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GENERALIZED WHITE NOISE FUNCTIONALS ON CLASSICAL WIENER SPACE

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ABSTRACT. In this note we reformulate the white noise calculus on the classical Wiener space $(\mathcal{C}', \mathcal{C})$. It is shown that most of the examples and operators can be redefined on $\mathcal{C}$ without difficulties except the Hida derivative. To overcome the difficulty, we find that it is sufficient to replace $\mathcal{C}$ by $L_2[0,1]$ and reformulate the white noise on the modified abstract Wiener space $(\mathcal{C}', L_2[0,1])$. The generalized white noise functionals are then defined and studied through their linear functional forms. For applications, we reprove the Itô formula and give the existence theorem of one-side stochastic integrals with anticipating integrands.

1. Introduction

The theory of generalized functions of infinite variables has been formulated in terms of Malliavin calculus and Hida calculus. The former, introduced by Malliavin[20,22], studies the calculus of generalized Wiener functionals and their applications on the classical Wiener space $(\mathcal{C}, \mathcal{B}(\mathcal{C}), w)$ while the later, also known as the white noise analysis initiated by T. Hida[6], investigates the calculus of generalized white noise functionals on the white noise space $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu)$ (see also [7, 17]), where $\mathcal{C}$ denotes the space of continuous functions which are defined on $[0,1]$ and vanish at 0 and w the Wiener measure on $\mathcal{C}$; $\mathcal{S}'$ denotes the space of tempered distributions and μ the standard Gaussian measure on $\mathcal{S}'$ and where $\mathcal{B}(\mathcal{C})$ and $\mathcal{B}(\mathcal{S}')$ denote respectively the Borel field of $\mathcal{C}$ and the Borel field of $\mathcal{S}'$. The connection between the two theory remains an interesting problem to be further studied. In this paper we are devoted

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to a reformulation of white noise analysis on the classical Wiener space and it is believed that the results might provide a major step toward the investigation on the connection between these two theories.

The basic machinery in white noise analysis is the so called S-transform which is used to define and study the generalized white noise functional (GWNF, for abbrev.). More precisely, the S-transform transforms a GWNF F in a unique way to a U-functional $U_F(\xi)$ defined on the Schwartz space $\mathcal{S}$. Then the GWNF F is defined by $S^{-1}U_F$. In our previous paper [19], it was shown that the GWNF's could be also defined and studied in terms of their linear functional forms without using inverse S-transform and, under this framework, the Hida calculus was reformulated on an arbitrary abstract Wiener space . As $\mathcal{C}$ is also regarded as an abstract Wiener space which has the space $\mathcal{C}'$ of Cameron-Martin functions as its reproducing kernel Hilbert space, it is desirable to reformulate Hida calculus on $\mathcal{C}$. It follows from the results in [19] that, without too much difficulties, most of the examples of GWNF's and their calculus can be redefined, the only difficulty that we have encountered so far is that the Hida derivative ∂_t can not be defined for all GWNF's. Notice that the Hida derivative is formally defined by $\partial_t{=}D_{1_{[t,1]}}$, the Frechét derivative in the direction of $1_{[t,1]}$, we show that this difficulty can be overcame if the abstract Wiener pair $(\mathcal{C}', \mathcal{C})$ is replaced by the new abstract Wiener pair $(\mathcal{C}', L_2[0,1])$ and choose the space $\mathcal{E}$ of mean of exponential type entire functions as test functionals defined on the complexification of $L_2[0,1]$. Then the theory of white noise analysis can be carried over here to formulate on the classical Wiener space.

2. Representation of Brownian motion and white noise

Let $\mathcal{C}$ be the collection of real-valued continuous functions f which is defined on [0,1] and satisfies $f(0) = 0$ and $\mathcal{C}'$ the subclass of $\mathcal{C}$ consisting of absolutely continuous functions f whose derivative $\dot{f}$ satisfies

$$\int_0^1 |\dot{f}(x)|^2 dx < \infty.$$

$\mathcal{C}'$ is a Hilbert space with norm $|\cdot|_0$ and inner product $< \cdot, \cdot >_0$ defined by

$$(2.1) \qquad < f, g >_0 = \int_0^1 \dot{f}(x)\dot{g}(x)dx,$$

members of which are also called Cameron-Martin functions. It is well-known that $(\mathcal{C}', \mathcal{C})$ forms an abstract Wiener space and the calculus on $\mathcal{C}$ is performed with respect to the Wiener measure w. The probability space $(\mathcal{C}, \mathcal{B}(\mathcal{C}), w)$ then serves as the underlying space in our investigations. The dual $\mathcal{C}^*$ of $\mathcal{C}$, identified as a subspace of $\mathcal{C}'$, may be characterized as follows:

FACT 2.1 [23]. $\mathcal{C}^* = \{f \in \mathcal{C}' : \dot{f}$ is a right continuous function of bounded variation and $\dot{f}(1) = 0\}$.

and, under this identification,the $(\mathcal{C}, \mathcal{C}^*)$ pairing $(\cdot, \cdot)$ is then given by

$$(f, g) = -\int_0^1 f(t)d\dot{g}(t),$$

for $f \in \mathcal{C}$ and $g \in \mathcal{C}^*$. If $f \in \mathcal{C}'$ and $g \in \mathcal{C}^*$, then we have $(f, g) = < f, g >_0$ For any $g \in \mathcal{C}^*$, $(\cdot, g)$ defines a random variable on $\mathcal{C}$ with mean zero and variance $|g|_0^2 = < g, g >_0$. In other words, $\mathbf{E}[(\cdot, g)^2] = |g|_0^2$. It follows from this isometry that if $g \in \mathcal{C}'$ and if (g_n) is a sequence in $\mathcal{C}^*$ such that $|g - g_n|_0 \to 0$ as n $\to \infty$, we define the function $< \cdot, g >_0$ as the $L^2(w)$-limit of $(\cdot, g_n)$, then $< \cdot, g >_0$ is a random variable with mean zero and variance $|g|_0^2$.

FACT 2.2. For each $t \in [0,1]$, define $b_t(s) = \min(t, s)$, then the Brownian motion $\{B(t) : t \in [0, 1]\}$ can be represented by

$$(2.2) \qquad B(t, x) = x(t) = (x, b_t),$$

for every $x \in \mathcal{C}$.

