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PRODUCTS OF WHITE NOISE FUNCTIONALS AND ASSOCIATED DERIVATIONS

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Abstract. Let the Gel'fand triple $(E)_\beta \subset (L^2) \subset (E)_\beta^*$ be the framework of white noise distribution theory constructed by Kondratiev and Streit. A new class of continuous multiplicative products on $(E)_\beta$ is introduced and associated continuous derivations on $(E)_\beta$ are discussed. Algebraic characterizations of first order differential operators on $(E)_\beta$ are proved. Some applications are also discussed.

1. Introduction

We take, as a framework of white noise distribution theory, a Gel'fand triple

$$(E)_\beta \subset (L^2) = L^2(E^*, \mu; \mathbb{C}) \subset (E)_\beta^*, \qquad 0 \le \beta < 1, \tag{1.1}$$

which was constructed in [18], where E^* is the space of tempered distributions and μ is the standard Gaussian measure associated with a Gel'fand triple $E \subset H \subset E^*$. In particular, if $\beta = 0$, (1.1) becomes $(E) \subset (L^2) \subset (E)^*$, which was constructed in [19]. The white noise distribution theory is an infinite dimensional analogue of the Schwartz distribution theory in which the role of Lebesgue measure on $\mathbb{R}^n$ is replaced by the Gaussian measure μ on E^*. This theory was initiated by Hida [11] and has been considerably developed with applications to

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stochastic analysis, Feynman path integral, infinite dimensional harmonic analysis, quantum probability and Cauchy problems in infinite dimensions and so on, see e.g., [1]–[9], [12]–[15], [22]–[26], [30].

It is well known [13,24] that the Wiener and Wick products are continuous multiplicative operations on $(E)_\beta$ and so $(E)_\beta$ becomes a topological algebra under the Wiener and Wick products. Furthermore, we note that the differential operator D_y is a first order differential operator as well as a continuous derivation on $(E)_\beta$. Motivated by this point of view, Obata [29] determined all the continuous derivations on (E) with respect to the Wiener products and then showed that they are first order differential operators with variable coefficients. In [1], Chung and Chung showed that the results in [29] can be extended to the Wick product case. Recently, Chung and Chung has introduced a class $\{\diamond_\gamma, \gamma \in \mathbb{C}\}$ of multiplicative products on (E) including the Wiener and Wick products and then have showed that with each $\diamond_\gamma$ and an (E)–valued distribution Φ on $\mathbb{R}$, we can associate a first order differential operator with variable coefficient Φ, which is indeed a continuous derivation on (E) with respect to $\diamond_\gamma$.

In this paper, we shall see that $\Xi_{0,m}(\kappa) + N$ is similar to N, i.e., there exists a linear homeomorphism $\mathcal{G}_\kappa \in GL((E)_{\beta(m)})$ such that $\Xi_{0,m}(\kappa) + N = \mathcal{G}_\kappa^{-1} N \mathcal{G}_\kappa$ (Corollary 3.7), where $\Xi_{0,m}(\kappa)$ and N are the integral kernel operator with kernel distribution $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$ and the number operator, respectively. Next, we define a product $\diamond_\kappa$ associated with $\mathcal{G}_\kappa$, and prove that a first order differential operator with variable coefficient associated with $\diamond_\kappa$ is indeed a continuous derivation on $(E)_\beta$ with respect to $\diamond_\kappa$ (Theorem 5.4), and then $\Xi_{0,m}(\kappa) + N$ is a continuous derivation on $(E)_\beta$ with respect to the product $\diamond_\kappa$. Finally, as applications, we shall study the eigenvalue problem, Cauchy problem and Poisson type equation associated with $\Xi_{0,m}(\kappa) + N$.

2. Preliminaries

Let H be the real Hilbert space of square-integrable functions on $\mathbb{R}$ with norm $|\cdot|_0$. Let $\mathcal{S}(\mathbb{R})$ be the Schwarz space consisting of rapidly decreasing C^∞-functions. Then we have a Gel'fand triple:

$$E \equiv \mathcal{S}(\mathbb{R}) \subset H \subset \mathcal{S}'(\mathbb{R}) \equiv E^*, \tag{2.1}$$

where E^* is the space of the tempered distributions. Note that the Gel'fand triple (2.1) is reconstructed by using a positive self–adjoint operator $A = 1 + t^2 - d^2/dt^2$ on H with Hilbert–Schmidt inverse (see [13], [24], [28]). In fact, E is a nuclear space equipped with the Hilbertian norms $|\xi|_p = |A^p\xi|_0$, $\xi \in E, p \in \mathbb{R}$. Let μ be the standard Gaussian measure on E^* whose characteristic function is given by

$$\int_{E^*} e^{i\langle x,\xi\rangle}\mu(dx) = \exp\left(-\frac{1}{2}|\xi|_0^2\right), \qquad \xi \in E,$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $E^* \times E$. The canonical $\mathbb{C}$-bilinear form on $(E_{\mathbb{C}}^{\otimes n})^* \times (E_{\mathbb{C}}^{\otimes n})$ is also denoted by the same symbol $\langle\cdot,\cdot\rangle$. We denote by $(L^2) \equiv L^2(E^*, \mu; \mathbb{C})$ the complex Hilbert space of square integrable functions on E^* with norm $\|\cdot\|_0$. By the Wiener-Itô decomposition theorem, each $\phi \in (L^2)$ admits an expression

$$\phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n\rangle, \qquad x \in E^*, \quad f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n}, \tag{2.2}$$

and $\|\phi\|_0^2 = \sum_{n=0}^{\infty} n!|f_n|_0^2$, where $H_{\mathbb{C}}^{\widehat{\otimes} n}$ is the n-fold symmetric tensor product of the complexification of H and $:x^{\otimes n}:$ denotes the Wick ordering of $x^{\otimes n}$.

