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LARGE TIME ASYMPTOTICS OF LÉVY PROCESSES AND RANDOM WALKS

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ABSTRACT. We consider a general class of real-valued Lévy processes $\{X(t),\ t \geq 0\}$, and obtain suitable large deviation results for the empiricals $L(t, A)$ defined by $t^{-1} \int_0^t 1_A(X(s)ds$ for $t > 0$ and a Borel subset A of $\mathbf{R}$. These results are used to obtain the asymptotic behavior of $P\{Z(t) \leq a\}$, where $Z(t) = \sup_{u \leq t} |x(u)|$, as $t \to \infty$, in terms of the rate function in the large deviation principle. A subclass of these processes is the Feller class: there exist nonrandom functions $b(t)$ and $a(t) > 0$ such that $\{(X(t) - b(t))/a(t) : t > 0\}$ is stochastically compact, i.e., each sequence has a weakly convergent subsequence with a nondegenerate limit. The stable processes are in this class, but it is much larger. We consider processes in this class for which $b(t)$ may be taken to be zero. For any $t > 0$, we consider the renormalized process $\{X(v\psi(t))/a(\psi(t)),\ v \geq 0\}$, where $\psi(t) = t(\log\log t)^{-1}$, and obtain large deviation probability estimates for $L_t(A) := (\log\log t)^{-1} \int_0^{\log\log t} 1_A(X(v\psi(t))/a(\psi(t)))dv$. It turns out that the upper and lower bounds are sharp and depend on the entire compact set of limit laws of $\{X(t)/a(t)\}$. The results extend to random walks in the Feller class as well. Earlier results of this nature were obtained by Donsker and Varadhan for symmetric stable processes and by Jain for random walks in the domain of attraction of a stable law.

1. Introduction

We consider the large time behavior of a Lévy process $\{X(t),\ t \geq 0\}$, i.e., a process with stationary independent increments. We will assume

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throughout that the process has right-continuous paths with left limits at every point. We will use the notation

$$P\{X(0)=0\}=1$$

and

$$P^x\{X(0)=x\}=1.$$

We will consider a real-valued process, but most of the results considered here hold in higher dimensions; the proofs require obvious modifications. If $\phi_t(u)$ denotes the characteristic function of $X(t)$, then

$$\phi_t(u)=\exp\{\psi_t(u)\},$$

where

$$\psi_t(u)=t[i\gamma u-\frac{\sigma^2}{2}u^2+\int_{\mathbf{R}-\{0\}}\{e^{iuy}-1-\frac{iuy}{1+y^2}\}d\nu(y)], \tag{1.1}$$

where ν denotes the Lévy measure, which satisfies the property

$$\int_{\mathbf{R}-\{0\}}\frac{y^2}{1+y^2}d\nu(y)<\infty. \tag{1.2}$$

It will be convenient at times to consider

$$d\hat{\nu}(y)=\frac{y^2}{1+y^2}d\nu(y), \tag{1.3}$$

so that $\hat{\nu}$ is a finite measure on $\mathbf{R}$ which assigns zero mass to the origin.

Among the questions that we consider, the simplest one to state is the asymptotic behavior of the distribution of $Z_t := \sup_{u\le t}|X(u)|$, as $t\to\infty$. For standard Brownian motion, the exact distribution of Z_t was discovered by Feller, see, e.g. [1]. However, the question becomes more complicated if we consider Brownian motion with a drift. This question and others that we consider here were answered for symmetric stable processes by Donsker and Varadhan [5] by using a large deviation principle for the empiricals. These results were extended to random walks for which the summands were in the domain of attraction of a strictly stable process in [9]. The process $X(t)$ is called strictly stable if $X(t)$ and $t^{1/\alpha}X(1)$ have the same distribution, where $0<\alpha\le 2$ denotes the index of the stable process.

We now introduce conditions that will often be required to be satisfied by the Lévy process.

CONDITION A. *If $\sigma^2 = 0$ in (1.1), then for every $a > 0$, $\nu[(-a,0)] \wedge \nu[(0,a)] > 0$.*

CONDITION B. *In addition to Condition A, the transition probability function $p(t,x,dy)$ has a density $p(t,x,y)$ with respect to Lebesgue measure m on $\mathbf{R}$ for $t > 0$.*

To give some description of the main results, we need to introduce further notation. We denote by C^∞ the class of infinitely differentiable functions on $\mathbf{R}$, and by C_K^∞ the class of compactly supported functions in C^∞. We write

$$\mathcal{E} = \{f \in C^\infty : f(x) = a > 0 \text{ for all } x \text{ outside of some compact set}\}.$$

D will denote the generator of the process on the Banach space of bounded continuous functions with the sup norm. If it is necessary to emphasize the process in question or its distribution at time 1, we will use a subscript on D to indicate that.

For $t > 0$, A a Borel set, we denote the normalized empirical of the process by

$$L(t,A) := \frac{1}{t}\int_0^t 1_A(X(s))ds, \tag{1.5}$$

where 1_A denotes the indicator of the set A. The dependence on ω, the sample path, will always be clear and will be omitted. We will use superscripts on L to indicate dependence on the process.

We will denote by $\mathcal{M}$ the class of probability measures on $\mathbf{R}$ with the topology of weak convergence. Also, M will denote the class of subprobability measures on $\mathbf{R}$ with the topology of vague convergence.

For Lévy processes the rate function in the large deviation principle for the empiricals is given by

$$I(\mu) := -\inf_{u\in\mathcal{E}} \int_{\mathbf{R}} \frac{Du}{u}(x)d\mu(x), \tag{1.6}$$

where we will use subscripts on I to indicate dependence on the process. I is lower semicontinuous with respect to the topologies of vague and weak convergence.

We will now describe the Feller class of distributions on $\mathbf{R}$. Let $\mu \in \mathcal{M}$, and define (for μ) for $x > 0$

$$G(x) := \mu\{y : |y| > x\}, \ K(x) := x^{-2}\int_{|y|\le x} y^2 d\mu(y),$$

$$Q(x) = G(x) + K(x).$$

The function Q is continuous and strictly decreasing for large x, see [8]. Let

$$\mathcal{F} = \{\mu \in \mathcal{M} : \limsup_{x\to\infty} \frac{G(x)}{K(x)} < \infty\}.$$

If μ is in the domain of attraction of a stable law of index α, then

$$\lim_{x\to\infty} \frac{G(x)}{K(x)} = \frac{2-\alpha}{\alpha}.$$

Feller showed that $\mu \in \mathcal{F}$ iff there exist a real sequence $\{b_n\}$ and a sequence of strictly positive numbers $\{a_n\}$ such that $a_n^{-1}(S_n - b_n)$ is *stochastically compact*, i.e., every sequence has a weakly convergent subsequence with a nondegenerate limit; here S_n denotes the sum of n i.i.d. random variables $X_1, X_2, \ldots, X_n$, with μ the distribution of X_1. The set of such limit laws was characterized by Pruitt [13]. We would like to remark here that one choice of $\{a_n\}$ is given by $n\, Q(a_n) = 1$. We assume here that b_n may be taken to be zero.

For an infinitely divisible law with Lévy measure ν, following [13] we define for $x > 0$

$$G^\nu(x) = \nu\{y : |y| > x\},\ K^\nu(x) = x^{-2}(\sigma^2 + \int_{|y|\le x} y^2 d\nu(y)),$$

and

$$Q^\nu(x) = G^\nu(x) + K^\nu(x).$$

Pruitt [13] showed that there exist $c_1 > 0$, $c_2 > 0$, $x_0 > 0$ such that $c_1 \le Q(x)/Q^\nu(x) \le c_2$ for all $x \ge x_0$, where Q is defined for μ as before, and μ is infinitely divisible with Lévy measure ν. In this case, if $\mu \in \mathcal{F}$, then we can define $a(t)$ by

$$t\, Q^\nu(a(t)) = 1$$

for $t \ge$ some t_0. Define $a(t) = 1$ for $t \le t_0$. Then if $X(t)$, $t \ge 0$, is the Lévy process with μ the distribution of $X(1)$, then (assuming that the centering constants $b(t)$ may be taken to be zero) we have that $\{X(t)/a(t)\}$ is stochastically compact. We will denote by K the set of limit laws, which is necessarily a compact set.

We now describe some of the main results. If $\{X(t),\ t \ge 0\}$ is a Lévy process, and $L(t,\cdot)$ is defined by (1.5), we obtain a large deviation principle for $L(t,\cdot)$ in Theorem 2.10 (upper bound) and Theorem 2.15 (lower bound). The lower bound is obtained in a form that is particularly useful for applications to renormalized Lévy processes in the Feller class and to random walks with summands in the Feller class.

