J. Korean Math. Soc. **35** (1998), No. 3, pp. 527–538

KERNEL OPERATORS ON FOCK SPACE

CHANGSOO BAHN, CHUL KI KO AND YONG MOON PARK

ABSTRACT. We study on kernel operators (Wick monomials) on symmetric Fock space. We give optimal conditions on kernels so that the corresponding kernel operators are densely defined linear operators on the Fock space. We try to formulate our results in the framework of white noise analysis as much as possible. The most of the results in this paper can be extended to anti-symmetric Fock space.

1. Introduction

Let $\mathcal{F}(L^2)$ be the symmetric Fock space over $L^2(\mathbb{R}^d, dx)$ and let $a(k)$ and $a^*(k), k \in \mathbb{R}^d$, be the annihilation and creation operators (bilinear forms) on $\mathcal{F}(L^2)$ respectively [1, 4, 8]. Consider the following type of integrals

(1.1)

$$T_W = \int_{\mathbb{R}^{d(n_1+n_2)}} w(k_1, \cdots, k_{n_1}, p_1, \cdots, p_{n_2})(\prod_{i=1}^{n_1} a^*(k_i))(\prod_{i=1}^{n_2} a(p_i))dkdp,$$

where $n_1, n_2 \in \mathbb{N}$. Such integrals, known as Wick monomials, are standard in quantum field theory [2, 8]. There have been many studies on sufficient conditions under which T_W becomes a densely defined (unbounded) operator on $\mathcal{F}(L^2)$ ([3], Proposition 1.2.3 and [8], Theorem X.44).

In white noise analysis [4, 6], the integrals (1.1) have been extensively studied under the name of white noise kernel operators of order (n_1, n_2).

Received February 24, 1998.

1991 Mathematics Subject Classification: 47A05, 47N50, 60G15.

Key words and phrases: Fock space, kernel operator, white noise analysis, Gelfand triple.

This is an invited paper to the International Conference on Probability Theory and its Applications.

Let

$$(1.2) \qquad (\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$$

be the Gelfand triple of white noise functions. If the kernel w belongs to $\mathcal{S}'(\mathbb{R}^{d(n_1+n_2)})$, T_W is a continuous linear operator from $(\mathcal{S})$ to $(\mathcal{S})^*$ ([4], Theorem 6.2). In application, it should be important to know conditions on the kernel w under which T_W becomes a continuous linear operators from $(\mathcal{S})$ to (L^2). We mention that in quantum probability theory the Wick monomials are defined in terms of stochastic integrals and called Maassen's kernels [5, 7].

The purpose of this paper is to give minimal conditions on the kernel $w \in \mathcal{S}'(\mathbb{R}^{d(n_1+n_2)})$ so that the corresponding operator T_W becomes a densely defined operator on $\mathcal{F}(L^2)$ (and (L^2)). We formulate our results in the language of white noise analysis. One of our main results can be described as follow: Let W be a continuous linear map from $\mathcal{S}(\mathbb{R}^{dn_2})$ to $L^2(R^{dn_1})$ defined by a kernel w. Then T_W formally defined as in (1.1) becomes a continuous linear operator from $(\mathcal{S})$ to (L^2). Thus T_W defines an unbounded operator on (L^2) with domain $(\mathcal{S})$. See Theorem 3.1 in Section 3.

We organize this paper as follows : In Section 2, we recall the definitions of the symmetric Fock space, and annihilation and creation operators. We then recall the known results ([8], Theorem X.44 and [3], Proposition 1.2.3). In section 3, we first recall Gelfand triple of white noise functions and then consider kernel operators associated to unbounded operators from $L^2(\mathbb{R}^{dn_2})$ to $L^2(\mathbb{R}^{dn_1})$ with domain $\mathcal{S}(\mathbb{R}^{dn_2})$. Finally we consider kernel operators associated to bounded operators from $L^2(\mathbb{R}^{dn_2})$ to $L^2(\mathbb{R}^{dn_1})$.