Let β be a given real number with $0 \le \beta < 1$. For each $p \ge 0$, define

$$\|\phi\|_{p,\beta}^2 = \sum_{n=0}^{\infty} (n!)^{1+\beta}|f_n|_p^2, \qquad \phi \in (L^2),$$

where ϕ is given as in (2.2). Let $(E_p)_\beta = \{\phi \in (L^2) : \|\phi\|_{p,\beta} < \infty\}$ and let $(E)_\beta$ be the projective limit of $\{(E_p)_\beta : p \ge 0\}$. Then $(E)_\beta$ is a nuclear space and we have a Gel'fand triple:

$$(E)_\beta \subset (L^2) \subset (E)_\beta^*, \tag{2.3}$$

where $(E)_\beta^*$ is the topological dual space of $(E)_\beta$. The triple (2.3) is called the *Kondratiev–Streit space* [18]. If $\beta = 0$, then (2.3) is called the *Hida–Kubo–Takenaka space* and denoted by $(E) \subset (L^2) \subset (E)^*$. An element in $(E)_\beta$ (and in $(E)_\beta^*$) is called a test (and generalized, resp.) white noise functional. We denote by $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ the canonical $\mathbb{C}$-bilinear form on $(E)_\beta^* \times (E)_\beta$. For each $\Phi \in (E)_\beta^*$, there exists a unique sequence $\{F_n\}_{n=0}^{\infty}$, $F_n \in (E_{\mathbb{C}}^{\otimes n})_{sym}^*$ such that

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \qquad \phi \in (E)_\beta,$$

where ϕ is given as in (2.2). In this case we use a formal expression for $\Phi \in (E)_\beta^*$:

$$\Phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, F_n \rangle, \qquad x \in E^*.$$

For each $\xi \in E_{\mathbb{C}}$, the function $\phi_\xi \in (E)_\beta$ given by

$$\phi_\xi(x) \equiv \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, \frac{\xi^{\otimes n}}{n!} \rangle = \exp\left(\langle x, \xi \rangle - \frac{1}{2} \langle \xi, \xi \rangle \right), \qquad x \in E^*$$

is called an *exponential vector.* Note that $\{\phi_\xi : \xi \in E_{\mathbb{C}}\}$ spans a dense subspace of $(E)_\beta$. The *S–transform* of $\Phi \in (E)_\beta^*$ is a function on $E_{\mathbb{C}}$ defined by

$$S\Phi(\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle, \qquad \xi \in E_{\mathbb{C}}.$$

In [18], Kondrative and Streit proved a characterization theorem for elements in $(E)_\beta^*$ and $(E)_\beta$ by analytic properties of the S–transforms. By using the characterization theorem, for each $\Phi, \Psi \in (E)_\beta^*$ the *Wick product* $\Phi \diamond \Psi \in (E)_\beta^*$ is defined by $S(\Phi \diamond \Psi) = S\Phi \cdot S\Psi$. In facts $\phi \diamond \psi \in (E)_\beta$, $\phi, \psi \in (E)_\beta$ and the Wick product is a continuous multiplicative operation on $(E)_\beta$ (see e.g., [24]).

Let $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ denote the space of all continuous linear operators from a locally convex space $\mathfrak{X}$ into another locally convex space $\mathfrak{Y}$. Also for notational convenience we write $\mathcal{L}(\mathfrak{X}) = \mathcal{L}(\mathfrak{X}, \mathfrak{X})$. For each $\Xi \in \mathcal{L}((E)_\beta, (E)_\beta^*)$, the $\mathbb{C}$-valued function $\widehat{\Xi}$ on $E_{\mathbb{C}} \times E_{\mathbb{C}}$ defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle, \qquad \xi, \eta \in E_{\mathbb{C}}$$

is called the *symbol* of operator Ξ. The following theorem is a characterization theorem for symbols of operator in $\mathcal{L}((E)_\beta, (E)_\beta^*)$ and in $\mathcal{L}((E)_\beta)$ ([24], [27], [28]).

THEOREM 2.1. *A $\mathbb{C}$-valued function $\Theta : E_{\mathbb{C}} \times E_{\mathbb{C}} \to \mathbb{C}$ is a symbol of an $\Xi \in \mathcal{L}((E)_\beta, (E)_\beta^*)$ if and only if Θ satisfies the following conditions:*

(O1) *For each $\xi, \xi', \eta, \eta' \in E_{\mathbb{C}}$, the function*

$$(z, w) \mapsto \Theta(z\xi + \xi', w\eta + \eta')$$

is an entire function on $\mathbb{C} \times \mathbb{C}$;

(O2) *There exist constants* $K \geq 0$, $C \geq 0$ *and* $p \geq 0$ *such that*

$$|\Theta(\xi,\eta)| \leq K \exp C \left(|\xi|_p^{\frac{2}{1-\beta}} + |\eta|_p^{\frac{2}{1-\beta}} \right), \qquad \xi,\eta \in E_{\mathbb{C}}.$$

Moreover, Θ *is the symbol of an operator in* $\mathcal{L}((E)_\beta)$ *if and only if it satisfies* $(O1)$ *and*

(O2′) *For any* $p \geq 0, \epsilon > 0$, *there exist* $q \geq 0$ *and* $K \geq 0$ *such that*

$$|\Theta(\xi,\eta)| \leq K \exp \epsilon \left(|\xi|_{p+q}^{\frac{2}{1-\beta}} + |\eta|_{-p}^{\frac{2}{1+\beta}} \right), \qquad \xi,\eta \in E_{\mathbb{C}}.$$