It is useful to know when a $\mu \in \mathcal{M}$ can be approximated weakly by compactly supported μ_n such that $I(\mu_n) \to I(\mu)$, where I denotes the rate function for a Lévy process. We show in Theorem 2.22 that this holds for all symmetric Lévy processes. In general, we do not know the answer. The key here is the much more important result of Donsker and Varadhan [3] where they obtain a closed form formula for $I(\mu)$ in the symmetric case. It would be interesting to obtain a closed form formula for $I(\mu)$ at least for a strictly stable process.

In section 3 we consider the asymptotic distribution of $Z_t = \sup_{u \le t} |X(u)|$. We show that, roughly speaking,

$$\lim_{t\to\infty} \frac{1}{t} \log P^x\{Z_t \le a\} = \varphi(a),$$

where $\varphi(a) := \inf\{I(\mu) : \mu([-a, a]) = 1\}$, and $0 < \varphi(a) < \infty$, provided the process satisfies Condition B. See Theorems 3.4 and 3.8 for precise statements.

We consider renormalized Lévy processes in the Feller class in section 4; the processes attracted to strictly stable processes are a subclass. Let $a(t)$ be the normalizer such that $\{X(t)/a(t),\ t > 0\}$ is stochastically compact. If $\{X(t)\}$ is a standard Brownian motion or, more generally, a symmetric stable process, then as shown in [5] and in many earlier papers by other authors, $a(\psi(t))$, where $\psi(t) = t/\log\log t$, $t > e$, is the right normalizer for Z_t, i.e., $\liminf_{t\to\infty} Z_t/a(\psi(t))$ is a.s. positive and finite. It turns out that if $\{X(t)\}$ is a Feller class process for which each $\theta \in K$, K being the weak limit set of $\{X(t)/a(t)\}$, satisfies Condition A, then $a(\psi(t))$ is the right normalizer here as well. We define for $t > e$,

$$L_t(A) := \frac{1}{\log\log t} \int_0^{\log\log t} 1_A\Big(\frac{X(v\psi(t))}{a(\psi(t))}\Big) dv.$$

To obtain analogues of sample paths of these processes and random walks in their "domain of attraction", as was done in [5] and [9], we obtain the large deviation principle for $L_t(\cdot)$ in section 4. The somewhat surprising fact here is that all I_θ, $\theta \in K$, play a role in the large deviation principle, and the probability bounds are sharp.

We indicate in section 5 the type of basic results one can obtain for the sample paths of such processes and random walks, which have many interesting applications.

2. Some general results for a Lévy process

We start with a lemma that will play an important role.

LEMMA 2.1. *Assume that the process $\{X(t)\}$ satisfies Condition A. Let $a > 0$, and let U be a nonempty open interval such that $\bar{U} \subset (-a, a)$. Then given $0 < a' < a$, there exists $h_o > 0$ such that for all $0 < h \leq h_o$*

$$\inf_{|x|\leq a'} P^x\{|X(t)| \leq a \text{ for all } t \leq h,\ X(h) \in U\} > 0. \tag{2.2}$$

Proof. We first consider the case when $\sigma^2 = 0$, i.e., the Brownian component is absent.

Let $U = (b, c)$, $-a < b < c < a$. Let $\epsilon > 0$ be such that

$$\epsilon < \{(c-b) \wedge (a-a')\}/4. \tag{2.3}$$

We can pick $\theta_1 > 0$, $\theta_2 > 0$ such that every open interval that contains either θ_1 or $-\theta_2$ has positive Lévy measure, and $\theta_1 + \theta_2 < \epsilon/8$. It is easily seen that there exists a positive integer k_0 and η, $0 < \eta < \theta_1 \wedge \theta_2$, such that given any $x \in [-a', a']$, for some nonnegative integer $k \leq k_0$ one of the following cases occurs: either

$$[x - \epsilon + k(\theta_1 - \eta),\ x + \epsilon + k(\theta_1 + \eta)] \subset U, \tag{2.4}$$

or

$$[x - \epsilon + k(-\theta_2 - \eta),\ x + \epsilon + k(-\theta_2 + \eta)] \subset U. \tag{2.5}$$

If $[x - \epsilon, x + \epsilon] \subset U$, then both (2.4) and (2.5) are satisfied with $k = 0$, otherwise (2.4) will occur when $x - \epsilon \leq b$ and (2.5) will occur when $x + \epsilon \geq c$.

We pick $h_1 > 0$ small such that

$$P\{X(t) \in (-\epsilon, \epsilon) \text{ for all } t \leq h_1\} = \delta_1 > 0. \tag{2.6}$$

If (2.4) occurs for the given $x \in [-a', a']$, we take $A = (\theta_1 - \eta, \theta_1 + \eta)$, and denote

$$X_A(t) := \sum_{u\leq t}(X(u) - X(u-))1_{\{v: X(v)-X(v-)\in A\}}(u),$$

$$\xi(t) := X(t) - X_A(t).$$

The processes $X_A(t)$ and $\xi(t)$ are mutually independent Lévy processes with right - continuous paths. There exists $h_2 > 0$ such that

$$P\{\xi(t) \in (-\epsilon, \epsilon) \text{ for all } t \leq h_2\} = \delta_2 > 0. \tag{2.7}$$

Also, for any $h > 0$, we have

(2.8) $$P\{X_A(h) = k\} = \exp\{-h\nu(A)\}(h\nu(A))^k/k!.$$

If (2.5) occurs for the given $x \in [-a', a']$, then we take $A = (-\theta_2 - \eta, -\theta_2 + \eta)$, and define $X_A(t)$ and $\xi(t)$ as above. Let $h_3 > 0$ be such that (2.7) holds in this case with h_3 replacing h_2. Let $\delta = \delta_1 \wedge \delta_2$, $h_0 = h_1 \wedge h_2 \wedge h_3$.

For all x that satisfy $[x - \epsilon, x + \epsilon] \subset U$, by (2.6) we have for all $0 < h \le h_0$,

$$P^x\{|X(t)| \le a \text{ for all } t \le h,\ X(h) \in U\} \ge \delta.$$

For any $|x| \le a'$, the event $\{|X(t) + x| \le a \text{ for all } t \le h, X(h) + x \in U\}$ contains the intersection of the events $\{\xi(t) + x \in (-\epsilon, \epsilon) \text{ for all } t \le h\}$ and $\{X_A(h) = k\}$. It follows that if $h \le h_0$, we have

$$\inf_{|x| \le a'} P^x\{|X(t)| \le a \quad \text{for all} \quad t \le h,\ X(h) \in U\}$$
$$\ge \delta \exp\{-h\nu(A)\}\ (h\nu(A))^{k_0}/k_0!.$$

This proves the lemma when $\sigma^2 = 0$. If $\sigma^2 > 0$, ν is zero, and $\gamma = 0$, i.e., we have Brownian motion with scale parameter σ^2, then the conclusion follows immediately from the fact that the measure

$$\mu(x, E) := P^x\{|X(t)| \le a \text{ for } t \le h,\ X(h) \in E\}$$

defined on the Borel subsets of $(-a, a)$, has a strictly positive density with respect to Lebesgue measure if $|x| < a$ (see, e.g. [1], p. 79, (11.10)). It is also known (Lemma 2.10 [5]) that $\mu(x, E)$ is a continuous function of $x \in (-a, a)$.

We now consider the general case, when $\sigma^2 > 0$ and the Lévy measure ν is nonzero. We have the decomposition

$$X(t) = B(t) + Y(t),$$

where $B(t)$ is continuous path Brownian motion with scale parameter σ^2, and $Y(t)$ is a Lévy process with right-continuous paths which satisfies Condition A; $Y(t)$ and $B(t)$ are independent processes. We choose $\epsilon > 0$ small so that $a' + \epsilon < a$ and $2\epsilon < c - b$. We pick h_o small such that for $h \le h_o$

$$\inf_{|x| \le a'} P^x\{|Y(t)| \le a - \epsilon \text{ for all } t \le h,\ Y(h) \in (b + \epsilon, c - \epsilon)\} > 0,$$

and

$$P\{|B(t)| \le \epsilon \text{ for all } t \le h_o\} > 0.$$

The result follows immediately from these inequalities. □

The lemma has the following corollary:

COROLLARY 2.9. *Given $a > 0$ and $0 < a' < a$, there exists $0 < \eta < \infty$ such that*

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{|x|\le a'} P^x\{|X(u)| \le a \text{ for all } u \le t\} \ge -\eta.$$

Proof. Let $U = (-a', a')$ in Lemma 2.1. Then we have $h_o > 0$ such that

$$\inf_{|x|\le a'} P^x\{|X(u)| \le a \text{ for } 0 \le u \le h_o,\ |X(h_o)| < a'\} \ge \delta > 0.$$

By the Markov property, we then have for any positive integer k,

$$\inf_{|x|\le a'} P^x\{|X(u)| \le a \text{ for } 0 \le u \le kh_o\} \ge \delta^k.$$

If $kh_o \le t < (k+1)h_o$, then

$$\inf_{|x|\le a'} P^x\{|X(u)| \le a \text{ for } 0 \le u \le t\} \ge \delta^k \ge \delta^{(t/h_o)},$$

which proves the corollary with $\eta = h_o^{-1} \log(1/\delta)$. □

We now state a large deviation upper bound of Donsker and Varadhan [4] which holds for any Feller semi-group.