In the remaining of this section, we shall define the white noise $\{W(t)\}$ as a generalized functional. Formally, the white noise is understood as the "time derivative" of Brownian motion. In notation we write $W(t) = \dot{B}(t)$, where $\dot{B}(t, x) = \dot{x}(t)$. Let φ be a sufficient smooth function. Then formally we have

$$\begin{aligned}
&\textstyle\int_{\mathcal{C}} \dot{B}(t,x)\varphi(x)w(dx) \\
(2.3) \qquad &= \lim_{\epsilon\to 0}\int_{\mathcal{C}}(x, \frac{1}{\epsilon}(b_{t+\epsilon}-b_t))\varphi(x)w(dx) \\
&= \lim_{\epsilon\to 0}(\frac{1}{\epsilon}(b_{t+\epsilon}-b_t), D(w*\varphi)(0)).
\end{aligned}$$

Notice that $\frac{1}{\epsilon}(b_{t+\epsilon}-b_t)$ converges in L_2 to $h_t = \mathbf{1}_{[t,1]}$ as $\epsilon \to 0$. The limit (2.3) tends formally to $\langle h_t, D(w*\varphi)(0)\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the $(L_2, L_2{}^*)$ pairing. This suggests that the definition of white noise should be defined according to the following rule:

$$\dot{B}(t)(\varphi) = \langle h_t, D(w*\varphi)(0)\rangle. \tag{2.4}$$

Notice that $h_t \in L_2[0,1]\setminus\mathcal{C}$, the definition of white noise (2.4) does not make sense in general if φ is merely a smooth function defined on $\mathcal{C}$. In order to define the white noise as a generalized functional, the test functionals should have smooth extensions to $L_2 = L_2[0,1]$. This leads us to consider the "new" abstract Wiener pair $(\mathcal{C}', L_2)$ in place of the classical Wiener pair $(\mathcal{C}',\mathcal{C})$. The Wiener measure w is then extended to a measure, still denoted by w, on the Borel field $\mathcal{B}(L_2)$ naturally by setting $w(E) = w(E\bigcap\mathcal{C})$ for all $E\in\mathcal{B}(L_2)$. Observe that the identity mapping is a continuous embedding from $\mathcal{C}$ into L_2, one can identify the dual space L_2^* of L_2 as a subspace of $\mathcal{C}^*$ in the sense that, for $y\in L_2^*$ and $x\in\mathcal{C}$, then $\langle x,y\rangle = (x,y)$, where $\langle\cdot,\cdot\rangle$ is the pairing of L_2-L_2^*. Furthermore, L_2^* can be characterized as follows:

FACT 2.3. $L_2^*=\{f\in\mathcal{C}^*\colon \dot{f}$ is absolutely continuous with $\ddot{f}\in L_2$ and $f(0)=\dot{f}(1)=0\}$.

Under the above identification, we have

$$\langle x, f\rangle = -\int_0^1 x(u)\ddot{f}(u)du, \tag{2.5}$$

for $x\in L_2$ and $f\in L_2^*$, where $\langle\cdot,\cdot\rangle$ denotes the (L_2, L_2^*) pairing. The random variable $<\cdot, k>_0$ is defined similarly for $k\in\mathcal{C}'$ when the AWS $(\mathcal{C}', L_2)$ is considered.

To conclude this section, we summary the above results into a Lemma for future applications.

LEMMA 2.4. (a) *If we identify the dual space of* $\mathcal{C}'$ *by itself, then* $\mathcal{C}^*$ *and* L_2^* *are identified as subspaces of* $\mathcal{C}'$ *according to* Fact 2.1 *and* Fact 2.3. *Moreover, we have*

$$L_2^* \subset \mathcal{C}^* \subset \mathcal{C}' \equiv \mathcal{C}' \subset \mathcal{C} \subset L_2,$$

where the inclusive relation are in fact densely embeddings.

(b) *The Brownian motion is represented by*

$$B(t,x) = (x, b_t),$$

for all $x \in \mathcal{C}$ *or by*

$$B(t,x) = < x, b_t >_0,$$

for almost all $x \in L_2$.

(c) *For* $x \in \mathcal{C}'$ *and* $y \in L_2^*$, *the following identity holds:*

$$(x,y) = \langle x,y \rangle = < x,y >_0.$$

(d) *The definition of white noise as a linear functional is given by*

$$\dot{B}(t)(\varphi) = \langle h_t, D(w * \varphi)(0) \rangle,$$

where $h_t = \mathbf{1}_{[t,1]}$ *and* φ *is a test functional, to be specified in the next section.*

NOTATIONS.

$\mathcal{L}(X,Y)$: the bounded linear operators from the Banach space X into the Banach space Y.

$\mathcal{L}^n(X)$: the continuous n-linear transform from $X^n = X \times \cdots \times X$ (n-fold) into $\mathbb{C}$.

$Tx_1 \dots x_n := T(x_1, \dots, x_n)(T \in \mathcal{L}^n(X)$.

$Tx^n := Tx \dots x$ (n-times).

3. Test and generalized functionals

Let (H,B) be a fixed but arbitrary abstract Wiener space (AWS, for abbrev.) Denote the norm of B by $\|\cdot\|_B$. Let $\mathcal{E}(B)$ be the class of functions f defined on B which satisfies the following condition:

(E-1) f has an analytic extension $\tilde{f}$ to the complexification B_c such that $|\tilde{f}(z)| \le c_f \exp(c_f' \|z\|_B)$ for some constants c_f, c_f' depending on f, where $\|z\|_B = (\|x\|_B^2 + \|y\|_B^2)^{\frac{1}{2}}$ for $z = x + iy$ $(x, y \in B)$.

For $m = 1, 2, 3, \dots$ and for $f \in \mathcal{E}(B)$ define the norms

$$\|f\|_m = \sup_{z \in B_c} \{|\tilde{f}(z)| e^{-m\|z\|_B}\}$$

and let $\mathcal{E}^m(B)$ be the class $\{f \in \mathcal{E}(B) : \|f\|_m < \infty\}$. Then $\{(\mathcal{E}^m(B), \|\cdot\|_m)\}$ forms an increasing sequence of Banach spaces such that $\bigcup_{m=1}^{\infty} \mathcal{E}^m(B) = \mathcal{E}(B)$. Endow $\mathcal{E}(B)$ with the inductive topology induced by $\mathcal{E}^m(B)$, $\mathcal{E}(B)$ becomes a locally convex topological algebra.

PROPOSITION 3.1 [19]. (a) *$\mathcal{E}(B)$ is Bornological.*

(b) *If $f \in \mathcal{E}^m(B)$ so is the function $g(x) = D^n f(x)y_1 \ldots y_n$ for $y_1, \ldots, y_n \in B$. Moreover,*

$$\|g\|_m \leq \|f\|^m \exp[m(\|y_1\|_B + \cdots + \|y_n\|_B]. \tag{3.1}$$

(c) *If $f \in \mathcal{E}(B)$, then $f_k(x) = \sum_{n=0}^{k}(1/n!)D^n f(0)x^n$ converges to f in $\mathcal{E}(B)$.*

(d) *Let $\Delta f(x) := \text{trace}[D^2 f(x)]$ and $\mathcal{N}f(x) := -\Delta f(x) + (x, Df(x))$. Then the mappings $f \to \Delta f$ and $f \to \mathcal{N}f$ are continuous on $\mathcal{E}(B)$.*

(e) *For any $f \in \mathcal{E}(B)$ the Wiener-Itô decomposition of f enjoys the following representation:*

$$f(x) = \sum_{n=0}^{\infty} \int_B D^n(w * f)(0)(x + iy)^n w(dy),$$

where the sum converges in $\mathcal{E}(B)$.

Now we turn to the AWS $(\mathcal{C}', \mathcal{C})$ and the AWS $(\mathcal{C}', L_2)$. For simplicity, let 's denote the $\mathcal{C}$-norm and the L_2-norm by $\|\cdot\|$ and $\|\cdot\|_2$, respectively.

DEFINITION 3.2. (a) Let $\mathcal{E}$ be the collection of functions f which is defined on $\mathcal{C}$ and there exist a function $g \in \mathcal{E}(L_2)$ such that $f(x) = g(x)$ for all $x \in \mathcal{C}$.

(b) Set $\mathcal{E}_0 = \mathcal{E}(\mathcal{C})$.