For each $y \in E_{\mathbb{C}}^*$ there exists a unique operator $D_y \in \mathcal{L}((E)_\beta)$ such that $D_y\phi_\xi = \langle y,\xi\rangle\phi_\xi$, $\xi \in E_{\mathbb{C}}$ which is called the *annihilation operator.* The adjoint operator $D_y^* \in \mathcal{L}((E)_\beta^*)$ of D_y is called the *creation operator.* By Theorem 2.1, we can show that for each $\kappa \in (E_{\mathbb{C}}{}^{\otimes(l+m)})^*$, there exists a unique operator $\Xi_{l,m}(\kappa) \in \mathcal{L}((E)_\beta,(E)_\beta^*)$ such that

$$\widehat{\Xi_{l,m}(\kappa)}(\xi,\eta) = \langle\!\langle \Xi_{l,m}(\kappa)\phi_\xi,\phi_\eta\rangle\!\rangle = \langle \kappa, \eta^{\otimes l} \otimes \xi^{\otimes m}\rangle e^{\langle \xi,\eta\rangle}, \ \xi,\eta \in E_{\mathbb{C}}.$$

This operator $\Xi_{l,m}(\kappa)$ is called the *integral kernel operator* with *kernel distribution* κ. In particular, $\Delta_G = \Xi_{0,2}(\tau)$ and $N = \Xi_{1,1}(\tau)$ are called the *Gross Laplacian* and *number operator*, respectively, where τ is the *trace* defined by $\langle \tau, \xi \otimes \eta\rangle$, $\xi,\eta \in E_{\mathbb{C}}$. It is well–known that for each $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$, $\Xi_{l,m}(\kappa) \in \mathcal{L}((E)_\beta)$ if and only if $\kappa \in (E_{\mathbb{C}}^{\otimes l}) \otimes (E_{\mathbb{C}}^{\otimes m})^*$.

3. Transformation group on white noise functionals

In this section we obtain a two-parameter transformation group $\mathcal{G}$ on $(E)_\beta$. Moreover, we shall prove that there exists an element $\mathcal{G}_\kappa \in \mathcal{G}$ such that $\mathcal{G}_\kappa(\Xi_{0,m}(\kappa) + N)\mathcal{G}_\kappa^{-1} = N$. For notational convenience, we define a function $\beta : \mathbb{N} \cup \{0\} \to [0,1)$ by

$$\beta(m) = \begin{cases} 0, & m = 0,1 \\ 1 - \frac{2}{m}, & m = 2,3,\cdots . \end{cases}$$

LEMMA 3.1. *Let $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$ and $a \in \mathbb{C}$. Then there exists a unique operator $\mathcal{G}_{\kappa,a} \in \mathcal{L}((E)_{\beta(m)}))$ such that*

$$\mathcal{G}_{\kappa,a}\phi_\xi = \exp\{\langle\kappa, \xi^{\otimes m}\rangle\}\phi_{a\xi}, \qquad \xi \in E_{\mathbb{C}}.$$

Proof. Let $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$ and $a \in \mathbb{C}$. For any $\xi, \eta \in E_{\mathbb{C}}$, we put

$$\Theta(\xi, \eta) = \exp\{\langle\kappa, \xi^{\otimes m}\rangle + a\langle\xi, \eta\rangle\}.$$

Then the function Θ satisfies conditions (O1) and (O2′) in Theorem 2.1 with $\beta = \beta(m)$. Hence by Theorem 2.1, there exists a unique operator $\mathcal{G}_{\kappa,a} \in \mathcal{L}((E)_{\beta(m)})$ such that

$$\widehat{\mathcal{G}_{\kappa,a}}(\xi, \eta) = \langle\langle \mathcal{G}_{\kappa,a}\phi_\xi, \phi_\eta \rangle\rangle = \Theta(\xi, \eta).$$

This complete the proof. □

Let $\mathbb{C}$ and $\mathbb{C}^* = \mathbb{C} - \{0\}$ be the additive and multiplicative groups of complex numbers, respectively. Let $GL(\mathfrak{X})$ denote the group of all linear homeomorphisms from a locally convex space $\mathfrak{X}$ onto itself.

THEOREM 3.2. *Let $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$ be fixed. And let $\mathcal{G} = \{\mathcal{G}_{a\kappa,b}\ ;\ a \in \mathbb{C}, b \in \mathbb{C}^*\}$. Then $\mathcal{G}$ is a subgroup of $GL((E)_{\beta(m)})$.*

Proof. By Lemma 3.1, we have $\mathcal{G}_{0\kappa,1}\phi_\xi = \phi_\xi$ for any $\xi \in E_{\mathbb{C}}$, and

$$\mathcal{G}_{a'\kappa,b'}(\mathcal{G}_{a\kappa,b}\phi_\xi) = \exp\{(a + a'b^m)\langle\kappa, \xi^{\otimes m}\rangle\}\phi_{b'b\xi} = \mathcal{G}_{(a+a'b^m)\kappa,b'b}\phi_\xi,$$

for any $a, a' \in \mathbb{C}$ and $b, b' \in \mathbb{C}^*$ and $\xi \in E_{\mathbb{C}}$. But $\{\phi_\xi : \xi \in E_{\mathbb{C}}\}$ spans a dense subspace of $(E)_{\beta(m)}$ and for any $a \in \mathbb{C}, b \in \mathbb{C}^*$, $\mathcal{G}_{a\kappa,b}$ is continuous. Hence it follows that for any $\phi \in (E)_{\beta(m)}$,

$$\mathcal{G}_{0\kappa,1}\phi = \phi \qquad \text{and} \qquad \mathcal{G}_{a'\kappa,b'}(\mathcal{G}_{a\kappa,b}\phi) = \mathcal{G}_{(a+a'b^m)\kappa,b'b}\phi.$$

Then $\mathcal{G}_{(-ab^{-m})\kappa,b^{-1}}$ is the inverse of $\mathcal{G}_{a\kappa,b}$ in $\mathcal{G}$. □

Let $\mathcal{F}_{\kappa,a} \in \mathcal{L}((E)^*_{\beta(m)})$ be the adjoint operator of $\mathcal{G}_{\kappa,a}$. Then by using similar arguments as in [5], we obtain explicit expressions $\mathcal{F}_{\kappa,a}\Phi$ and $\mathcal{G}_{\kappa,a}\phi$ for $\Phi \in (E)^*_{\beta(m)}$ and $\phi \in (E)_{\beta(m)}$.