THEOREM 2.10. *If $\{X(t)\}$ is a Lévy process, $\mu \in M$, and $c < I(\mu)$, then there exists a vague neighborhood N of μ such that*

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_x P^x\{L(t,\cdot) \in N\} \le -c,$$

where $I(\mu)$ is given by (1.6) with D denoting the generator of $\{X(t)\}$. Furthermore, if $C \subset M$ is closed, then

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_x P^x\{L(t,\cdot) \in C\} \le -\inf_{\mu\in C} I(\mu).$$

The next result is a corollary of this theorem and Corollary 2.9.

LEMMA 2.11. *If the Lévy process $\{X(t)\}$ satisfies Condition A, then given $a > 0$, there exists $\mu \in \mathcal{M}$ with support in $[-a, a]$ such that $I(\mu) < \infty$.*

Proof. Let

$$C = \{\mu \in \mathcal{M} : \text{ support of } \mu \subset [-a, a]\}.$$

Since the process has right-continuous paths, we have $L(t, \cdot) \in C$ if and only if $|X(u)| \leq a$ for all $u \leq t$. By Corollary 2.9 and Theorem 2.10 there exists $\eta < \infty$ such that $-\eta \leq -\inf\{I(\mu) : \mu \in C\}$. Since I is lower semicontinuous, and C is compact, there exists $\mu_o \in C$ such that $I(\mu_o) = \inf\{I(\mu) : \mu \in C\}$, and we have $I(\mu_o) \leq \eta$, which proves the lemma. □

We now prove the large deviation lower bound for Lévy processes that satisfy the Condition B. We will need the analogue of Lemma 2.11 [5] under our more general condition.

LEMMA 2.12. *If the process satisfies Condition B, then for any $a > 0$, any Borel set $E \subset [-a, a]$ of positive Lebesgue measure, and $|x| < a$, we have*

$$\psi(x, E) := \int_0^\infty \hat{p}(t, x, E)e^{-t}dt > 0, \tag{2.13}$$

where

$$\hat{p}(t, x, E) := P^x\{|X(u)| \leq a \text{ for all } u \leq t,\ X(t) \in E\}. \tag{2.14}$$

Furthermore, $\hat{p}(t, x, E)$ is jointly continuous in $t > 0$ and $|x| < a$; consequently $\psi(\cdot, E)$ is a continuous function.

Proof. The joint continuity of $\hat{p}(t, x, E)$ in $t > 0$ and $|x| < a$ is proved in Lemma 2.10 [5] under the only assumption that the process satisfies Condition B. This immediately implies the continuity of $\psi(\cdot, E)$. For the proof of (2.13), we follow the proof of Lemma 2.11 [5] until the very end where one must check that for any $x \in (-a, a)$, any nonempty open interval U with $\bar{U} \subset (-a, a)$, there exists $h > 0$ such that

$$P^x\{|X(u)| \leq a \text{ for all } u \leq h,\ X(h) \in U\} > 0.$$

A stronger form of this statement is proved in Lemma 2.1 above under the weaker requirement of Condition A. This proves the lemma. □

THEOREM 2.15. *Suppose the process satisfies Condition B. Then for any $a > 0$, any $0 < a' < a$, any $\mu \in \mathcal{M}$ with topological support in*

$(-a, a)$, and any weak neighborhood N_μ of μ, we have

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{|x|\le a'} P^x\{L(t,\cdot) \in N_\mu,\ |X(u)| \le a \text{ for all } u \le t, |X(t)| \le a'\} \ge -I(\mu).$$

Proof. We can use the results of Donsker and Varadhan [4] together with Lemma 2.1 proved above to derive this result; it will, however, be more convenient for us to use Theorem 4.4 [10]. We need to check (2.13) and the joint continuity of $\hat{p}(t, x, E)$, which are already proved in Lemma 2.12 above, and another condition, namely that $I(\mu) < \infty$ implies that $\mu << m$, where m denotes Lebesgue measure. As shown in [10], this condition is satisfied if $m(A) = 0$ implies that for every $x \in \mathbf{R}$, there exists a $t > 0$ such that

$$\int_t^\infty p(u, x, A)e^{-u} du = 0.$$

This condition is clearly implied by Condition B ($p(u, x, dy)$ is absolutely continuous with respect to Lebesgue measure). By Lemma 2.12, the function $\psi(\cdot, E)$ is continuous and Theorem 4.4 [10] then implies that

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{|x|\le a'} P^x\{L(t,\cdot) \in N_\mu,\ |X(u)| \le a \text{ for all } u \le t\} \ge -I(\mu).$$

To finish the proof of the theorem, without any loss of generality, we may assume $I(\mu) < \infty$. Let a'' be such that $a' < a'' < a$ and support $\mu \subset (-a'', a'')$. Then given $\epsilon > 0$, there exists $t_0 > 0$ such that for all $t \ge t_0$ we have

$$\frac{1}{t} \log \inf_{|x|\le a'} P^x\{L(t,\cdot) \in N_\mu,\ |X(u)| \le a'' \text{ for all } u \le t\} \ge -(I(\mu) + \epsilon).$$

By Lemma 2.1 we also have for some $h > 0$

$$\inf_{|x|\le a''} P^x\{|X(u)| \le a, \text{ for all } u \le h,\ |X(h)| \le a'\} \ge \delta > 0.$$

By the Markov property, we have for all $|x| \le a'$, $t \ge t_0$, h as above,

$$P^x\{L(t+h,\cdot) \in N_\mu,\ |X(u)| \le a \text{ for all } u \le t+h,\ |X(t+h)| \le a'\}$$
$$\ge P^x\{L(t+h,\cdot) \in N_\mu, |X(u)| \le a'' \text{ for all } u \le t\}.$$
$$\inf_{|y|\le a''} P^y\{|X(u)| \le a, \text{ for all } u \le h,\ |X(h)| \le a'\} \ge \exp\{-t(I(\mu)+\epsilon)\}\cdot\delta.$$

The theorem follows from this immediately. □

The next result is of independent interest, which was observed above in the proof of Theorem 2.15. See [10] for the proof in a more general context.

THEOREM 2.16. *If $\{X(t)\}$ satisfies Condition B, then $I(\mu) < \infty \Longrightarrow \mu << m$.*

For a different application, we would like to state below another lower bound result which is an immediate corollary of Theorem 4.4 [10] whose conditions have already been checked; one needs to observe $p(t, x, E)$ is a continuous function of x for $t > 0$ if $p(t, x, dy)$ has density $p(t, x, y)$.

THEOREM 2.17. *Suppose the process satisfies Condition B. Then for any $a > 0$, any $\mu \in \mathcal{M}$ and any weak neighborhood N_μ of μ, we have*

$$\liminf_{t \to \infty} \frac{1}{t} \log \inf_{|x| \le a} P^x\{L(t, \cdot) \in N_\mu\} \ge -I(\mu).$$

We now give some auxiliary results which are of independent interest. If $\mu \in \mathcal{M}$ and $I(\mu) < \infty$, can one find $\mu_n \in \mathcal{M}$ with compact support such that $\mu_n \to \mu$ weakly and $I(\mu_n) \to I(\mu)$ as $n \to \infty$? This is a nontrivial question. For standard Brownian motion, this is proved in [2]. We will prove this result below for any symmetric Lévy process.

For a Lévy process, Lebesgue measure is the unique invariant measure (modulo a constant multiplier). By Lemma 2.5 [3], there exists no $\mu \in \mathcal{M}$ such that $I(\mu) = 0$, since that would imply that μ is an invariant probability measure for the process. In general, it is not true that $\inf\{\mu \in \mathcal{M} : I(\mu)\} = 0$ for a Lévy process; in fact, Brownian motion with constant drift is a counterexample [4]. However, for a symmetric Lévy process we will also show below that there exist compactly supported $\mu_n \in \mathcal{M}$ such that $I(\mu_n) \to 0$. This result is also true for a strictly stable process (symmetric or not) which satisfies Condition A.

Let D denote the infinitesimal generator of the process for the strongly continuous semigroup on C, the Banach space of bounded continuous functions on $\mathbf{R}$ with the sup norm. Let $\mathcal{D}$ denote the domain of D. We will denote by C_0^∞ the space of infinitely differentiable functions on $\mathbf{R}$ which together with derivatives of all orders vanish at infinity, and by C_K^∞ its subset of those functions that have compact support.