Clearly $\mathcal{E} \subset \mathcal{E}_0$ and the injection from $\mathcal{E}$ into $\mathcal{E}_0$ is continuous so that their dual spaces enjoy the following relation:

$$\mathcal{E}_0^* \subset \mathcal{E}^*. \tag{3.2}$$

Both $\mathcal{E}$ and $\mathcal{E}_0$ will serve as the test functionals in our investigations and members of their dual spaces will be called generalized white noise functionals (GWNF, for abbrev.). For notational convenience, we shall use "$\langle\langle \cdot, \cdot \rangle\rangle$" to denote the pairing for both $\mathcal{E}^* - \mathcal{E}$ and $\mathcal{E}_0^* - \mathcal{E}_0$.

The space of GWNF's will be endowed with weak*-topology. Thus a sequence $\{F_n\}$ of GWNF's is said to converge to a GWNF F if $\langle\langle F_n, \varphi \rangle\rangle \to \langle\langle F, \varphi \rangle\rangle$ for all $\varphi \in \mathcal{E}$.

4. Examples of GWNF

Unlike Hida's original approach, examples of GWNF's given in this section will be defined in their linear functional form instead of their S-transforms, for more details we refer the reader to [7–13,17,19].

EXAMPLE 4.1. (a) Suppose that f is a measurable function that satisfies the following condition:

$$\int_C |f(x)|e^{m\|x\|}w(dx) < \infty \tag{4.1a}$$

for all $m = 1, 2, 3, \cdots$. Then f defines a GWNF f in $\mathcal{E}_0^*$ given by

$$\langle\langle f, \varphi\rangle\rangle = \int_C f(x)\varphi(x)w(dx), \tag{4.1b}$$

for $\varphi \in \mathcal{E}_0$.

(b) Suppose that f is function that satisfies the condition (4.1a) with the $\mathcal{C}$-norm $\|x\|$ being replaced by the L_2-norm $\|x\|_2$ for all $m = 1, 2, 3, \cdots$. Then f generates a GWNF in $\mathcal{E}^*$ defined also by (4.1b) for $\varphi \in \mathcal{E}$. In particular, $L_p \in \mathcal{E}_0^*$, for $p > 1$.

EXAMPLE 4.2. (a) Suppose that μ is a Borel measure that satisfies the following condition:

$$\int_C e^{m\|x\|}\mu(dx) < \infty \tag{4.2a}$$

for all $m = 1, 2, 3, \cdots$. Then μ generates a GWNF μ in $\mathcal{E}_0^*$ given by

$$\langle\langle f, \varphi\rangle\rangle = \int_C \varphi(x)\mu(dx), \tag{4.2b}$$

for $\varphi \in \mathcal{E}_0$.

(b) Suppose that μ is a measure that satisfies the condition (4.1a) with the $\mathcal{C}$-norm $\|x\|$ replaced by the L_2-norm $\|x\|_2$ for all $m = 1, 2, 3, \ldots$. Then μ generates a GWNF in $\mathcal{E}^*$ given by the same form as (4.2b) for $\varphi \in \mathcal{E}$. For example, w and $O_t(x, dy) = w(dy - e^{-t}x)$ (the transition measure of $\mathcal{C}$ or L_2-valued Ornstein-Uhlenbeck process) are GWNF's in $\mathcal{E}_0^*$.

EXAMPLE 4.3. First define

$$\tilde{y}(x) = \begin{cases} (x, y) & \text{if } y \in \mathcal{C}^* \\ \langle x, y\rangle & \text{if } y \in L_2^*. \end{cases}$$

Clearly $\tilde{y} \in \mathcal{E}_0$ if $y \in \mathcal{C}^*$ and $\tilde{y} \in \mathcal{E}$ if $y \in L_2^*$.

If $y \in \mathcal{C}'$, we define $\tilde{y}(x) =< x, y >_0$, and, in this case, $\tilde{y}$ is defined only a.e.(w) for $x \in L_2$.

One naturally ask what if $y \in \mathcal{C}$ (or $y \in L_2$). Suppose that $y \in \mathcal{C}$. We choose a sequence $\{y_n\} \subset \mathcal{C}^*$ such that $\|y_n - y\| \to 0$ and for any $\varphi \in \mathcal{E}_0$, since $D(w * \varphi)(0) \in \mathcal{C}^*$, we have

$$\begin{aligned} \langle\langle \tilde{y}_n, \varphi \rangle\rangle &= \textstyle\int_{\mathcal{C}} (x, y_n)\varphi(x) w(dx) \\ &= (y_n, D(w * \varphi)(0)) \\ &\to (y, D(w * \varphi)(0)) \end{aligned}$$

as $n \to \infty$. This leads to the following definition:

$$\langle\langle \tilde{y}, \varphi \rangle\rangle = (y, D(w * \varphi)(0)), \tag{4.3a}$$

for all $\varphi \in \mathcal{E}_0$.

If $y \in L_2^*$ then the similar arguments yield the following definition:

$$\langle\langle \tilde{y}, \varphi \rangle\rangle = \langle y, D(w * \varphi)(0) \rangle, \tag{4.3b}$$

where $\varphi \in \mathcal{E}$.

As a consequence of (4.3b), the white noise $\dot{B}(t)$ now may be defined rigorously by

$$\langle\langle \dot{B}(t), \varphi \rangle\rangle = \langle h_t, D(w * \varphi)(0) \rangle, \tag{4.3c}$$

for all $\varphi \in \mathcal{E}$. Equivalently, $\dot{B}(t) = \tilde{h}_t$. It follows from (4.3c) that $\dot{B}(t) \in \mathcal{E}^*$.

EXAMPLE 4.4 (construction of a GWNF by additive renormalization). Suppose that $k_1, k_2, \ldots, k_n$ are elements of L_2^*. Then $\prod_{j=1}^n \tilde{k}_j(x) \in \mathcal{E} \subset \mathcal{E}^*$. Let $\varphi \in \mathcal{E}$ and apply integration by parts formula[13], we obtain

$$\begin{aligned} \langle\langle \prod_{j=1}^n \tilde{k}_j(x), \varphi(x) \rangle\rangle &= \int_{\mathcal{C}} \prod_{j=1}^n \tilde{k}_j(x)\varphi(x) w(dx) \\ &= \int_{\mathcal{C}} Q_n(x)\varphi(x) w(dx) + D^n(w * \varphi)(0)k_1, \ldots, k_n, \\ &= \langle\langle Q_n(x), \varphi(x) \rangle\rangle + D^n(w * \varphi)(0)k_1, \ldots, k_n, \end{aligned} \tag{4.4a}$$

where $Q_n(x) = \prod_{j=1}^n \tilde{k}_j(x) + \int_{\mathcal{C}} \prod_{j=1}^n \tilde{k}_j(x + iy) w(dy)$. If we let $k_j \to y_j$

in L_2^* for all j, then (4.4a) does not tend to a limit in general. Now, subtracts the first term of (4.4a) from both sides of the above identity, one obtains

$$\langle\langle \prod_{j=1}^{n} \tilde{k}_j(x) - Q_n(x), \varphi(x)\rangle\rangle = D^n(w * \varphi)(0)k_1, \ldots, k_n. \tag{4.4b}$$

The last term does tend to $D^n(w * \varphi)(0)y_1, \ldots, y_n$ which determines a GWNF $\{: \tilde{y}_1\tilde{y}_2 \ldots \tilde{y}_n :\}$ defined as follows:

$$\langle\langle : \tilde{y}_1\tilde{y}_2 \ldots \tilde{y}_n :, \varphi\rangle\rangle = D^n(w * \varphi)(0)y_1, \ldots, y_n. \tag{4.4c}$$

This GWNF is also called the (additive) renormalization of $\tilde{y}_1\tilde{y}_2 \ldots \tilde{y}_n$.