THEOREM 3.3. *Let* $\Phi(x) = \sum_{n=0}^{\infty}\langle : x^{\otimes n} :, F_n\rangle \in (E)^*_{\beta(m)}$, $F_n \in (E_{\mathbb{C}}^{\otimes n})^*_{\mathrm{sym}}$. *Then we have, for any positive integer* m,

$$\mathcal{F}_{\kappa,a}\Phi(x) = \sum_{n=0}^{\infty}\langle : x^{\otimes n} :, \sum_{k=0}^{[\frac{n}{m}]}\frac{1}{k!}a^{n-mk}F_{n-mk}\widehat{\otimes}\kappa^{\otimes k}\rangle.$$

THEOREM 3.4. *For* $\phi(x) = \sum_{n=0}^{\infty}\langle : x^{\otimes n} :, f_n\rangle \in (E)_{\beta(m)}, f_n \in E_{\mathbb{C}}^{\widehat{\otimes} n}$, *we have, for any positive integer* m,

$$\mathcal{G}_{\kappa,a}\phi(x) = \sum_{n=0}^{\infty}\langle : x^{\otimes n} :, \sum_{k=0}^{\infty}\frac{(n+mk)!}{n!k!}a^n\kappa^{\otimes k}\widehat{\otimes}_{mk}f_{n+mk}\rangle.$$

PROPOSITION 3.5. *The operator* $\mathcal{G}_{\kappa,a}$ *is expressed by*

$$\mathcal{G}_{\kappa,a} = e^{(\log a)N} \circ e^{\Xi_{0,m}(\kappa)}.$$

Proof. Let $\phi(x) = \sum_{n=0}^{\infty}\langle : x^{\otimes n} :, f_n\rangle$. Then by Theorem 3.4, it holds that $\mathcal{G}_{0\kappa,a}\phi = e^{(\log a)N}\phi$ and $\mathcal{G}_{\kappa,1}\phi = e^{\Xi_{0,m}(\kappa)}\phi$. Clearly, $\mathcal{G}_{0\kappa,a} \circ \mathcal{G}_{\kappa,1} = \mathcal{G}_{\kappa,a}$. Hence we complete the proof. □

PROPOSITION 3.6. *For any* $\alpha_1, \alpha_2 \in \mathbb{C}$, *we have*

$$\mathcal{G}_{\kappa,a}(\alpha_1\Xi_{0,m}(\kappa) + \alpha_2 N) = \big((\alpha_1 + \alpha_2 m)e^{-m\log a}\Xi_{0,m}(\kappa) + \alpha_2 N\big)\mathcal{G}_{\kappa,a}.$$

Proof. Note that for any $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$,

$$[N, \Xi_{0,m}(\kappa)] = -m\Xi_{0,m}(\kappa).$$

Therefore, we have

$$e^{\Xi_{0,m}(\kappa)}N = \big(N + m\Xi_{0,m}(\kappa)\big)e^{\Xi_{0,m}(\kappa)}$$

and

$$e^{(\log a)N}\Xi_{0,m}(\kappa) = e^{-m\log a}\Xi_{0,m}(\kappa)e^{(\log a)N}.$$

Hence by Proposition 3.5, we obtain that

$$\mathcal{G}_{\kappa,a}(\alpha_1\Xi_{0,m}(\kappa) + \alpha_2 N) = \big((\alpha_1 + \alpha_2 m)e^{-m\log a}\Xi_{0,m}(\kappa) + \alpha_2 N\big)\mathcal{G}_{\kappa,a}.$$

Thus we complete the proof. □

COROLLARY 3.7. *Let* $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$ *and let* $\mathcal{G}_{\kappa} = \mathcal{G}_{-\frac{1}{m}\kappa,1}$. *Then we have*

$$\mathcal{G}_{\kappa}(\Xi_{0,m}(\kappa) + N) = N\mathcal{G}_{\kappa}.$$

Proof. The proof is straightforward from Proposition 3.6. □

4. Products on white noise functionals

In this section, we introduce a new class of continuous multiplicative products $\diamond_\kappa$ on $(E)_\beta$ indexed by $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$.

Let $\mathcal{H}$ be a linear homeomorphism on $(E)_\beta$. Then we can define a continuous multiplicative operation $\circ_{\mathcal{H}}$ on $(E)_\beta$ associated with $\mathcal{H}$ by

$$\phi \circ_{\mathcal{H}} \psi = \mathcal{H}^{-1}(\mathcal{H}\phi \diamond \mathcal{H}\psi), \qquad \phi, \psi \in (E)_\beta,$$

where $\diamond$ is the Wick product.