The next lemma is well-known. For a function u, as usual, $u', u'', \ldots, u^{(k)}$ denote the first, second, ..., k-th order derivatives of u.

LEMMA 2.18. *The space C_0^∞ is contained in $\mathcal{D}$, and if $u \in C_0^\infty$, then*

$$(2.19) \qquad Du(x) = \gamma u'(x) + \frac{\sigma^2}{2}u''(x) + \int_{\mathbf{R}-\{0\}} h_u(x,y)d\nu(y),$$

where ν is the Lévy measure corresponding to D,

$$h_u(x,y) := u(x+y) - u(x) - \frac{y}{1+y^2}u'(x),$$

and

$$\lim_{y\to 0} \frac{1+y^2}{y^2}h_u(x,y) := \frac{1}{2}u''(x).$$

The Lévy process also generates a strongly continuous semigroup $\{\tilde{T}_t\}$ on $L^2(m) := L^2$. Let $\tilde{D}$ denote the corresponding generator on L^2, and let $\tilde{\mathcal{D}}$ denote its domain. If $\tilde{T}_t$ is self-adjoint for each $t > 0$, we say that the process is symmetric. If Condition B holds, then this simply means that $p(t,x,y) = p(t,y,x)$ for all x, y and $t > 0$. We will also denote

$$\mathcal{D}_o = \{f \in C_0^\infty : f^{(k)} \in L^2 \text{ for all } k \geq 0\}$$

where $f^{(0)} = f$, and $f^{(k)}$ denotes the k-th derivative of f.

LEMMA 2.20. *For a Lévy process C_K^∞ is a core for the generator $\tilde{D}$ on L^2.*

Proof. For $f \in \mathcal{D}_o$ we have

$$\begin{aligned} \tilde{T}_t f(x) &= \textstyle\int_{\mathbf{R}} f(y)p(t,x,dy) \\ &= \textstyle\int_{\mathbf{R}} f(x+y)p(t,0,dy). \end{aligned}$$

By dominated convergence

$$(\tilde{T}_t f)^{(k)}(x) = \int_{\mathbf{R}} f^{(k)}(x+y)p(t,0,dy),$$

therefore

$$\|(\tilde{T}_t f)^{(k)}\|_2 \leq \|f^{(k)}\|_2, \ k \geq 0.$$

This shows that $\tilde{T}_t$ takes $\mathcal{D}_o$ into $\mathcal{D}_o$. Since $\mathcal{D}_o$ is dense in L^2, by Proposition 3.3, Chapter 1 [7] the space $\mathcal{D}_o$ is a core for $\tilde{D}$.

To show that C_K^∞ is a core, it suffices to show that for any $f \in \mathcal{D}_o$, there exists a sequence $\{f_n\}$ in C_K^∞ such that $f_n \to f$ in L^2 and $\tilde{D}f_n \to \tilde{D}f$ in L^2, as $n \to \infty$.

Let $f \in \mathcal{D}_o$ be given. For $n \geq 1$, let $h_n \in C_K^\infty$ be such that $\|h_n\|_\infty \leq 1$, and $h_n(x) = 1$ for $x \in [-n, n]$. Let $f_n(x) := h_n(x)f(x), \ x \in \mathbf{R}$. Then

$\|f_n^{(k)} - f^{(k)}\|_2 \to 0$ for all $k \geq 0$. Let $g_n := f_n - f$. We will show that $\|\tilde{D}g_n\|_2 \to 0$ as $n \to \infty$. Let

$$\tilde{D}_1 g_n(x) = \int_{\mathbf{R}-\{0\}} \left\{ g_n(x+y) - g_n(x) - \frac{y}{1+y^2} g_n'(x) \right\} d\nu(y).$$

Let $\theta_n(x,y)$ denote the integrand. By Taylor's theorem

$$\theta_n(x,y) = \frac{y^3}{1+y^2} g_n'(x) + \frac{y^2}{2} g_n''(x) + \frac{1}{2}\int_0^y (y-t)^2 g_n^{(3)}(x+t)dt.$$

Therefore

$$|\theta_n(x,y)| \leq \frac{|y|^3}{1+y^2}|g_n'(x)| + \frac{y^2}{2}|g_n''(x)| + \frac{1}{2}\int_0^{|y|} y^2 |g_n^{(3)}(x+t)|dt. \tag{2.21}$$

This estimate will be used for $|y| \leq 1$. By Schwarz inequality

$$\left|\int_{|y|\geq 1} \theta_n(x,y) d\nu(y)\right|^2 \leq \nu\{y : |y| \geq 1\}$$
$$\times \int_{|y|\geq 1} \{|g_n(x+y)|^2 + |g_n(x)|^2 + |g_n'(x)|^2\} d\nu(y),$$

hence

$$\left\|\int_{|y|\geq 1} \theta_n(x,y) d\nu(y)\right\|_2^2 \leq \{2\|g_n\|_2^2 + \|g_n'\|_2^2\}(\nu\{y : |y| \geq 1\})^2,$$

and the right side tends to zero as $n \to \infty$. Also, by (2.21) we have

$$\left|\int_{|y|<1} \theta_n(x,y) d\nu(y)\right|$$
$$\leq \int_{|y|\leq 1} \{|g_n'(x)| + \frac{1}{2}|g_n''(x)| + \frac{1}{2}\int_0^1 |g_n^{(3)}(x+t)|dt\} y^2 d\nu(y).$$

Therefore by Schwarz inequality

$$\left\|\int_{|y|<1} \theta_n(x,y) d\nu(y)\right\|_2^2$$
$$\leq \int_{|y|\leq 1} y^2 d\nu(y) \int_{|y|\leq 1} \{\|g_n'\|_2^2 + \frac{1}{4}\|g_n''\|_2^2 + \frac{1}{4}\|g_n^{(3)}\|_2^2\} y^2 d\nu(y),$$

and the right side again tends to zero as $n \to \infty$. This is clearly enough, and the lemma is proved. □

The following theorem is a corollary of the above lemma and a theorem of Donsker and Varadhan [3], Theorem 5.

THEOREM 2.22. *If the process is symmetric and $\mu \in \mathcal{M}$ is such that $I(\mu) < \infty$, then there exists a sequence $\{\mu_n\} \subset \mathcal{M}$, each μ_n having compact support, such that $\mu_n \to \mu$ weakly and $I(\mu_n) \to I(\mu)$.*

Proof. By Theorem 5 [3], $I(\mu) < \infty$ if and only if $\mu << m$ and $f := (d\mu/dm)^{1/2}$ belongs to the domain of $(-\tilde{D})^{1/2}$. Furthermore, if $I(\mu) < \infty$, then $I(\mu) = \|(-\tilde{D})^{1/2} f\|_2^2$.

If f belongs to the domain of $\tilde{D}$, then by Lemma 2.19, there exist $f_n \in C_K^\infty$ such that $f_n \to f$ in L^2, and $\tilde{D} f_n \to \tilde{D} f$ in L^2. Since $f \in$ domain of $(-\tilde{D})^{1/2}$, denoting inner product in L^2 by $\langle \cdot, \cdot \rangle$, it follows that

$$\begin{aligned} I(\mu) &= \|(-\tilde{D})^{1/2} f\|_2^2 = \langle -\tilde{D} f, f \rangle \\ &= \lim_{n\to\infty} \langle -\tilde{D} f_n, f_n \rangle = \lim_n I(\mu_n). \end{aligned}$$

Thus the conclusion holds for $f \in$ domain of $-\tilde{D}$. We now borrow a trick from Donsker and Varadhan [2]. If $f \in$ domain of $(-\tilde{D})^{1/2}$, we define

$$g_n(x) = \int f^2(x-y) h_n(y) dm(y),$$

where h_n is the density of a Gaussian distribution with zero mean and variance $1/n$. Then $g_n^{1/2}$ belongs to $\mathcal{D}_o$. By the lower semicontinuity of I, we have

$$\liminf_{n\to\infty} I(g_n^2 dm) \geq I(f^2 dm),$$

since $g_n^2 dm \to f^2 dm$ weakly. On the other hand, by convexity, we have

$$I(g_n^2 dm) \leq I(f^2 dm), \ n \geq 1.$$

Therefore

$$\lim_{n\to\infty} I(g_n^2 dm) = I(f^2 dm) = I(\mu).$$

From the first part, for each n we can approximate $I(g_n^2 dm)$ by $I(\mu_{n,k})$, where $\mu_{n,k} \in \mathcal{M}$ has compact support and $\mu_{n,k} \to g_n^2 dm$ weakly as $k \to \infty$. This proves the theorem. □

The next result shows that for a symmetric Lévy process there exist $\mu_n \in \mathcal{M}$ with compact supports such that $I(\mu_n) \to 0$.