Before closing the introduction, we remark that most of the results in this paper can be extended to kernel operators on anti-symmetric (fermion) Fock space.

2. Preliminary : Fock spaces

In this section, we recall the construction of symmetric Fock spaces, and the theory of creation and annihilation operators on Fock space. We then provide some known facts on kernel operators on Fock space. For more details, we refer the reader to the references [1, 8].

Let $\mathcal{H}$ be a complex Hilbert space. In this paper we are mainly interested in the case of $\mathcal{H} = L^2(\mathbb{R}^d)$. Denote by $\mathcal{H}^{(n)}$ the n-fold tensor product of $\mathcal{H}$: $\mathcal{H}^{(n)} = \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}(n$ times). Let S_n be the symmetrization operator on $\mathcal{H}^{(n)}$ defined by

$$S_n(\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n) = \frac{1}{n!} \sum_{\pi \in P_n} \psi_{\pi(1)} \otimes \psi_{\pi(2)} \otimes \cdots \otimes \psi_{\pi(n)}, \tag{2.1}$$

where P_n is the permutation group on n elements. Put

$$\mathcal{F}^{(n)}(\mathcal{H}) = S_n \mathcal{H}^{(n)}. \tag{2.2}$$

The symmetric Fock space over $\mathcal{H}$ is defined by

$$\mathcal{F}(\mathcal{H}) = \oplus_{n=0}^{\infty} \mathcal{F}^{(n)}(\mathcal{H}), \tag{2.3}$$

where $\mathcal{F}^{(0)}(\mathcal{H}) = \mathbb{C}$. Thus, for $\phi = \{\phi^{(n)}\} \in \mathcal{F}(\mathcal{H})$ and $\psi = \{\psi^{(n)}\} \in \mathcal{F}(\mathcal{H})$, the inner product of ϕ and ψ is given by

$$(\phi, \psi)_{\mathcal{F}(\mathcal{H})} = \sum_{n=0}^{\infty} (\phi^{(n)}, \psi^{(n)})_{\mathcal{H}^{(n)}}. \tag{2.4}$$

Let us introduce another Fock space over $\mathcal{H}$, namely white noise Fock space, as follows. Let $\Gamma^{(n)}(\mathcal{H})$ be the Hilbert space of $S_n\mathcal{H}^{(n)}$ equipped with the inner product

$$(\phi^{(n)}, \psi^{(n)})_{\Gamma^{(n)}(\mathcal{H})} = n!(\phi^{(n)}, \psi^{(n)})_{\mathcal{H}^{(n)}}, \quad \phi^{(n)}, \psi^{(n)} \in S_n\mathcal{H}^{(n)}. \tag{2.5}$$

The white noise Fock space over $\mathcal{H}$ is defined by

$$\Gamma(\mathcal{H}) = \oplus_{n=0}^{\infty} \Gamma^{(n)}(\mathcal{H}). \tag{2.6}$$

Notice that for $\phi = \{\phi^{(n)}\} \in \Gamma(\mathcal{H})$ and $\psi = \{\psi^{(n)}\} \in \Gamma(\mathcal{H})$ we have

$$\begin{aligned} (\phi, \psi)_{\Gamma(\mathcal{H})} &= \sum_{n=0}^{\infty} (\phi^{(n)}, \psi^{(n)})_{\Gamma^{(n)}(\mathcal{H})} \\ &= \sum_{n=0}^{\infty} n!(\phi^{(n)}, \psi^{(n)})_{\mathcal{H}^{(n)}}. \end{aligned} \tag{2.7}$$

For the details we refer to [4], Section A.2.