PROPOSITION 4.1. *Let $\Xi \in \mathcal{L}((E)_\beta)$. Then Ξ is a derivation with respect to the multiplication $\circ_{\mathcal{H}}$, i.e., $\Xi(\phi \circ_{\mathcal{H}} \psi) = \Xi\phi \circ_{\mathcal{H}} \psi + \phi \circ_{\mathcal{H}} \Xi\psi$, $\phi, \psi \in (E)_\beta$ if and only if $\mathcal{H}\Xi\mathcal{H}^{-1}$ is a derivation with respect to the Wick product.*

Proof. The proof is clear from the definition of $\circ_{\mathcal{H}}$. □

From now on we only consider the case $\mathcal{H} = \mathcal{G}_\kappa$ for a fixed kernel $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$ with integer $m \geq 2$, where $\mathcal{G}_\kappa$ is given in Corollary 3.7. We put $\diamond_\kappa = \circ_{\mathcal{G}_\kappa}$. That is, $\diamond_\kappa$ is a continuous multiplicative operation on $(E)_{\beta(m)}$ defined by

$$\phi \diamond_\kappa \psi = \mathcal{G}_\kappa^{-1}(\mathcal{G}_\kappa\phi \diamond \mathcal{G}_\kappa\psi), \qquad \phi, \psi \in (E)_{\beta(m)}. \tag{4.1}$$

PROPOSITION 4.2. *Let $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$. Then there exists a unique operator $T_\kappa \in \mathcal{L}((E)_{\beta(m)}, (E)^*_{\beta(m)})$ such that the following hold:*

$$\phi_\xi \diamond_\kappa \phi_\eta = \widehat{T_\kappa}(\xi, \eta)\phi_{\xi+\eta}, \qquad \xi, \eta \in E_{\mathbb{C}}, \tag{4.2}$$

and

$$\langle\langle \phi \diamond_\kappa \phi_\xi, \phi_\eta \rangle\rangle = e^{\langle \xi, \eta \rangle} \langle\langle T_\kappa \phi_\xi \diamond \phi_\eta, \phi \rangle\rangle, \phi \in (E)_{\beta(m)}, \xi, \eta \in E_{\mathbb{C}}. \tag{4.3}$$

Proof. By using the equation (4.1) we can easily see that

$$\phi_\xi \diamond_\kappa \phi_\eta = \exp\left\{ \frac{1}{m} \langle \kappa, (\xi + \eta)^{\otimes m} - \xi^{\otimes m} - \eta^{\otimes m} \rangle \right\} \phi_{\xi+\eta}, \qquad \xi, \eta \in E_{\mathbb{C}}.$$

Note that the function $\Theta_\kappa(\xi,\eta) = \exp\left\{\frac{1}{m}\langle\kappa, (\xi+\eta)^{\otimes m} - \xi^{\otimes m} - \eta^{\otimes m}\rangle\right\}$ satisfies (O1) and (O2) in Theorem 2.1. So, there exists a unique operator $T_\kappa \in \mathcal{L}((E)_{\beta(m)}, (E)^*_{\beta(m)})$ such that (4.2) holds. Now for any $\xi, \eta, \zeta \in E_{\mathbb{C}}$ we observe that

$$\begin{aligned}\langle\!\langle \phi_\zeta \diamond_\kappa \phi_\xi, \phi_\eta \rangle\!\rangle &= \langle\!\langle T_\kappa \phi_\zeta, \phi_\xi \rangle\!\rangle \langle\!\langle \phi_{\zeta+\xi}, \phi_\eta \rangle\!\rangle \\ &= e^{\langle \xi,\eta\rangle} \langle\!\langle T^*_\kappa \phi_\xi, \phi_\zeta \rangle\!\rangle \langle\!\langle \phi_\zeta, \phi_\eta \rangle\!\rangle \\ &= e^{\langle \xi,\eta\rangle} \langle\!\langle T^*_\kappa \phi_\xi \diamond \phi_\eta, \phi_\zeta \rangle\!\rangle.\end{aligned}$$

Since Θ_κ is symmetric, we have $T^*_\kappa = T_\kappa$, and hence (4.3) holds for all $\phi = \phi_\zeta, \zeta \in E_{\mathbb{C}}$. Thus the proof follows from the fact that $\{\phi_\zeta\,;\, \zeta \in E_{\mathbb{C}}\}$ spans a dense linear subspace of $(E)_{\beta(m)}$. □

EXAMPLE 4.3. Let $\kappa \in (E_{\mathbb{C}}^{\otimes 2})^*$ and $\widetilde{\kappa} \in \mathcal{L}(E_{\mathbb{C}}, E^*_{\mathbb{C}})$ be given by

$$\langle \widetilde{\kappa}\xi, \eta\rangle = \langle \kappa, \xi \otimes \eta\rangle, \qquad \xi, \eta \in E_{\mathbb{C}}.$$

Then we have

$$\phi_\xi \diamond_\kappa \phi_\eta = e^{\langle \kappa, \xi \widehat{\otimes} \eta\rangle} \phi_{\xi+\eta} = e^{\frac{1}{2}\langle (\widetilde{\kappa}+\widetilde{\kappa}^*)\xi, \eta\rangle} \phi_{\xi+\eta}, \qquad \xi, \eta \in E_{\mathbb{C}}.$$

Hence we obtain that $T_\kappa = \Gamma((\widetilde{\kappa}+\widetilde{\kappa}^*)/2)$, where $\Gamma(A)$ is the second quantization operator of $A \in \mathcal{L}(E_{\mathbb{C}}, E^*_{\mathbb{C}})$. In this case, the equation (4.3) becomes

$$\langle\!\langle \phi \diamond_\kappa \phi_\xi, \phi_\eta \rangle\!\rangle = e^{\langle \xi,\eta\rangle} \langle\!\langle \phi_{\frac{1}{2}(\widetilde{\kappa}+\widetilde{\kappa}^*)\xi+\eta}, \phi \rangle\!\rangle, \qquad \phi \in (E)_{\beta(m)}, \xi, \eta \in E_{\mathbb{C}}.$$

Moreover, if $\kappa \in (E_{\mathbb{C}}^{\otimes 2})^*_{sym}$, then we have

$$\langle\!\langle \phi \diamond_\kappa \phi_\xi, \phi_\eta \rangle\!\rangle = e^{\langle \xi,\eta\rangle} \langle\!\langle \phi_{\widetilde{\kappa}\xi+\eta}, \phi \rangle\!\rangle, \qquad \phi \in (E)_{\beta(m)}, \xi, \eta \in E_{\mathbb{C}}.$$

In particular, $\diamond_{\gamma\tau}$ becomes the γ-product $\diamond_\gamma$ studied in [2]. Furthermore, $\diamond_0$ is the Wick product and $\diamond_\tau$ is the Wiener product (see [2], [24]).