LEMMA 2.23. *For a symmetric Lévy process there exist $\mu_n \in \mathcal{M}$, $n \geq 1$, each μ_n with compact support, such that $I(\mu_n) \to 0$, as $n \to \infty$.*

Proof. Let $f \in C_K^\infty$ be such that $\int f^2 dm = 1$. Let

$$f_n(x) := \frac{1}{\sqrt{n}} f(x/n), \ n \geq 1.$$

Then $\int f_n^2 dm = 1$. Let $\mu_n = f_n^2 dm$. Since f_n belongs to the domain of $-\tilde{D}$, it belongs to the domain of $(-\tilde{D})^{1/2}$, and we have by Theorem 5 [3],

$$I(\mu_n) =< -\tilde{D} f_n, f_n > .$$

It suffices to show that $\|\tilde{D} f_n\|_2 \to 0$, as $n \to \infty$. In the symmetric case, the Lévy part of the generator is given by

$$\tilde{D}_1 f_n(x) := \int \{f_n(x+y) + f_n(x-y) - 2f_n(x)\} d\nu(y),$$

and we can follow the argument given in the proof of Lemma 2.20, with obvious changes, to conclude that $\|\tilde{D} f_n\| \to 0$, as $n \to \infty$. This proves the theorem. □

For a strictly stable process one does not need symmetry because of the scaling property. The next lemma states this result.

LEMMA 2.24. *If the process is strictly stable and satisfies the Condition A, then there exist $\mu_n \in \mathcal{M}$, each with compact support, such that $I(\mu_n) \to 0$ as $n \to \infty$.*

Proof. By Lemma 2.11, there exists $\mu \in \mathcal{M}$ with compact support such that $I(\mu) < \infty$. Let α, $0 < \alpha \leq 2$, denote the index of the process, and define

$$\mu_n(A) := \mu(A/n^{1/\alpha}).$$

By the scaling property, as $n \to \infty$

$$I(\mu_n) = I(\mu)/n \to 0,$$

and the lemma is proved. □

The following corollary of these lemmas is useful.

COROLLARY 2.25. *Suppose the Lévy process satisfies the Condition A. Assume that the process is either symmetric or strictly stable. Then given any $\mu \in M$, there exist $\mu_n \in \mathcal{M}$ such that $\mu_n \to \mu$ vaguely, $I(\mu_n) < I(\mu)$, and $I(\mu_n) \to I(\mu)$, as $n \to \infty$.*

Proof. Let $\mu(\mathbf{R}) = \theta \leq 1$. By Lemmas 2.23 and 2.24, there exist $\gamma_n \in \mathcal{M}$ such that $I(\gamma_n) \to 0$, as $n \to \infty$. Define $\mu_\theta = \theta^{-1}\mu$, and

$$\mu_n := \theta\mu_\theta + (1-\theta)\gamma_n.$$

Then $\mu_n \in \mathcal{M}$, and

$$\begin{aligned} I(\mu_n) &\leq \theta I(\mu_\theta) + (1-\theta)I(\gamma_n) \\ &= I(\mu) + (1-\theta)I(\gamma_n) \to I(\mu) \end{aligned}$$

as $n \to \infty$. On the other hand, $\mu_n \to \mu$ vaguely, so $\liminf I(\mu_n) \geq I(\mu)$. The corollary is proved. □

3. The asymptotic distribution of $\sup_{u \leq t} |X(u)|$

We need to introduce some notation first. Let

$$Z(t) := \sup_{u \leq t} |X(u)|, \quad t \geq 0.$$

For $a > 0$, we write

$$\begin{aligned} C(a) &:= \{\mu \in \mathcal{M} : \mu([-a,a]) = 1\}, \\ \hat{C}(a) &:= \{\mu \in \mathcal{M} : \text{topological support of } \mu \subset (-a,a)\}, \end{aligned}$$

(3.1)
$$\varphi(a) := \inf\{I(\mu) : \mu \in C(a)\},$$

(3.2)
$$\hat{\varphi}(a) := \inf\{I(\mu) : \mu \in \hat{C}(a)\}.$$

Since $\hat{C}(a) \subset C(a)$, we have

(3.3)
$$\varphi(a) \leq \hat{\varphi}(a), \quad a > 0.$$

The following theorem is the main result of this section.

THEOREM 3.4. *If the process satisfies Condition B, then for any a and a' such that $0 < a' < a$, we have*

$$\begin{aligned} -\hat{\varphi}(a) &\leq \liminf_{t\to\infty} \frac{1}{t} \log \inf_{|x| \leq a'} P^x\{Z(t) < a\} \\ &\leq \limsup_{t\to\infty} \frac{1}{t} \log \sup_x P^x\{Z(t) \leq a\} \leq -\varphi(a). \end{aligned} \tag{3.5}$$

Furthermore, $\varphi(a) = \hat{\varphi}(a)$ except for countably many values of a. Consequently, all inequalities in (3.5) are actually equalities except for countably many values of a.

Proof. If $0 < a < b$, then $\hat{C}(b) \supset C(a)$, hence

$$\varphi(a) \geq \hat{\varphi}(b). \tag{3.6}$$

If $c_n \uparrow c$, and φ is continuous at c, then by (3.6) we have $\varphi(c_n) \geq \hat{\varphi}(c)$, and since φ and $\hat{\varphi}$ are decreasing functions, this implies that $\varphi(c-) \geq \hat{\varphi}(c)$. By the assumed continuity of φ at c and by (3.3), this implies that $\varphi(c) = \hat{\varphi}(c)$.

The set $C(a)$ is compact, and $\{Z(t) \leq a\} = \{L(t, \cdot) \in C(a)\}$ a.s. by the right-continuity of the paths of the process. Therefore, the upper bound in (3.5) follows from Theorem 2.10. The lower bound in (3.5) immediately follows from Theorem 2.15.

Finally, since φ is decreasing, it is discontinuous at only countably many points. The theorem is therefore proved. □

REMARK 3.7. One can check that φ is right-continuous and $\hat{\varphi}$ is left-continuous. We have not been able to show that either function is continuous if the process is assumed to satisfy Condition B alone. However, if the process is strictly stable, then $\varphi = \hat{\varphi}$. We state it below as a theorem.

THEOREM 3.8. *If the process is strictly stable and satisfies Condition A, then for any a and a', $0 < a' < a$, we have*

$$\begin{aligned} \liminf_{t\to\infty} \frac{1}{t} \log \inf_{|x|\leq a'} P^x\{Z(t) < a\} &= \limsup_{t\to\infty} \frac{1}{t} \log \sup_x P^x\{Z(t) \leq a\} \\ &= -\varphi(a). \end{aligned}$$

Proof. If $\mu \in \mathcal{M}$, then by the scaling property, if α is the index of stability, we have for any $\theta > 0$

$$\theta I(\mu_\theta) = I(\mu),$$

where $\mu_\theta(A) := \mu(\theta^{-1/\alpha} A)$ for any Borel set A. Let $a > 0$ be given. By the lower semicontinuity of I and the compactness of $C(a)$, there exists $\beta \in C(a)$ such that $\varphi(a) = I(\beta)$. Let $\theta_n \downarrow 1$, then the topological support of β_{θ_n} is contained in $(-a, a)$, so $I(\beta_{\theta_n}) \geq \hat{\varphi}(a)$. On the other hand, $I(\beta_{\theta_n}) = \theta_n^{-1} I(\beta) \to I(\beta) = \varphi(a)$, as $n \to \infty$. Therefore, $\varphi(a) \geq \hat{\varphi}(a)$, which together with (3.3) implies that $\varphi(a) = \hat{\varphi}(a)$. The theorem is thus proved. □

For applications it will be useful to state the analogue of Theorem 3.8 for a random walk whose summands are in the domain of attraction of a stable law for which the conditions of the above theorem hold.

Let $Y_1, Y_2, \ldots$ be independent, identically distributed random variables with a common distribution F. We assume that F is in the domain of attraction of a stable law G with index α, $0 < \alpha \leq 2$, such that the Lévy measure of G satisfies Condition A. Let

$$S_n := Y_1 + \cdots + Y_n, \quad n \geq 1$$

and let $a(n)$ be a sequence tending to infinity such that $S_n/a(n)$ converges weakly to G. This means that if $\alpha > 1$, then Y_1 has mean zero, and for $\alpha = 1$, we simply take Y_1 to be symmetric; for $\alpha < 1$, the centering does not matter.

Let $k(n)$ and $d(n)$ be positive integer sequences tending to infinity and let $k(n)d(n) = r(n)$. Then combining Theorems 3.2 and 4.1 from [9] with the above theorem we immediately have:

THEOREM 3.9. *We have for any* $0 < a' < a$,

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{k(n)} \log \inf_{|x|\leq a'} P\{|x + (S_j/a(d(n)))| \leq a,\ 1 \leq j \leq r(n)\} \\ = &\liminf_{n\to\infty} \frac{1}{k(n)} \log \sup_x P\{|x + (S_j/a(d(n)))| \leq a,\ 1 \leq j \leq r(n)\} \\ = &-\varphi(a), \end{aligned}$$

where φ *is defined by* (3.1) *in terms of the* I*-function of* G.

