + \psi(t)dt).$$

According to the uniqueness of the representation of Itô process $(\widetilde{S}_t)$ and the invariance of the stochastic integral under a change of probability, from (5.3) we know that $\sigma^*(t) = \widetilde{S}_t a(t)$, $dt \times d\mathbf{P}$-a.e., a.s., and consequently, $a(t)\psi(t) = b(t)$, $dt \times d\mathbf{P}$-a.e., a.s. So $(\psi(t))$ is a solution of equation (5.4).

Now assume that a satisfies (5.5), ψ is a solution of (5.4) and verifies (5.6). By the Novikov theorem $\mathcal{E}(-\psi.B)$ is a $\mathbf{P}$-martingale. So we can define a probability measure $\mathbf{Q}$ such that $\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathcal{E}(-\psi.B)_T$. In order to prove that $(\widetilde{S}_t)$ is a $\mathbf{Q}$-martingale, it suffices to prove that the product $\mathcal{E}(-\psi.B)\widetilde{S}$ is a $\mathbf{P}$-martingale. By (5.3) we have

$$\widetilde{S}^i_t = \widetilde{S}^i_0 \exp\left\{\int_0^t [a^i(s)dB_s + b^i(s)ds] - \frac{1}{2}\int_0^t |a^i(s)|^2 ds\right\}.$$

Thus from (5.4) we know that

$$\mathcal{E}(-\psi.B)_t\widetilde{S}^i_t = \widetilde{S}^i_0 \exp\left\{\int_0^t (a^i(s) - \psi(s))dB_s - \frac{1}{2}\int_0^t |a^i(s) - \psi(s)|^2 ds\right\}.$$

Once again by the Novikov Theorem $\mathcal{E}(-\psi.B)\widetilde{S}^i$ is a $\mathbf{P}$-martingale. □

REMARK. Without condition (5.5) $\mathbf{Q}$ is an equivalent local martingale measure but not necessarily a martingale measure. We refer the reader to Ansel and Stricker (1993) for an investigation on this subject.

The following theorem provides a sufficient condition for the existence of a unique equivalent martingale measures.

THEOREM 5.2. *Assume that $m \geq d$, a satisfies (5.5) and $a^T(t)a(t)$ are non-degenerated for a.e. t, where $a^T(t)$ stands for the transpose of $a(t)$. Put $\psi(t) = (a^T(t)a(t))^{-1}a^T(t)b(t)$. If ψ satisfies (5.4) and (5.6), then there exists a unique equivalent martingale measure $\mathbf{P}^*$ for the market. Moreover, we have*

$$\mathbf{E}\Big[\frac{d\mathbf{P}^*}{d\mathbf{P}}\Big|\mathcal{F}_t\Big] = \exp\Big\{-\int_0^t \psi(s)dB_s - \frac{1}{2}\int_0^t |\psi(s)|^2 ds\Big\},\ 0 \leq t \leq T.$$

Proof. By Theorem 5.1 there exists an equivalent martingale measure. To prove the uniqueness, let $\mathbf{Q}$ be an equivalent martingale measure. There exists a $\theta \in (\mathcal{L}^2)^d$ such that $\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathcal{E}(-\theta.B)_T$. By Theorem 5.1, we have $a(t)\theta(t) = b(t)$. Consequently, applying $(a^T(t)a(t))^{-1}a^T(t)$ to the both sides of this equation we get $\theta(t) = \psi(t)$. The uniqueness is thus proved. □

REMARK. If $m = d$, then ψ satisfies (5.4) automatically. In this case if (5.5)-(5.6) are satisfied, then there exists a unique equivalent martingale measure if and only if $a(t,\omega)$ is non-singular, for $(t,\omega) \in [0,T] \times \Omega$, a.e., a.s. See Karatzas (1997).

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