5. First order differential operators

In [2], Chung and Chung discussed the *first order γ-differential operator* $\Xi \in \mathcal{L}((E))$ with coefficient $\Phi \in E_{\mathbb{C}}^* \otimes (E)$, where Ξ is given by

$$\Xi = \int_{\mathbb{R}} \Phi(t) \diamond_\gamma \partial_t \, dt$$

as a formal integral expression. We now introduce a first order differential operator associated with the $\diamond_\kappa$-product.

PROPOSITION 5.1. *Let $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$ and let $\Phi \in E_{\mathbb{C}}^* \otimes (E)_{\beta(m)}$. Then there exists a unique $\Xi \in \mathcal{L}((E)_{\beta(m)})$ such that*

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \langle \Phi, \xi \rangle \diamond_\kappa \phi_\xi, \phi_\eta \rangle\rangle, \qquad \xi, \eta \in E_{\mathbb{C}},$$

where $\langle \Phi, \xi \rangle \in (E)_{\beta(m)}$ is given by

$$\langle\langle \langle \Phi, \xi \rangle, \phi \rangle\rangle = \langle\langle \Phi, \xi \otimes \phi \rangle\rangle, \qquad \phi \in (E)_{\beta(m)}.$$

Proof. The proof is immediately from that the function

$$\Theta(\xi, \eta) = \langle\langle \langle \Phi, \xi \rangle \diamond_\kappa \phi_\xi, \phi_\eta \rangle\rangle, \qquad \xi, \eta \in E_{\mathbb{C}}$$

satisfies (O1) and (O2$'$) in Theorem 2.1 with $\beta = \beta(m)$. □

DEFINITION 5.2. Let $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$ and $\Phi \in E_{\mathbb{C}}^* \otimes (E)_{\beta(m)}$. The operator Ξ defined in Proposition 5.1 is called a *first order κ-differential operator* with coefficient Φ and denoted by

$$\Xi = \int_{\mathbb{R}} \Phi(t) \diamond_\kappa \partial_t \, dt$$

as a formal integral expression.

THEOREM 5.3. *Let $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$ and $\Phi \in E_{\mathbb{C}}^* \otimes (E)_{\beta(m)}$. Then for $\Xi \in \mathcal{L}((E)_{\beta(m)})$ the following statements are equivalent:*

(i) *Ξ is a first order κ-differential operator with coefficient Φ.*

(ii) *For any $\xi \in E_{\mathbb{C}}$ and $n \geq 0$, we have*
$\Xi(\langle :\cdot^{\otimes n}:, \xi^{\otimes n} \rangle) = n \langle :\cdot^{\otimes (n-1)}:, \xi^{\otimes (n-1)} \rangle \diamond_\kappa \langle \Phi, \xi \rangle.$

(iii) *For any $\xi \in E_{\mathbb{C}}$ and $n \geq 0$, $\Xi(\langle \cdot, \xi \rangle^{\diamond_\kappa n}) = n \langle \cdot, \xi \rangle^{\diamond_\kappa (n-1)} \diamond_\kappa \langle \Phi, \xi \rangle$.*

Proof. (i) $\Rightarrow$ (ii) Since Ξ is a first order κ-differential operator with coefficient Φ, the symbol of Ξ is given by

$$\widehat{\Xi}(\xi,\eta) = \langle\langle \Xi\phi_\xi, \phi_\eta\rangle\rangle = \langle\langle \langle\Phi,\xi\rangle \diamond_\kappa \phi_\xi, \phi_\eta\rangle\rangle, \qquad \xi,\eta \in E_{\mathbb{C}}.$$

Hence we obtain that for any $\xi \in E_{\mathbb{C}}$, $\Xi\phi_\xi = \langle\Phi,\xi\rangle \diamond_\kappa \phi_\xi$. Therefore for any $\xi \in E_{\mathbb{C}}$, $\phi \in (E)$ and $z \in \mathbb{C}$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{1}{n!} \langle\langle \Xi(\langle :\cdot^{\otimes n}:, \xi^{\otimes n}\rangle), \phi\rangle\rangle z^n \\ &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \langle\langle \langle\Phi,\xi\rangle \diamond_\kappa \langle :\cdot^{\otimes(n-1)}:, \xi^{\otimes(n-1)}\rangle, \phi\rangle\rangle z^n.\end{aligned}$$

Thus the proof follows.

(ii) $\Rightarrow$ (i) The proof is obvious.