4. Large deviations for renormalized Lévy processes of Feller type

A Lévy process $\{X(t)\}$ is in the Feller class if there exists a nonrandom normalizing function $a(t)$ such that $a(t) > 0$ for all $t > 0$, $a(t) \to \infty$ as $t \to \infty$, and $\{X(t)/a(t),\ t > 0\}$ is a stochastically compact set in which no subsequence converges weakly to a degenerate limit law. This class contains all strictly stable processes (with $a(t) = t^{1/\alpha}$, α being the index of stability), but it is much larger. For more information on such distributions, we refer to [8],[12],[13].

We denote

$$\psi(t) := t/\log\log t,\ t > e, \tag{4.1}$$

and for $t > 0$

$$L_t(A) := \frac{1}{t}\int_0^t 1_A(X(s)/a(\psi(t)))ds, \tag{4.2}$$

which is the empirical measure for the normalized process. By a change of variable, we have

$$L_t(A) = \frac{1}{\log\log t}\int_0^{\log\log t} 1_A(\frac{X(v\psi(t))}{a(\psi(t))})dv. \tag{4.3}$$

If the process is strictly stable, then $X(v\psi(t))/a(\psi(t))$ has the same distribution as $X(v)$, and in that case, for a fixed t the distribution of $L_t(\cdot)$ in the space $\mathcal{M}$ is the same as that of

$$\tilde{L}_t(\cdot) := \frac{1}{\log\log t}\int_0^{\log\log t} 1_{\cdot}(X(v))dv. \tag{4.4}$$

We will be interested in the large deviations of $L_t(\cdot)$ for processes described above. For strictly stable processes, this is the same as studying the large deviations of the original process with a different time scale, and the I-function for the stable process governs the large deviation principle. However, in the new situation a whole class of limit laws enter the picture and we will see that all of them play a role in the large deviation principle.

For $t > e$, we write

$$Z_t(s) := \frac{X(s\psi(t))}{a(\psi(t))}, \ s \geq 0. \tag{4.5}$$

For a fixed t, $Z_t(s)$, $s \geq 0$, is a Lévy process.

Since $\{X(t)\}$ is assumed to belong to the Feller class, $\{Z_t(1),\ t > e\}$ is a stochastically compact family, i.e., every sequence has a subsequence that converges weakly as $t \to \infty$; furthermore, all limit laws are nondegenerate. Let K denote the set of these limit laws, which is a compact subset of $\mathcal{M}$. It is known [13] that every law in K has an infinitely differentiable density. It also follows from the main result in [13] that if $\{X(t)\}$ satisfies Condition A, and $\{X(t)\}$ is symmetric, then each law in K satisfies Condition A, hence also satisfies Condition B. Instead of assuming $\{X(t)\}$ to be symmetric, we only need to assume that every (infinitely divisible) law in K satisfies Condition A. We write

$$L_t(v, A) := \frac{1}{v}\int_0^v 1_A(Z_t(s))ds,$$

so that $L_t(\log\log t, A) = L_t(A)$.

We now prove the upper bound in the large deviation principle for $L_t(\cdot)$. Let θ denote $(\gamma, \sigma^2, \hat{\nu})$ in the Lévy representation of the infinitely divisible law θ, and let

$$D_\theta f(x) = \gamma f'(x) + \frac{\sigma^2}{2} f''(x) + \int_{\mathbf{R}-\{0\}} \left(f(x+y) - f(x) - \frac{y}{1+y^2} f'(x)\right) \frac{1+y^2}{y^2} d\hat{\nu}(y)$$

be the generator of the Lévy process with parameters $\gamma, \sigma^2, \hat{\nu}$.

LEMMA 4.5. *If $f \in \mathcal{E}$, then $D_\theta f$ is a bounded continuous function that vanishes at infinity. Furthermore, if $\beta \in \mathcal{M}$, then*

$$H_f(\theta;\beta) := \int D_\theta f(x) d\beta(x) \tag{4.6}$$

is a continuous function of γ, σ^2, $\hat{\nu}, \beta$ on $\mathbf{R} \times \mathbf{R}^+ \times \hat{\mathcal{M}} \times \mathcal{M}$, where $\mathbf{R}^+ = [0, \infty)$, $\hat{\mathcal{M}}$ = set of finite measures on $\mathbf{R}$ with the topology of weak convergence, and $\mathbf{R} \times \mathbf{R}^+ \times \hat{\mathcal{M}} \times \mathcal{M}$ is given the product topology. The conclusion also holds if $\mathcal{M}$ is replaced by M.

Proof. The first assertion immediately follows from the definition of D_θ. The second assertion is an immediate consequence of the first. □

COROLLARY 4.6. *The I-function corresponding to D_θ, denoted by I_θ, i.e.,*

$$I_\theta(\beta) := -\inf_{f \in \mathcal{E}} \int \frac{D_\theta f}{f}(x) d\beta(x),$$

is lower semicontinuous in γ, σ^2, $\hat{\nu}$ and β with respect to the product topology. It is lower semi-continuous in β with respect to vague convergence as well.

In the following, we will use the convention that for $t > 0$

$$P^x\{Z_t(0) = x\} = 1,$$

and

$$P^x\{Z_t(v) \in A\} = P\{Z_t(v) + x \in A\}.$$

We will now prove the following lemma.

LEMMA 4.7. *Let $\mu \in M$, $\theta \in K$, and $c < I_\theta(\mu)$ be given. Then there exists a weak neighborhood V_θ of θ, a vague neighborhood N_μ of μ, and $0 < c_1 < c_2$, such that for all $v > 0$, all $t > e$ for which the distribution of $Z_t(1)$ belongs to V_θ, we have*

$$\frac{1}{v}\log\sup_x P^x\{L_t(v,\cdot)\in N_\mu\} \le \frac{\log c_2 - \log c_1}{v} - c.$$

Proof. Let $g \in \mathcal{E}$ be such that $0 < c_1 \le g \le c_2$. For $t > e$, let D^t denote the generator of the Lévy process $Z_t(\cdot)$. Following the argument of Donsker and Varadhan [4], using the Feynman-Kac formula, we have for $C \subset M$, measurable,

$$\begin{aligned} g(x) &= E^x\{g(Z_t(v))\exp[-\int_0^v \frac{D^t g}{g}(Z_t(s))ds]\} \\ &\ge c_1 E^x\{\exp[-\int_0^v \frac{D^t g}{g}(Z_t(s)ds]\} \\ &= c_1 E^x\{\exp[-v\int_{\mathbf{R}} \frac{D^t g}{g}(x)L_t(v,dx)]\} \\ &\ge c_1 E^x\{\exp[-v\int_{\mathbf{R}} \frac{D^t g}{g}(x)L_t(v,dx)]1_{[L_t(v,\cdot)\in C]}\} \\ &\ge c_1\exp\{-v\sup_{\mu'\in C}\int_{\mathbf{R}} \frac{D^t g}{g}(x)d\mu'(x)\}P^x\{L_t(v,\cdot)\in C\}, \end{aligned}$$

hence

$$\frac{1}{v}\log\sup_x P^x\{L_t(v,\cdot)\in C\} \le \frac{1}{v}(\log c_2 - \log c_1) + \sup_{\mu'\in C}\int \frac{D^t g}{g}d\mu'. \tag{4.8}$$

If $I_\theta(\mu) > c$, then there exists some $f \in \mathcal{E}$ such that $0 < c_1 \le f \le c_2$ for some c_1 and c_2, and

$$-\int \frac{D_\theta f}{f}(x)d\mu(x) > c.$$

By Lemma 4.5, there exists a weak neighborhood V_θ of θ and a vague neighborhood N_μ of μ such that

$$-\int \frac{D_{\theta'} f}{f}(x)d\mu'(x) > c$$

for $\theta' \in V_\theta$ and $\mu' \in N_\mu$. It follows that

$$\sup_{\theta' \in V_\theta} \sup_{\mu' \in N_\mu} \int \frac{D_{\theta'} f}{f}(x) d\mu'(x) \le -c,$$

hence if t is such that the distribution of $Z_t(1)$ belongs to V_θ, we have

$$\sup_{\mu' \in N_\mu} \int \frac{D^t f}{f}(x) d\mu'(x) \le -c.$$

The result now follows from (4.8) if we take $C = N_\mu$, $g = f$. The lemma is proved. □

The following corollary of the lemma is obvious.