In particulr, for any positive times $\{t_1, t_2, \ldots, t_n\} \subset [0, 1]$, not necessarily distinct, the renormalization of $\dot{B}(t_1)\dot{B}(t_2) \ldots \dot{B}(t_n)$ is given by

$$\langle\langle : \dot{B}(t_1)\dot{B}(t_2) \ldots \dot{B}(t_n) :, \varphi\rangle\rangle = D^n(w * \varphi)(0)h_{t_1}h_{t_2} \ldots h_{t_n}. \tag{4.4d}$$

It is easy to see that $: \dot{B}(t_1)\dot{B}(t_2) \ldots \dot{B}(t_n) : \in \mathcal{E}^*$.

EXAMPLE 4.5. If $T \in \mathcal{L}(\mathcal{C}', \mathcal{C}')$ is symmetric, then the function $< Tx, x >_0$ does not make sense for $x \in L_2$, however, the renormalization $< Tx, x >_0$ makes sense and defines a GWNF given by

$$\langle\langle :< Tx, x >_0:, \varphi\rangle\rangle = \text{trace}_{\mathcal{C}'}[TD^2(w * \varphi)(0)]. \tag{4.5a}$$

Formally $:< Tx, x >_0 :=< Tx, x >_0 - \text{trace}_{\mathcal{C}'}[T]$ (neither $< Tx, x >_0$ nor $\text{trace}_{\mathcal{C}'}[T]$ exits, but their difference make sense).

Apply (4.5a) with $T = I$, we have

$$\langle\langle : \int_0^1 \dot{B}(t)^2 dt :, \varphi\rangle\rangle = \text{trace}_{\mathcal{C}'}[D^2(w * \varphi)(0)]. \tag{4.5b}$$

EXAMPLE 4.6 (construction of GWNF by multicative renormalization). Suppose that $h \in \mathcal{C}'$ and α is a complex number. Then $\exp(\alpha\tilde{h}) \in L^2(w)$ and

$$\int_{\mathcal{C}} e^{\alpha\tilde{h}(x)}\varphi(x)w(dx) = e^{\frac{1}{2}\alpha^2|h|_0^2} w * \varphi(\alpha h). \tag{4.6a}$$

Multiply both sides of (4.6a) by $\exp(-\frac{1}{2}\alpha^2|h|_0^2)$, one obtains

$$\langle\langle e^{(-\frac{1}{2}\alpha^2|h|_0^2)}e^{(\alpha\tilde{h})}, \varphi\rangle\rangle = w * \varphi(\alpha h). \tag{4.6b}$$

Since $w * \varphi(\alpha y)$ exists for any $y \in L_2$, and the mapping $\varphi \to w * \varphi(\alpha y)$ is a continuous functional on $\mathcal{E}$, (4.6b) suggest that we define

$$\langle\langle : \exp(\alpha \tilde{y}) :, \varphi \rangle\rangle = w * \varphi(\alpha y), \tag{4.6c}$$

for all $\varphi \in \mathcal{E}$. Then $: \exp(\alpha\tilde{y}) : \in \mathcal{E}^*$. Formally, we may write $: \exp(\alpha\tilde{y}) : = \exp(-\frac{1}{2}\alpha^2|y|_0^2)\exp(\alpha\tilde{y})$. Thus we call $: \exp(\alpha\tilde{y}) :$ a multiplicative renormalization of $\exp(\alpha\tilde{y})$.

Apply (4.6c) with $y = h_t$, we have

$$\langle\langle : \exp(\alpha \dot{B}(t)) :, \varphi \rangle\rangle = w * \varphi(\alpha h_t). \tag{4.6d}$$

EXAMPLE 4.7. Let $\alpha \neq 1$ be a complex number. The multiplicative renormalization of $\exp\{\frac{\alpha}{2}\int_0^1 \dot{B}(t)^2 dt\}$ defines a GWNF given by

$$\langle\langle : \exp\{\frac{\alpha}{2}\int_0^1 \dot{B}(t)^2 dt\} :, \varphi \rangle\rangle = \int_C \varphi((1-\alpha)^{-\frac{1}{2}}x)w(dx). \tag{4.7}$$

EXAMPLE 4.8 (Donsker's delta function). Donsker's delta function may be formally represented by $\delta_u(B(t))$. As a GWNF, it is defined as follows:

$$\langle\langle \delta_u(B(t), \varphi \rangle\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ius - \frac{1}{2}s^2 t) w * \varphi(isb_t) ds.$$

For more examples of GWNF's, we refer the reader to [7–13,17,19].

5. Calculus of GWNF's

Multiplication by functions

Let ψ is a fixed but arbitrary function in $\mathcal{E}$ (resp. $\mathcal{E}_0$). Then the mapping $\varphi \to \varphi\psi$ is continuous on $\mathcal{E}$ (resp. $\mathcal{E}_0$). This fact leads to the following

DEFINITION 5.1. Let $F \in \mathcal{E}^*$ and $\varphi \in \mathcal{E}$. Define

$$\langle\langle \psi F, \varphi \rangle\rangle = \langle\langle F\psi, \varphi \rangle\rangle = \langle\langle F, \psi\varphi \rangle\rangle.$$

Then $\psi F \in \mathcal{E}^*$.

On the other hand, if $F \in \mathcal{E}_0^*$ and $\varphi \in \mathcal{E}_0$, then we define

$$\langle\langle \psi F, \varphi \rangle\rangle = \langle\langle F, \psi\varphi) \rangle\rangle.$$

The later implies that $\psi F \in \mathcal{E}_0^*$.

Translation

DEFINITION 5.2. Let $F \in \mathcal{E}^*$ and $z \in \mathcal{C}^*$. Then we define

$$\langle\langle \tau_z F, \varphi \rangle\rangle := \langle\langle F, e^{\tilde{z}} \tau_{-z} \varphi \rangle\rangle,$$

where $\tau_z \varphi(x) = \varphi(x - z)$.

Differentiation

DEFINITION 5.3. Let $F \in \mathcal{E}^*$.
(a) For any $x \in L_2$, define

$$\langle\langle D_x^* F, \varphi \rangle\rangle := \langle\langle F, D_x \varphi \rangle\rangle,$$

for all $\varphi \in \mathcal{E}$, where $D_x \varphi(y) = (x, D\varphi(y)) = D\varphi(y)x$ $(x, y \in L_2)$.
(b) For any $z \in L_2^*$ and for any $\varphi \in \mathcal{E}$, define

$$\langle\langle D_z F, \varphi \rangle\rangle := \langle\langle F, \tilde{z} \varphi \rangle\rangle - \langle\langle F, D_z \varphi \rangle\rangle,$$

for all $\varphi \in \mathcal{E}$.