(ii) $\Leftrightarrow$ (iii) Note that for any $\xi \in E_{\mathbb{C}}$, $\mathcal{G}_\kappa^{-1}(\langle\cdot,\xi\rangle) = \langle\cdot,\xi\rangle$. Hence by the definition of $\diamond_\kappa$, we obtain that for any $\xi \in E_{\mathbb{C}}$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle\cdot,\xi\rangle^{\diamond_\kappa n} = \mathcal{G}_\kappa^{-1}\phi_\xi = e^{\frac{1}{m}\langle\kappa,\xi^{\otimes m}\rangle}\phi_\xi.$$

Therefore (ii) implies that for any $\xi \in E_{\mathbb{C}}$

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n!} \Xi(\langle\cdot,\xi\rangle^{\diamond_\kappa n}) &= e^{\frac{1}{m}\langle\kappa,\xi^{\otimes m}\rangle} \sum_{n=0}^{\infty} \frac{1}{n!} \Xi(\langle :\cdot^{\otimes n}:, \xi^{\otimes n}\rangle) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \langle\cdot,\xi\rangle^{\diamond_\kappa n} \diamond_\kappa \langle\Phi,\xi\rangle.\end{aligned}$$

Hence (ii) implies that for any $\xi \in E_{\mathbb{C}}$ and $z \in \mathbb{C}$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \Xi(\langle\cdot,\xi\rangle^{\diamond_\kappa n}) z^n = \sum_{n=0}^{\infty} \frac{1}{n!} \langle\cdot,\xi\rangle^{\diamond_\kappa n} \diamond_\kappa \langle\Phi,\xi\rangle z^{n+1}.$$

Similarly (iii) implies that for any $\xi \in E_{\mathbb{C}}$ and $z \in \mathbb{C}$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \Xi(\langle :\cdot^{\otimes n}:, \xi^{\otimes n}\rangle) z^n = \sum_{n=0}^{\infty} \frac{1}{n!} \langle :\cdot^{\otimes n}:, \xi^{\otimes n}\rangle \diamond_\kappa \langle\Phi,\xi\rangle z^{n+1}.$$

Thus we complete the proof □

THEOREM 5.4. *Let $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$ and $\Xi \in \mathcal{L}((E)_{\beta(m)})$. Then Ξ is a derivation with respect to $\diamond_\kappa$ if and only if Ξ is a first order κ-differential operator with some coefficient $\Phi \in E_{\mathbb{C}}^* \otimes (E)_{\beta(m)}$.*

Proof. Let Ξ be a first order κ-differential operator with coefficient $\Phi \in E_{\mathbb{C}}^* \otimes (E)_{\beta(m)}$. Then we note that $\mathcal{G}_\kappa\langle\cdot,\xi\rangle = \mathcal{G}_\kappa^{-1}\langle\cdot,\xi\rangle = \langle\cdot,\xi\rangle$ for all $\xi \in E_{\mathbb{C}}$. Then the operator $\Xi' = \mathcal{G}_\kappa \Xi \mathcal{G}_\kappa^{-1}$ is a first order Wick differential operator with coefficient Φ', where $\Phi' \in E_{\mathbb{C}}^* \otimes (E)_{\beta(m)}$ is given by $\langle \Phi', \xi\rangle = \mathcal{G}_\kappa\langle\Phi,\xi\rangle$ for each $\xi \in E_{\mathbb{C}}$. In fact, we have

$$\begin{aligned}\Xi'(\langle :\cdot^{\otimes n}:, \xi^{\otimes n}\rangle) &= \mathcal{G}_\kappa \Xi \mathcal{G}_\kappa^{-1}(\langle\cdot,\xi\rangle^{\diamond n})\\ &= \mathcal{G}_\kappa \Xi(\langle\cdot,\xi\rangle^{\diamond_\kappa n})\\ &= \mathcal{G}_\kappa(n\langle\cdot,\xi\rangle^{\diamond_\kappa (n-1)} \diamond_\kappa \langle\Phi,\xi\rangle)\\ &= n\langle :\cdot^{\otimes(n-1)}:, \xi^{\otimes(n-1)}\rangle \diamond \langle\Phi',\xi\rangle.\end{aligned}$$

So, by Theorem 4.5 in [1], Ξ' is a derivation with respect to $\diamond$ and hence by Proposition 4.1, Ξ is a derivation with respect to $\diamond_\kappa$.

Conversely, let Ξ be a derivation with respect to $\diamond_\kappa$. Define a map $\widetilde{\Phi} : E_{\mathbb{C}} \to (E)_{\beta(m)}$ by $\widetilde{\Phi}(\xi) = \Xi(\langle\cdot,\xi\rangle)$, $\xi \in E_{\mathbb{C}}$. Then $\widetilde{\Phi} \in \mathcal{L}(E_{\mathbb{C}}, (E)_{\beta(m)})$. Hence there exists a unique $\Phi \in E_{\mathbb{C}}^* \otimes (E)_{\beta(m)}$ such that

$$\langle\Phi,\xi\rangle = \Xi(\langle\cdot,\xi\rangle), \qquad \xi \in E_{\mathbb{C}}.$$

Since Ξ is a derivation with respect to $\diamond_\kappa$, for any $\xi \in E_{\mathbb{C}}$ and $n \geq 0$ we have

$$\Xi(\langle\cdot,\xi\rangle^{\diamond_\kappa n}) = n\langle\cdot,\xi\rangle^{\diamond_\kappa (n-1)} \diamond_\kappa \Xi(\langle\cdot,\xi\rangle) = n\langle\cdot,\xi\rangle^{\diamond_\kappa (n-1)} \diamond_\kappa \langle\Phi,\xi\rangle.$$

Thus by Proposition 5.3, Ξ is a first order κ-differential operator with coefficient Φ. □

EXAMPLE 5.5. For each $y \in E_{\mathbb{C}}^*$ the differential operator D_y is a first order κ-differential operator with coefficient $y \otimes 1$.

EXAMPLE 5.6. Let $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$. Then by Corollary 3.7 and Proposition 4.1, $\Xi_{0,m}(\kappa) + N$ is a derivation with respect to $\diamond_\kappa$. Moreover, by Theorem 5.4, $\Xi_{0,m}(\kappa) + N$ is a first order κ-differential operator with coefficient Φ_0, where $\langle\Phi_0,\xi\rangle = (\Xi_{0,m}(\kappa) + N)(\langle\cdot,\xi\rangle) = \langle\cdot,\xi\rangle$. In particular, $\gamma\Delta_G + N$ is the first order $\gamma\tau$-differential operator with coefficient Φ_0 (see [2]).