COROLLARY 4.9. *If $t_j \nearrow \infty$ is such that $Z_{t_j}(1)$ converges weakly to $\theta \in K$, then given $\mu \in M$, given $c < I_\theta(\mu)$, there exists a vague neighborhood N_μ of μ such that*

$$\limsup_{t_j \to \infty} \frac{1}{\log \log t_j} \log \sup_x P^x\{L_{t_j}(\cdot) \in N_\mu\} \le -c.$$

The next theorem follows from this corollary.

THEOREM 4.10. *Let $\hat{K}$ be a closed subset of M. Then*

$$\limsup_{t \to \infty} \frac{1}{\log \log t} \log \sup_x P^x\{L_t(\cdot) \in \hat{K}\} \le - \inf_{\substack{\theta \in K \\ \mu \in \hat{K}}} I_\theta(\mu). \tag{4.11}$$

Proof. Let $t_n \nearrow \infty$ be a sequence along which the limsup in (4.11) is achieved as a limit. Since $\{Z_{t_n}(1)\}$ is weakly precompact, dropping to a subsequence, if necessary, we may assume that $\{Z_{t_n}(1)\}$ converges weakly to some $\theta \in K$. For any $\mu \in \hat{K}$, by the previous corollary, given $c_\mu < I_\theta(\mu)$, there exists a vague neighborhood N_μ of μ such that

$$\limsup_{j \to \infty} \frac{1}{\log \log t_j} \log \sup_x P^x\{L_{t_j}(\cdot) \in N_\mu\} \le -c_\mu.$$

Since $\hat{K}$ is compact, finitely many neighborhoods $N_{\mu_1}, \ldots, N_{\mu_r}$ cover $\hat{K}$, and writing c_j for c_{μ_j}, $1 \le j \le r$, we get

$$\limsup_{j \to \infty} \frac{1}{\log \log t_j} \log \sup_x P^x\{L_{t_j}(\cdot) \in \hat{K}\} \le - \min_{1 \le j \le r} c_j.$$

This implies that the left side in (4.11) is dominated by $-\min\{c_j : 1 \le j \le r\}$, and (4.11) follows from this fact easily. □

We now consider the lower bounds.

The next theorem is an analogue of Theorem 2.15.

THEOREM 4.12. *If $t_n \nearrow \infty$ is a sequence such that $\{Z_{t_n}(1)\}$ converges weakly to θ, and θ satisfies Condition B, then given $\mu \in \mathcal{M}$ with* supp $\mu \subset (-a,a)$, *given N_μ, a weak neighborhood of μ, and $0 < a' < a$, we have*

$$\liminf_{n\to\infty} \frac{1}{\log\log t_n} \log \inf_{|x|\le a'} P^x\{L_{t_n}(\cdot) \in N_\mu, |Z_{t_n}(v)| \le a \text{ for } v \le \log\log t_n\} \ge -I_\theta(\mu).$$

Proof. Let $Y(t),\ t \ge 0$, be the Lévy process with $Y(1)$ having distribution θ. Then by Theorem 2.15 we have

$$\liminf_{v\to\infty} \frac{1}{v} \log \inf_{|x|\le a'} P^x\{L^Y(v,\cdot) \in N_\mu,\ |Y(u)| \le a,\ 0 \le u \le v,\ |Y(v)| \le a\} \ge -I_\theta(\mu). \tag{4.13}$$

Since $Z_{t_n}(1) \to Y(1)$ weakly, we can use Lemma 4.4 [9] to show that given $\delta > 0$, given $v_o > 0$, there exists t_o such that for $t_n \ge t_o$

$$\inf_{|x|\le a'} P^x\{L_{t_n}(v_o,\cdot) \in N_\mu,\ |Z_{t_n}(u)| \le a,\ 0 \le u \le v_o,\ |Z_{t_n}(v_o)| \le a'\} \ge \exp[-v_o(1+\delta)I_\theta(\mu)]. \tag{4.14}$$

Since N_μ contains a convex neighborhood of μ, without any loss of generality, we may assume that N_μ is convex. Then for any integer $k \ge 1$, we get from (4.14) via iteration and the Markov property that

$$\inf_{|x|\le a'} P^x\{L_{t_n}(kv_o,\cdot) \in N_\mu,\ |Z_{t_n}(u)| \le a,\ 0 \le u \le kv_o\} \ge \exp[-kv_o(1+\delta)I_\theta(\mu)]. \tag{4.15}$$

It is very easy to conclude from this that the conclusion of the theorem holds. □

The theorem has the following simple corollary. We omit the proof.

COROLLARY 4.16. *If every $\theta \in K$ satisfies Condition B and μ, a, a' are as above, then*

$$\liminf_{t\to\infty} \frac{1}{\log\log t} \log \inf_{|x|\leq a'} P^x\{L_t(\cdot) \in N_\mu,\ |Z_t(u)| \leq a,\ 0 \leq u \leq \log\log t\}$$
$$\geq -\sup_{\theta\in K} I_\theta(\mu).$$

The next theorem is a bit more subtle in that μ is no longer assumed to be compactly supported. It needs a somewhat different argument.

THEOREM 4.17. *If $t_n \nearrow \infty$ is a sequence such that $\{Z_{t_n}(1)\}$ converges weakly to θ, and θ satisfies Condition A, then given $\mu \in \mathcal{M}$, given N_μ, a weak neighborhood of μ, given $a > 0$, we have*

$$\liminf_{t_n\to\infty} \frac{1}{\log\log t_n} \log \inf_{|x|\leq a} P^x\{L_{t_n}(\cdot) \in N_\mu\} \geq -I_\theta(\mu).$$

Proof. The ideas are similar to those used in the proof of Theorem 4.4 [10], so we will sketch the proof. Let Q be a stationary probability measure on the Skorohod space $D[0,\infty)$ of real-valued functions, and let $H(Q)$ denote the entropy function with respect to the Markov process $Y(t)$, which is a Lévy process with $Y(1)$ having distribution θ. See [6],[10] for definitions and more information. By Theorem 6.1 [6],

$$\inf\{H(Q) : Q \text{ with marginal } \mu\} = I_\theta(\mu).$$

We may assume $I_\theta(\mu) < \infty$. Clearly $I_\theta(\mu) > 0$ since there is no invariant probability measure for $Y(t)$. Given $\delta > 0$, there exists Q_δ such that $H(Q_\delta) < (1+\delta)I_\theta(\mu)$. By Lemma 2.5 [10], we can find a convex combination $\hat{Q}_\delta = \sum_{p=1}^r \gamma_p Q_p$ of ergodic $Q_1, \ldots, Q_r$ such that $H(\hat{Q}_\delta) < (1+2\delta)I_\theta(\mu)$. Note also that we can take $\hat{Q}_\delta \to Q$ weakly as $\delta \to 0$.

Let μ_p be the marginal of Q_p. Then $\sum_{p=1}^n \lambda_p\mu_p := \mu^\delta$ is close to μ for δ small. We now define some weak neighborhoods. Let $f_1, \ldots, f_k$ be bounded, uniformly continuous functions on $\mathbf{R}$, and let $\epsilon > 0$ be given. For $\beta \in \mathcal{M}$, let

$$N(\beta,\epsilon) = \{\nu \in \mathcal{M} : |\int f_j d\beta - \int f_j d\nu| < \epsilon,\ 1 \leq j \leq k\}.$$

We can take δ small so that $N(\mu^\delta, 3\epsilon/4) \subset N(\mu,\epsilon)$.

Let $a > 0$ be such that $\mu_p([-a,a]) > 0$ for $1 \leq p \leq r$. By Proposition 4.1 [10], we have for μ_p-a.e. (x), $1 \leq p \leq r$, any $\epsilon' > 0$ ($L^Y(t,\cdot)$ is given by (1.5) with Y replacing X)

$$(4.18)\ \liminf_{t\to\infty} \frac{1}{t} \log P^x\{L^Y(t,\cdot) \in N(\mu_p, \frac{\epsilon}{4r}), \\ |Y(u)| \le a \text{ for some } t \le u \le (1+\epsilon')t\} \ge -(1+\epsilon')H(Q_p).$$