When $F \in \mathcal{E}_0^*$, $x \in \mathcal{C}$ and $z \in \mathcal{C}^*$, the differential operators D_x^* and D_z are defined similarly.

PROPOSITION 5.4. *Let $F \in \mathcal{E}^*$(resp. $\mathcal{E}_0^*$).*
(a) *If $x \in L_2$(resp. $\mathcal{C}$), $D_x^* F \in \mathcal{E}^*$(resp. $\mathcal{E}_0^*$).*
(b) *If $z \in L_2^*$(resp. $\mathcal{C}^*$), then $D_z F \in \mathcal{E}^*$(resp. $\mathcal{E}_0^*$) and we have*

$$\tilde{z} F = D_z F + D_z^* F.$$

Proof. The proof follows immediately from Proposition 3.1. □

DEFINITION 5.5. (a) Let $\partial_t = D_{h_t}$. (Recall that $h_t = \mathbf{1}_{[t,1]}$.)
(b) For $F \in \mathcal{E}^*$ and for $\varphi \in \mathcal{E}$, define

$$\langle\langle \partial_t^* F, \varphi \rangle\rangle = \langle\langle F, \partial_t \varphi \rangle\rangle$$

PROPOSITION 5.6. (a) *For $z \in L_2^*$ and $x \in \mathcal{C}'$, we have*

$$\tilde{z}(x) = \int_0^1 \dot{x}(t) \dot{z}(t) dt.$$

(b) *For $z \in L_2^*$ and for $\varphi \in \mathcal{E}$, we have*

$$D_z \varphi(x) = \int_0^1 \partial_t \varphi(x) \dot{z}(t) dt \ (x \in \mathcal{C}). \tag{5.1}$$

(c) *For $z \in L_2^*$ and for $\varphi \in \mathcal{E}$, we have*

$$D_z^* \varphi(x) = \int_0^1 \partial_t^* \varphi(x) \dot{z}(t) dt \ (x \in \mathcal{C}). \tag{5.2}$$

(d) *For $\varphi \in \mathcal{E}$, then we have*

$$\dot{B}(t)\varphi = \partial_t \varphi + \partial_t^* \varphi. \tag{5.3}$$

REMARK 5.7. In Proposition 5.5, if the test functional φ is replaced a GWNF $F \in \mathcal{E}^*$, then (5.2) and (5.3) hold only symbolically. It can be interpreted in the following way: Suppose that $Q \in L(L_2^*, \mathcal{E}^*)$. Then one can realize $Q(t)$ as an $\mathcal{E}^*$-valued function in t-variable and interpret the formal $\mathcal{E}^*$-valued integral

$$-\text{“}\int_0^1 Q(t)\ddot{z}(t)dt\text{”} = Q(z) \ \ (z \in L_2^*). \tag{5.4}$$

For example, the derivative DF of F is an element in $L(L_2^*, \mathcal{E}^*)$ if we define $DF(z) := D_z F$ for $z \in L_2^*$. Then $DF(t)$ is realized as an $\mathcal{E}^*$-valued generalized function such that

$$D_z F = -\int_0^1 DF(t)\ddot{z}(t)dt. \tag{5.5a}$$

The identity (5.5a) may rewritten as

$$D_z F = \int_0^1 \dot{DF}(t)\dot{z}(t)dt. \tag{5.5b}$$

Apply (5.5b) with $z = h_t$ formally, we have $\partial_t F = \dot{DF}(t)$.

EXAMPLE 5.8. (a) For any $t, s \in [0, 1]$, we have

$$\partial_t \dot{B}(s) = \delta_s(t). \tag{5.6a}$$

Identity (5.6a) is in fact a symbolic representation of the following identity:

$$D_z \dot{B}(s) = \dot{z}(s). \tag{5.6b}$$

Since $\dot{B}(s, x) =< x, h_s >_0$, we have $D_z \dot{B}(s, x) =< h_s, z >_0 = \dot{z}(s)$. This verifies (5.6b) and hence proves (5.6a)

(b) Let $f \in L_2$. Then $:\int_0^1 f(u)\dot{B}(u)^2 du:$ is a GWNF which can also be represented by $:< Tx, x >_0:$, where T is a symmetric bounded linear operator of the form $Tx(t) = \int_0^t f(s)dx(s)$. Then we have

$$\partial_t\left(: \int_0^1 f(u)\dot{B}(u)^2 du :\right) = 2f(t)\dot{B}(t). \tag{5.7}$$

The proof follows immediately from the fact that $D_z(:\langle Tx, x\rangle_0) = 2\langle Tx, z\rangle_0$.

(c) For any $t, s \in [0,1]$, we have

$$\partial_t\colon \dot{B}(s)^n : = n\delta_s(t)\colon \dot{B}(s)^{n-1} :. \tag{5.8}$$

Rewrite $: \dot{B}(s)^n :=: \tilde{h}_t^n :$. It follow that $D_z : \tilde{h}_s^n : = n < h_s, z >_0 : \tilde{h}_s^{n-1} :$ which, in turns, implies (5.8).

(d) For $t_1, t_2, t_3, \ldots, t_n$ we have

$$\partial_{t_1}^*\partial_{t_2}^* \ldots \partial_{t_n}^* \mathbf{1} =: \dot{B}(t_1)\dot{B}(t_2)\ldots\dot{B}(t_n) :$$

(see also [7, 17, 19]).

(e) For any $t, s \in [0,1]$,

$$\partial_t^*\colon \dot{B}(s)^n : =: \dot{B}(s)^n\dot{B}(t) :$$

(see also [7, 17, 19]).

DEFINITION 5.9. Let $F \in \mathcal{E}^*$. Then we define

(a) $\langle\langle NF, \varphi\rangle\rangle = \langle\langle F, N\varphi\rangle\rangle$ for all $\varphi \in \mathcal{E}$;

(b) $\langle\langle \Delta^* F, \varphi\rangle\rangle = \langle\langle F, \Delta\varphi\rangle\rangle$ for all $\varphi \in \mathcal{E}$.

PROPOSITION 5.10 [7, 17, 19].

(a) For $\varphi \in \mathcal{E}$, we have $\Delta\varphi = \int_0^1 \partial_t^2 \varphi dt$.

(b) For $F \in \mathcal{E}^*$, we have $NF = \int_0^1 \partial_t^* \partial_t F dt$ and $\Delta^* F = \int_0^1 \partial_t^* \partial_t^* F dt$.

Fourier transform

LEMMA 5.11 [19]. *For any $\varphi \in \mathcal{E}$ and any pair of complex number α and β, define*

$$\mathcal{F}_{\alpha,\beta}\varphi(y) = \int_C f(\alpha x + \beta y) w(dx).$$

Then $\mathcal{F}_{\alpha,\beta}\mathcal{E} \subset \mathcal{E}$ and $\mathcal{F}_{\alpha,\beta}\mathcal{E}_0 \subset \mathcal{E}_0$. Moreover, $\mathcal{F}_{\alpha,\beta}$ is continuous on $\mathcal{E}$ and on $\mathcal{E}_0$.