6. Applications

In this section, we shall discuss the eigenvalue problem, Cauchy problem and Poisson type equation associated with $\Xi_{0,m}(\kappa) + N$, $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$.

We now consider the eigenvalue problem associated with $\Xi_{0,m}(\kappa)+N$, i.e., we consider

$$(\Xi_{0,m}(\kappa) + N)\psi = \lambda\psi, \tag{6.1}$$

where $\psi \in (E)_{\beta(m)}$ and $\lambda \in \mathbb{C}$ are unknown.

By using the fact that $\Xi_{0,m}(\kappa) + N = \mathcal{G}_\kappa^{-1} N \mathcal{G}_\kappa$, we easily have the following proposition:

PROPOSITION 6.1.

(i) *λ is an eigenvalue of $\Xi_{0,m}(\kappa)+N$ if and only if λ is an eigenvalue of N.*
(ii) *ϕ is an eigenfunction of $\Xi_{0,m}(\kappa) + N$ if and only if $\mathcal{G}_\kappa\phi$ is an eigenfunction of N.*
(iii) *The set of all eigenvalues of $\Xi_{0,m}(\kappa) + N$ is $\{0, 1, 2, \cdots\}$.*

Now, we consider the following Cauchy problem:

$$\frac{du_t}{dt} = -(\Xi_{0,m}(\kappa) + N)u_t, \qquad u_0 = \phi \in (E)_{\beta(m)}. \tag{6.2}$$

Note that $\widetilde{u}_t = \mathcal{G}_{0,e^{-t}}$ is the one-parameter subgroup of $GL((E)_{\beta(m)})$ with infinitesimal generator $-N$ (see [5], [17], [24]). Hence we can easily check that $u_t = \mathcal{G}_\kappa^{-1}\mathcal{G}_{0,e^{-t}}\mathcal{G}_\kappa$ is the one-parameter subgroup of $GL((E)_{\beta(m)})$ with infinitesimal generator $-(\Xi_{0,m}(\kappa) + N)$. Thus we have the following theorem.

THEOREM 6.2. *Let $\phi \in (E)_{\beta(m)}$. Then $u_t = \mathcal{G}_\kappa^{-1}\mathcal{G}_{0,e^{-t}}\mathcal{G}_\kappa\phi \in (E)_{\beta(m)}$ is a unique solution of the equation (6.2).*

Finally, we consider the following Poisson type equation:

$$(\Xi_{0,m}(\kappa) + N + \lambda I)u = \phi, \tag{6.3}$$

where $\phi \in (E)_{\beta(m)}$ and $\lambda > 0$.

The λ-*potential* ($\lambda > 0$) of test functional $\phi \in (E)_{\beta(m)}$ is defined by

$$H_\lambda \phi = \int_0^\infty e^{-\lambda t} \mathcal{G}_{0,e^{-t}} \phi dt,$$

where the integral is a white noise integral (see [17], [24]). For the case $\lambda = 0$, define *the normalized potential* of $\phi \in (E)_{\beta(m)}$ by

$$G\phi = \int_0^\infty \mathcal{G}_{0,e^{-t}} (\phi - E(\phi)) dt,$$

where $E(\phi)$ is the expectation of ϕ.

THEOREM 6.3 [17]. *Let $\phi \in (E)_{\beta(m)}$. Then we have*

$$NG\phi = \phi - E(\phi) \qquad \text{and} \qquad (N + \lambda I) H_\lambda \phi = \phi.$$

THEOREM 6.4. *Let $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$ and $\phi \in (E)_{\beta(m)}$. Then $u = \mathcal{G}_\kappa^{-1} H_\lambda \mathcal{G}_\kappa \phi \in (E)_{\beta(m)}$ is a solution of the equation* (6.3).

Proof. Let $\phi \in (E)_{\beta(m)}$. Then by Theorem 6.3, $v = H_\lambda \mathcal{G}_\kappa \phi$ is a solution of the equation $(N + \lambda I)v = \mathcal{G}_\kappa \phi$. Hence we obtain that

$$(\mathcal{G}_\kappa^{-1} (N + \lambda I) \mathcal{G}_\kappa) \mathcal{G}_\kappa^{-1} v = \phi.$$

Thus by Corollary 3.7, we have

$$(\Xi_{0,m}(\kappa) + N + \lambda I) \mathcal{G}_\kappa^{-1} v = \phi,$$

That is, $u = \mathcal{G}_\kappa^{-1} H_\lambda \mathcal{G}_\kappa \phi$ satisfies the equation (6.3). □

THEOREM 6.5. *Let $\phi \in (E)_{\beta(m)}$. Then we have*

$$(\Xi_{0,m}(\kappa) + N) \mathcal{G}_\kappa^{-1} G \mathcal{G}_\kappa \phi = \phi - E(\mathcal{G}_\kappa \phi).$$

Proof. Let $\phi \in (E)_{\beta(m)}$. Then by Theorem 6.3, we have

$$NG\mathcal{G}_\kappa \phi = \mathcal{G}_\kappa \phi - E(\mathcal{G}_\kappa \phi).$$

Hence we have

$$(\mathcal{G}_\kappa^{-1} N \mathcal{G}_\kappa) \mathcal{G}_\kappa^{-1} G \mathcal{G}_\kappa \phi = \phi - \mathcal{G}_\kappa^{-1} E(\mathcal{G}_\kappa \phi) = \phi - E(\mathcal{G}_\kappa \phi).$$

Thus by Corollary 3.7, we complete the proof. □

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