Let $a < c < b$. Let A_p be a Borel subset of $\mathbf{R}$ such that $\mu_p(A_p) > 0$ and (4.18) holds for all $x \in A_p$, $1 \le p \le r$. Since $I_\theta(\mu_p) < \infty$ implies $\mu_p << m$, we have $m(A_p) > 0$, $1 \le p \le r$. Hence by Lemma 2.10 [10], there exists $\eta > 0$, $s_0 > 0$ such that given $x \in [-b, b]$, given p, for some $s \le s_0$

$$(4.19)\qquad p(s, x, A_p) \ge \eta.$$

By using the strong Markov property along with (4.18) and (4.19), we see that given $\epsilon' > 0$, $\delta > 0$, there exists $v_p > \epsilon' s_0$ such that for all $x \in [-b, b]$

$$P^x\{L^Y(v_p,\cdot) \in N(\mu_p, \frac{\epsilon}{3r}),\ |Y(u)| \le a \text{ for some } v_p \le u \le (1+2\epsilon')v_p\} \\ (4.20)\qquad \ge \exp\{-v_p(1+\epsilon')(1+\delta)H(Q_p)\},$$

where $v_1 + \cdots + v_r = v$ and $v_p = \lambda_p v$, $1 \le p \le r$. We use the strong Markov property again to combine the inequalities in (4.20) to get for all $x \in [-b, b]$

$$P^x\{L^Y(v,\cdot) \in N(\mu^\delta, \frac{\epsilon}{3}),\ |Y(u)| \le a \text{ for some } v \le u \le (1+2\epsilon')v\} \\ (4.21)\qquad \ge \exp\{-v(1+\epsilon')(1+\delta)H(\hat{Q}_\delta)\}.$$

The sequence $Z_{t_n}(1)$ converges weakly to θ as $n \to \infty$, and we now use the argument in Lemma 4.4 [9] to conclude that for $n \ge n_0$, $x \in [-b, b]$, recall that $a < c < b$,

$$P^x\{L_{t_n}(v,\cdot) \in N(\mu^\delta, \frac{\epsilon}{2}),\ |Y(u)| \le c \text{ for some } v \le u \le (1+2\epsilon')v\} \\ (4.22)\qquad \ge \exp\{-v(1+\epsilon')(1+\delta)H(\hat{Q}_\delta)\}.$$

We now iterate this inequality by using the strong Markov property to conclude that given $b > 0$, given $\delta > 0$, $n \ge$ some n_o

$$(4.23)\ \liminf_{v\to\infty} \frac{1}{v} \log \inf_{|x|\le b} P^x\{L_{t_n}(v,\cdot) \in N(\mu, \epsilon)\} \ge -(1+\delta)(1+2\delta)I_\theta(\mu).$$

This implies that for any $\delta > 0$
(4.24)
$$\liminf_{n\to\infty} \frac{1}{\log\log t_n} \log \inf_{|x|\le b} P^x\{L_{t_n}(\cdot) \in N(\mu,\epsilon)\} \ge -(1+\delta)(1+2\delta)I_\theta(\mu).$$
Since the left-side does not depend on δ, the result follows from this. □

We would like to state a corollary of this result. We omit the proof which is straightforward.

COROLLARY 4.25. *Assume that each $\theta \in K$ satisfies Condition A. Then given $\mu \in \mathcal{M}$, given N_μ, a weak neighborhood of μ, given $a > 0$, we have*
$$\liminf_{t\to\infty} \frac{1}{\log\log t} \log \inf_{|x|\le a} P^x\{L_t(\cdot) \in N_\mu\} \ge -\sup_{\theta\in K} I_\theta(\mu).$$

REMARK 4.26. The local upper and lower bound results show that the bounds are sharp for each μ.

The results of this section also extend to random walks $S_n = Y_1 + \cdots + Y_n$, where the distribution of Y_1 belongs to the Feller class, in general, and to the domain of attraction of a strictly stable law, in particular.

Assume that Y_1 has distribution in the Feller class, and that there exists a sequence $a(n) \nearrow \infty$ such that $\{S_n/a(n)\}$ is a tight sequence, and no subsequence converges to a degenerate limit. Let $d(n) \nearrow \infty$, $k(n) \nearrow \infty$ be integer sequences and let $r(n) = d(n)k(n)$. Define
$$L_n^x(A) = \frac{1}{r(n)} \sum_{j=0}^{r(n)-1} 1_A\Big(x + \frac{S_j}{a(d(n))}\Big).$$
Then the following upper and lower bound results hold. We assume that Condition A holds for θ.

THEOREM 4.27. *Assume that $\{S_{d(n)}/a(d(n))\}$ converges weakly to θ along $\{n_j\}$. Let $\mu \in M$, and let $c < I_\theta(\mu)$. Then there exists a vague neighborhood N_μ of μ such that*
$$\limsup_{j\to\infty} \frac{1}{k(n_j)} \log \sup_x P\{L_{n_j}^x(\cdot) \in N_\mu\} \le -I_\theta(\mu).$$

THEOREM 4.28. *Assume that $\{S_{d(n)}/a(d(n))\}$ converges weakly to θ along $\{n_j\}$. Let $\mu \in \mathcal{M}$ with topological support in $(-a,a)$. Let N_μ be* *a weak neighborhood of μ. Then for $a' < a$*

$$\liminf_{j\to\infty} \frac{1}{k(n_j)} \log \inf_{|x|\le a'} P\{L^x_{n_j}(\cdot) \in N_\mu, \\ |x + \frac{S_r}{a(d(n_j))}| \le a,\ 1 \le r \le r(n_j) - 1\} \ge -I_\theta(\mu).$$

THEOREM 4.29. *Assume that $\{S_{d(n)}/a(d(n))\}$ converges weakly to θ along $\{n_j\}$. Let $\mu \in \mathcal{M}$, and let N_μ be a weak neighborhood of μ. Then for any $a > 0$*

$$\liminf_{j\to\infty} \frac{1}{k(n_j)} \log \inf_{|x|\le a} P\{L^x_{n_j}(\cdot) \in N_\mu\} \ge -I_\theta(\mu).$$

REMARK 4.30. If Y_1 is such that $\{S_n/a(n)\}$ converges weakly to a stable law which satisfies Condition A, then $\{n_j\}$ is replaced by $\{n\}$ in the above theorems.

The proofs of the results for random walks follow the same ideas as in [9].

5. The limit points of $\{L_t(\cdot)\}$

We can essentially follow the proof of Theorem 5.1 in [9] to prove the theorem stated below. We assume that each $\theta \in K$ satisfies Condition A. The results needed to prove the theorem are already contained in the previous sections.

THEOREM 5.1. *Assume that $\{X(t)\}$ belongs to the Feller class. For $\theta \in K$, let*

$$C_\theta = \{\beta \in M : I_\theta(\beta) \le 1\}.$$

Let $L_t(\cdot)$ be defined by (4.2). Then for each x, P^x-a.s.

$$\cap_{\theta\in K} C_\theta \subset \cap_{s>0} \overline{\cup_{t\ge s}\{L_t(\cdot)\}} \subset \cup_{\theta\in K} C_\theta,$$

where the closure is taken in M with respect to the vague topology.

REMARK 5.2. In the proof of this theorem, if we use Corollary 4.25 to establish the left containment, then we avoid the major difficulty of having to find compactly supported μ_n, weakly converging to a given μ, such that $I(\mu_n) \to I(\mu)$.

REMARK 5.3. At least in the symmetric case, we can show, by using the results of Section 2, that $\cap_{\theta\in K} C_\theta$ is nontrivial. Indeed, each density in C_K^∞ can be rescaled to belong to this set. We have not been able to identify the limit set in Theorem 5.1 in terms of the $C_\theta, \theta \in K$, in this general situation. In most of the sample path properties that can be obtained, the limit constant can only be shown to be trapped between two constants, each of which is determined in terms of the C_θ, $\theta \in K$. The statements of such properties are easy to formulate, see [5], [9] and [11].

REMARK 5.4. If the random variable $X(1)$ is in the domain of attraction of a stable law, i.e., $X(t)/a(t)$ converges weakly to a stable law θ_0 (necessarily strictly stable), then K consists of a singleton, and we have $\cap_{\theta\in K} C_\theta = \cup_{\theta\in K} C_\theta = C_{\theta_o}$.

As observed in [5], we get the following useful corollary of the theorem, which has many applications, see [5], [9].

COROLLARY 5.5. *If Φ is a functional on M which is lower semicontinuous in the vague topology, then P^x-a.s.*

$$\limsup_{t\to\infty} \Phi(L_t(\cdot)) \geq \sup\{\Phi(\beta) : \beta \in \cap_{\theta\in K} C_\theta\}$$

and if Φ is a functional on M which is upper semicontinuous in the vague topology, then P^x-a.s.

$$\limsup_{t\to\infty} \Phi(L_t(\cdot) \leq \sup\{\Phi(\beta) : \beta \in \cup_{\theta\in K} C_\theta\}.$$

REMARK 5.6. If $\{S_n\}$ is a random walk, where $S_n = Y_1 + \cdots + Y_n$, $n \geq 1$, the Y_j's being i.i.d. random variables, and Y_1 belongs to the Feller class such that $\{S_n/a(n)\}$ is a stochastically compact sequence and each $\theta \in K$ satisfies Condition A, then the corresponding results formulated for $\{S_n\}$ also hold, see [9].

REMARK 5.7. One can also formulate and prove results for local times for processes in the Feller class and for random walks as in [5], [9], but we have not done that here. See also [12].

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