DEFINITION 5.12. For a GWNF $G \in \mathcal{E}^*$, define the Fourier transform $\mathcal{F}G$ by

$$\langle\langle \mathcal{F}G, \varphi \rangle\rangle = \langle\langle F, \mathcal{F}_{1,-i}\varphi \rangle\rangle,$$

for all $\varphi \in \mathcal{E}$. The inverse Fourier transform is given by

$$\langle\langle \mathcal{F}^{-1}G, \varphi \rangle\rangle = \langle\langle G, \mathcal{F}_{1,i}\varphi \rangle\rangle,$$

for all $\varphi \in \mathcal{E}$.

It follows from the Lemma 5.11 that the Fourier transform is well-defined and is continuous on $\mathcal{E}^*$. It can be shown that the Fourier transform generalizes the finite dimensional Fourier transform defined in the following form:

$$\mathcal{F}\psi(y) = (\sqrt{2\pi})^{-n} \int_{R^n} \psi(x) \exp(-i < y, x >) dx.$$

An another definition of Fourier transform was given by Kuo[17]. It is worth to remark that, unlike the finite dimensional case, the Fourier transform send "good" whit noise functional to "bad" one. This might be the reason why the Fourier transform does not play a significant role in white noise calculus so far as we know, and for this reason we omit the discussions of properties of Fourier transform in this paper. For interested reader we refer to [17,19].

Conditional expectation

Let $\mathcal{B}_n$ denote the σ-field generates by $\{B(t_j) : 0 \leq t_1 < t_2 \cdots < t_n \leq 1\}$. We shall define the conditional expectation $\mathbf{E}[\Phi|\mathcal{B}_n]$ for a GWNF Φ. As before we start with the case that Φ is a test white noise functional. I is easy to verify the following Lemma.

LEMMA 5.13. (a) *Given any $\varphi \in \mathcal{E}_0$ and $0 \leq t_1 < t_2 \cdots < t_n \leq 1$ we have*

$$\begin{aligned} &\mathbf{E}[\varphi|\mathcal{B}_n](x) \\ &= \int_C \varphi(\sum_{j=1}^{n-1} (t_{j+1} - t_j)^{-1} (x - y, b_{t_{(j+1)}} - b_{t_j})(b_{t_{(j+1)}} - b_{t_j}) + y) w(dy). \end{aligned}$$

(b) *Define the mapping $\Lambda(\varphi) = \mathbf{E}[\varphi|\mathcal{B}_n]$. Then $\Lambda(\mathcal{E}_0) \subset \mathcal{E}_0$ and Λ is continuous on $\mathcal{E}_0$. Moreover, $\langle\langle \Lambda\varphi, \psi \rangle\rangle = \langle\langle \varphi, \Lambda\psi \rangle\rangle$, for all $\varphi, \psi \in \mathcal{E}_0$.*

Apply Lemma 5.13, we define the generalized conditional expectation as follows:

DEFINITION 5.14. For a GWNF $\Phi \in \mathcal{E}_0^*$ and $t > 0$, we define

$$\langle\langle \mathbf{E}[\Phi|\mathcal{B}_n], \varphi \rangle\rangle = \langle\langle \Phi, \mathbf{E}[\varphi|\mathcal{B}_n] \rangle\rangle.$$

The above definition of generalized conditional expectation will provide a scheme for studying the generalized stochastic processes, we shall discuss this subject in another paper systematically.

6. Generalized Itô formula

After [19], for $f \in \mathcal{S}'$ and a nonzero $h \in \mathcal{C}'$, we define the composition $f(\tilde{h})$ of f and $\tilde{h}$ as follows:

DEFINITION 6.1. For any $\varphi \in \mathcal{E}$, define

$$\langle\langle f(\tilde{h}), \varphi \rangle\rangle = ((\hat{G}_{h,\varphi}, f)),$$

where $((\cdot, \cdot))$ denotes the $\mathcal{S} - \mathcal{S}'$ pairing and $\hat{G}_{h,\varphi}(u) = (\sqrt{2}\pi)^{-1} e^{(-\frac{1}{2}u^2\|h\|_0^2)}$ $\mathcal{F}_{1,i}\varphi(uh)$.

A consequence of Definition 6.1, for $t \neq 0$, $f(B(t))$ is then defined by

$$\langle\langle f(B(t)), \varphi \rangle\rangle = ((\hat{G}_{t,\varphi}, f)), \tag{6.1}$$

where $G_{t,\varphi}(u) = (\sqrt{2}\pi)^{-1} e^{(-\frac{1}{2}u^2 t)} \mathcal{F}_{1,i}\varphi(ub_t)$.

Differentiate $\langle\langle f(B(t)), \varphi \rangle\rangle$ with respect to t and employ the same arguments as given in [19], it is not hard to verify that

$$\frac{d}{dt}\langle\langle f(B(t)), \varphi \rangle\rangle = \langle\langle f'(B(t)), \partial_t \varphi \rangle\rangle + \frac{1}{2}\langle\langle f''(B(t)), \varphi \rangle\rangle.$$

This proves the following

THEOREM 6.2. (Itô formula). *For $F \in \mathcal{E}^*$, and for $b > a > 0$, we have*

$$f(B(b)) = f(B(a)) + \int_a^b \partial_t^* f'(B(t))dt + \frac{1}{2}\int_a^b f''(B(t))dt. \tag{6.2}$$

REMARK 6.3. A different definition of $f(B(t)$ was defined by I. Kubo[9], there the generalized Itô formula (6.2) was first announced.

The integral $\int_a^b \partial_t^* f'(B(t))dt$ appear in (6.2) is also called the Hitsuda-Skorohod integral. In [9,13], Kubo and Takenaka has also shown that if $\{X_t\}$ is nonanticipating and $\mathbf{E}[\int_a^b |X_t|^2]dt < \infty$, then

$$\int_a^b X_t dB(t) = \int_a^b \partial_t^* X_t dt.$$

The generalized process $\{f(B(t)\}$ is understood as a nonanticipating process in the generalized sense. Thus Theorem 6.2 indeed generalize the Itô formula. We shall discuss the stochastic integration in more details in the next section. For more systematical investigation for this subject we refer the reader to [17].

What happen if $\{f(B(t)\}$ is replaced by a anticipating process, say, $\{f(B(t), B(1))\}$. The corresponding Itô formula was also known as Hitsuda's formula (see [17]). We find that in the generalized sense the idea for proving Theorem 6.2 also provides an easy proof for Hitsuda's formula. Let's start with the definition of $f(B(t), B(1))$ for $f \in \mathcal{S}'(\mathbb{R}^2)$.

DEFINITION 6.4. For any $\varphi \in \mathcal{E}$ and for $f \in \mathcal{S}'(\mathbb{R}^2)$, define

$$\langle\langle f(B(t), B(1)), \varphi \rangle\rangle = (\!(\hat{Q}_{h,\varphi}, f)\!),$$

where $(\!(\cdot, \cdot)\!)$ denotes the $\mathcal{S}(\mathbb{R}^2) - \mathcal{S}'(\mathbb{R}^2)$ pairing and

$$\begin{aligned} Q_{h,\varphi}(u) &= (\sqrt{2}\pi)^{-2} e^{-\frac{1}{2}|ub_t + vb_1|^2} \mathcal{F}_{1,i}\varphi(ub_t + vb_1) \\ &= (\sqrt{2}\pi)^{-2} e^{-\frac{1}{2}u^2 t - uvt - \frac{1}{2}v^2} \mathcal{F}_{1,i}\varphi(ub_t + vb_1). \end{aligned}$$

Differentiating $\langle\langle f(B(t), B(1)), \varphi \rangle\rangle$, we obtain immediatedly the following Hitsuda's formula:

THEOREM 6.5 (Hitsuda's formula) [8, 17].

$$\begin{aligned} & f(B(b), B(1)) - f(B(a), B(1)) \\ = & \int_a^b \partial_t^* f_x(B(t), B(1))dt + \int_a^b f_{xy}(B(t), B(1))dt \\ & + \frac{1}{2} \int_a^b f_{xx}(B(t), B(1))dt. \end{aligned}$$

7. Generalized stochastic integrals

In this section we shall define and study the stochastic integral for an anticipating integrand which is in general a generalized stchastic processes. Being motivated by H.-H. Kuo and A. Russek's work [16], we shall define and study the so called one-side integrals in a direct setting without using S-transform and Wiener-Itô decomposition of white noise functionals.

DEFINITION 7.1. Let $\mathcal{X} = \{X_t : t \geq 0\}$ be a $\mathcal{E}^*$-valued process.
(a) Define

$$\partial_{t^+} X_t := \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \langle DX_t, b_{t+\epsilon} - b_t \rangle$$

if the limit exists in $\mathcal{E}^*$.
(b) Define

$$\partial_{t^-} X_t := \lim_{\epsilon \to 0^-} \frac{1}{\epsilon} (DX_t, b_{t+\epsilon} - b_t)$$

if the limit exists in $\mathcal{E}_0^*$.

EXAMPLE 7.2. (a) For $f \in \mathcal{S}'$, we have

$$\partial_{t^+} f(B(t)) = 0 \text{ and } \partial_{t^-} f(B(t)) = f'(B(t)).$$

In particular, we have $\partial_{t^+} B(t) = 0$ and $\partial_{t^-} B(t) = 1$.
(b) Let $X_t(x) = B(1, x) = (x, b_1)$. Then we have

$$\partial_{t^+} B(1) = \mathbf{1}_{[0,1)}(t) \quad \text{and} \quad \partial_{t^-} B(1) = \mathbf{1}_{(0,1]}(t).$$

If we let $X_t(x) = B(1-t, x) = (x, b_{1-t})$ then we have

$$\partial_{t^+} B(1-t) = \mathbf{1}_{[0,\frac{1}{2})}(t) \quad \text{and} \quad \partial_{t^-} B(1-t) = \mathbf{1}_{(0,\frac{1}{2}]}(t).$$

We are now ready to define the one side stochastic integral (see [16, 17]).

DEFINITION 7.3. (a) Let $\{X_t\}$ be a $\mathcal{E}^*$-valued continuous process and, for each n, let $\Gamma_n = \{a = t_{0n} < t_{1n} < \ldots t_{kn} = b\}$ be a partition of $[a, b] \subset [0, 1]$ such that the mesh $|\Gamma_n| = \max_{1 \leq j \leq k}\{t_{jn} - t_{(j-1)n}\} \to 0$ as $n \to \infty$.

Define the "right"-integral by

$$\int_a^b X_t dB(t^+) := \lim_{n \to \infty} \sum_{j=1}^k X_{t_{(j-1)n}} (B(t_{jn}) - B(t_{(j-1)n})$$

provided that the limit exists in $\mathcal{E}^*$.

(b) Next suppose that $\{X_t\}$ is a $\mathcal{E}^*$-valued right continuous process with only finitely many discontinuities $\{t_1, t_2, \ldots, t_{n-1}\}$ such that $\{a = t_0 \leq t_1 < t_2 < \cdots < t_{n-1} \leq t_n = b\}$. Then we define the "right"-integral by

$$\int_a^b X_t dB(t^+) := \sum_{j=1}^n \int_{t_{j-1}}^{t_j} X_t dB(t^+).$$

Similarly we define the "left"-integral as follows:

DEFINITION 7.4. (a) Let $\{X_t\}$ and Γ_n be the same as given in Definition 7.3(a). Then we define the "left"-integral by

$$\int_a^b X_t dB(t^-) := \lim_{n\to\infty} \sum\nolimits_{j=1}^k X_{t_{jn}}(B(t_{jn}) - B(t_{(j-1)n}).$$

(b) Let $\{X_t\}$ be a left continuous process with only finitely many discontinuities as given in Definition 7.3(b). Then we define the "left"-integral by

$$\int_a^b X_t dB(t^-) := \sum_{j=1}^n \int_{t_{j-1}}^{t_j} X_t dB(t^-).$$

LEMMA 7.5. *Let $\Gamma = \{a = t_0 < t_1 < \ldots t_k = b\}$ be any partition of $[a,b]$ and $\{X_t\}$ be a $\mathcal{E}^*$-valued continuous process. Then we have*

$$\begin{aligned}&\sum_{j=1}^n X_{t_{j-1}}(B(t_j) - B(t_{j-1})) - \sum_{j=1}^n \langle DX_{t_{j-1}}, b_{t_j} - b_{t_{j-1}}\rangle \\ = &\sum_{j=1}^n D^*_{(b_{t_j} - b_{t_{j-1}})} X_{t_{j-1}}.\end{aligned}$$

Moreover, as $|\Gamma| \to 0$, the last term in (7.1) tends to $\int_a^b \partial_t^* X_t dt$.

It follows from Lemma 7.5 that if the first sum of (7.1) tends to $\int_a^b X_t dB(t^+)$ then the second sum will converge to a limit which will be denoted by $\mathbf{I}^+_{[a,b]}(X_t)$. This leads to the following

DEFINITION 7.6. Let $\{\Gamma_n\}$ be a sequence of partitions of $[a,b]$ such that $|\Gamma_n| \to 0$. Then we define

$$\mathbf{I}^+_{[a,b]}(X_t) := \lim_{n\to\infty} \sum_{j=1}^k \langle DX_{t_{(j-1)n}}, b_{t_{jn}} - b_{t_{(j-1)n}}\rangle. \tag{7.2a}$$

Similarly we define

$$\mathbf{I}^{-}_{[a,b]}(X_t) := \lim_{n\to\infty}\sum_{j=1}^{k}\langle DX_{t_{jn}}, b_{t_{jn}} - b_{t_{(j-1)n}}\rangle. \tag{7.2b}$$

PROPOSITION 7.7. *Let $\{\varphi_t\}$ be a continuous $\mathcal{E}$-valued process. Then we have*

(a) *$\partial_t\varphi_t$ is continuous function of t and*

$$\mathbf{I}^{+}_{[a,b]}(\varphi_t) = \int_a^b \partial_t\varphi_t dt.$$

(b) *$\int_a^b \varphi_t dB(t^+)$ exists and $\int_a^b \varphi_t dB(t^+) = \int_a^b \partial_t\varphi_t + \int_a^b \partial_t^*\varphi_t dt.$*

It is natural to ask under what conditions

$$\mathbf{I}^{+}_{[a,b]}(X_t) = \int_a^b \partial_{t^+}X_t dt?$$

or

$$\mathbf{I}^{-}_{[a,b]}(X_t) = \int_a^b \partial_{t^-}X_t dt?$$

If $\{X_t\}$ is a continuous $\mathcal{E}$-valued process, the answer is positive as we have shown in Lemma 7.7. To state the next Proposition which answers the above question firmly and guarantee the existence of ' "right-integral" (or "left-integral"), we need the concept of strongly continuity $\mathcal{E}^*$-valued functions introduced in the following

DEFINITION 7.8. An $\mathcal{E}^*$-valued function $F(t)$ defined on $[0,1]$ is said to be strongly continuous if, for each m, the function $F : [0,1] \to \mathcal{E}^*_m$ is strongly continuous. A strongly continuous $\mathcal{E}^*_0$-valued function is defined similarly.

The main result of this section is given as follows:

PROPOSITION 7.9. *Let $\{X_t : t \in [0,1]\}$ be an $\mathcal{E}^*_0$-valued process.*

(a) *Suppose that X_t and $\partial_{t^+}X_t$ are strongly continuous. In addition we assume that $\frac{1}{\epsilon}(DX_t, b_{(t+\epsilon)} - b_t) \to \partial_{t^+}X_t$ uniformly in t as $\epsilon \to 0^+$. Then we have*

$$\int_a^b X_t dB(t^+) = \int_a^b \left(\partial_{t^+}X_t + \partial_t^*X_t\right)dt.$$

(b) *Suppose that* X_t *and* $\partial_{t^-}X_t$ *are strongly continuous. In addition we assume that* $\frac{1}{\epsilon}(DX_t, b_{(t+\epsilon)} - b_t) \to \partial_{t^+}X_t$ *uniformly in* t *as* $\epsilon \to 0^-$. *Then we have*

$$\int_a^b X_t dB(t^-) = \int_a^b \Big(\partial_{t^-}X_t + \partial_t^* X_t\Big) dt.$$

Proof. We sketch the proof only for (a).
Let $\Gamma_n = \{a = t_{0n} < t_{1n} < \cdots < t_{kn} = b\}$ be a sequence of partitions of $[a, b]$ such that the mesh $|\Gamma_n| \to 0$ as $n \to \infty$. Denote $\Delta t_{jn} = t_{(j+1)n} - t_{jn}$. Since X_t and $\partial_{t^+}X_t$ are strongly continuous, we have, for each m,

$$\mathcal{E}_m^* - \lim_{n\to\infty} \sum_{j=1}^n \int_{t_{(j-1)n}}^{t_{jn}} \partial_{t^+_{(j-1)n}} X_{t_{(j-1)n}} dt = \int_a^b \partial_{t^+} X_t dt$$

and

$$\mathcal{E}_m^* - \lim_{n\to\infty} \sum_{j=1}^n \int_{t_{(j-1)n}}^{t_{jn}} \partial^*_{t_{(j-1)n}} X_{t_{(j-1)n}} dt = \int_a^b \partial_t^* X_t dt.$$

Also since

$$\begin{aligned} &\sum_{j=1}^n X_{t_{(j-i)n}}(B(t_{jn}) - B(t_{(j-1)n})) \\ = &\sum_{j=1}^n \frac{1}{\Delta t_{jn}}(DX_{t_{(j-1)n}}, B(t_{jn}) - B(t_{(j-1)n}))\Delta t_{jn} \\ &+ \sum_{j=1}^n \int_{t_{(j-1)n}}^{t_{jn}} \partial^*_{t_{(j-1)n}} X_{t_{(j-1)n}} dt, \end{aligned}$$

it follows from the assumption that $\lim_{n\to\infty} \sum_{j=1}^n X_{t_{(j-i)n}}(B(t_{jn}) - B(t_{(j-1)n}))$ exists and we have

$$\int_a^b X_t dB(t^+) = \int_a^b \Big(\partial_{t^+}X_t + \partial_t^* X_t\Big) dt. \qquad \square$$

COROLLARY 7.10. Proposition 7.10 *remain true if "continuous" is replaced by "piecewise continuous".*

EXAMPLE 7.11. (a) Suppose that X_t is a "constant" $\mathcal{E}^*$-valued random variable, say $X_t = F$ $(F \in \mathcal{E}^*)$. Then we have

$$\int_a^b X_t dB(t^+) = \int_a^b X_t dB(t^-) = F(B(b) - B(a))$$

and, by the definitions given above, we have

$$\mathbf{I}^{+}_{[a,b]}(X_t) = \mathbf{I}^{-}_{[a,b]}(X_t) = \int_a^b \partial_t^* X_t dt - F(B(b) - B(a)). \tag{7.3}$$

But in general $\partial_{t^-} F$ does not exist unless that F is sufficient smooth, say $F \in \mathcal{E}$ or F is measurable with respect to the σ- field $\mathcal{B}_a$ generated by $\{B(u) : 0 \leq u \leq a\}$.

Take $F = B(1)$, for example. It follows from (7.3) that we obtain $\mathbf{I}^{+}_{[0,1]}(X_t) = \mathbf{I}^{-}_{[0,1]}(X_t) = 1$. On the other hand, note that $F(x) = (x, b_1) \in \mathcal{E}_0$ and recall that $\partial_{t^+} = \mathbf{1}_{[0,1)}$ and $\partial_{t^-} = \mathbf{1}_{(0,1]}$ (see Example 7.2), we conclude that $\mathbf{I}^{+}_{[a,b]}(X_t) = \int_0^1 \partial_{t^+} B(1)dt$ and $\mathbf{I}^{-}_{[a,b]}(X_t) = \int_0^1 \partial_{t^-} B(1)dt$.

(b) It follows by direct computation, we obtain

$$\begin{aligned} \int_a^b B(t)dB(t^+) &= \frac{1}{2}(B^2(b) - B^2(a)) - \frac{1}{2}(b-a) \\ \text{and} \quad \int_a^b B(t)dB(t^-) &= \frac{1}{2}(B^2(b) - B^2(a)) + \frac{1}{2}(b-a). \end{aligned}$$

(c) If $\{X_t\}$ is nonanticipating such that $\mathbf{E}[\int_a^b |X_t|^2]dt < \infty$, then we have

$$\begin{aligned} \partial_{t^+} X_t &= 0 \\ \text{and} \quad \int_a^b X_t dB(t^+) &= \int_a^b \partial_t^* X_t dt. \end{aligned}$$

(d) It follows from Example 7.2(a) that if $f \in \mathcal{S}$ we have

$$\begin{aligned} \int_a^b f(B(t))dB(t^+) &= \int_a^b \partial_t^* f(B(t))dt \\ \text{and} \quad \int_a^b f(B(t))dB(t^-) &= \int_a^b \Big(f'(B(t)) + \partial_t^* f(B(t))\Big)dt. \end{aligned}$$

In particular, we have

$$\begin{aligned} \int_a^b B(t)dB(t^+) &= \int_a^b \partial_t^* B(t)dt \\ &= \frac{1}{2}\Big((B^2(b) - B^2(a) - (b-a)\Big) \\ \text{and} \quad \int_a^b B(t)dB(t^-) &= \int_a^b \Big(1 + \partial_t^* B(t)\Big)dt \\ &= \frac{1}{2}\Big((B^2(b) - B^2(a) + (b-a)\Big). \end{aligned}$$

REMARK 7.12. The stochastic integration for anticipating integrands had been studied by many authors (see, for example, [1, 3, 7, 15, 16, 21]) by different methods. Our approach which depends very much on the calculus on $\mathcal{C}$ provides a direct way to study generalized stochastic integration. To see how far this approach can reach, it is desirable to investigate this subject more systematically along this line.

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