From now on we confine ourselves to the case of $\mathcal{H} = L^2(\mathbb{R}^d; dx)$. If $\mathcal{H} = L^2(\mathbb{R}^d)$, then $\otimes_{j=1}^n L^2(\mathbb{R}^d) = L^2(\mathbb{R}^{dn})$ and $S(\otimes^n L^2(\mathbb{R}^d)) = L_s^2(\mathbb{R}^{dn})$, where L_s^2 is the set of functions in L^2 which are invariant under permutations of the variables. Let $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ be the Schwartz space and the space of tempered distributions [8]. For $f \in \mathcal{S}(\mathbb{R}^d)$ and

$\psi = \{\psi^{(n)}\} \in \mathcal{F}(L^2)$ $(\Gamma(L^2))$, the annihilation operator $a(f)$ and the creation operator $a^*(f)$ are defined by

$$(2.8)\ (a(f)\psi)^n(k_1, \cdots, k_n) = \sqrt{n+1}\int_{\mathbb{R}^d} \overline{f(k)}\psi^{(n+1)}(k, k_1, \cdots, k_n)dk,$$

$$(2.9)(a^*(f)\psi)^n(k_1, \cdots, k_n) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} f(k_i)\psi^{(n-1)}(k_1, \cdots, \hat{k_i}, \cdots, k_n),$$

where $\hat{k_i}$ means that k_i is omitted.

A vector $\psi = \{\psi^{(n)}\} \in \mathcal{F}(L^2)$ for which $\psi^{(n)} = 0$ for all but finitely many n is called a finite particle vector. Denote by $\mathcal{F}_0$ the set of finite particle vectors. Put

$$D_{\mathcal{S}} = \{\psi : \psi = \{\psi^{(n)}\} \in \mathcal{F}_0, \psi^{(n)} \in \mathcal{S}(\mathbb{R}^{dn}) \text{ for any } n\}.$$

For each $p \in \mathbb{R}^d$, we define an operator $a(p)$ on $\mathcal{F}(L^2)(\Gamma(L^2))$ with domain $D_{\mathcal{S}}$ by

$$(a(p)\psi)^{(n)}(k_1, \cdots, k_n) = \sqrt{n+1}\,\psi^{(n+1)}(p, k_1, \cdots, k_n).$$

The adjoint of the operator $a(p)$ is given formally by

$$(a^*(p)\psi)^{(n)}(k_1, \cdots, k_n) = \frac{1}{\sqrt{n}}\sum_{l=1}^{n} \delta(p-k_l)\psi^{(n-1)}(k_1, \cdots, k_{l-1}, k_{l+1}, \cdots, k_n),$$

which is a well-defined quadratic form on $D_{\mathcal{S}} \times D_{\mathcal{S}}$. One can check that, if w belongs to $\mathcal{S}'(\mathbb{R}^{d(n_1+n_2)})$, the integral (1.1) is defined as a quadratic forms on $D_{\mathcal{S}} \times D_{\mathcal{S}}$. See [8], Section X.7 for the details.

Let A be any self-adjoint operator on $\mathcal{H}$ with domain of essential self-adjointness D. Let $D_A = \{\psi \in \mathcal{F}_0 : \psi^{(n)} \in \otimes^n D \text{ for each } n\}$ and define $d\Gamma(A)$ on $D_A \cap \mathcal{F}^{(n)}(\mathcal{H})$ as

$$\begin{aligned}(2.10)\ & d\Gamma(A) \\ = {} & A \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes A \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes A.\end{aligned}$$

Then $d\Gamma(A)$ is essentially self-adjoint on D_A [8]. $d\Gamma(A)$ is called the second quantization of A. The number operator N is defined to be $N = d\Gamma(\mathbf{1})$, which can be written as

$$N = \int_{\mathbb{R}^d} a^*(p)a(p)\,dp. \tag{2.11}$$

For $\tau \in \mathbb{R}$, let

$$N_\tau = \int_{\mathbb{R}^d} \mu(p)^\tau a^*(p)a(p)\,dp, \tag{2.12}$$

where $\mu(p) = (p^2+1)^{1/2}$. One can check that N_τ is the second quantization of the multiplication operator $\mu(p)$ on $L^2(\mathbb{R}^d)$ [3, 8].

We now collect some more or less well-known results about kernel operators on Fock space. The followings are the results on kernel operators associated to L^2-kernels:

THEOREM 2.1 ([8], Theorem X.44). *Let n_1 and n_2 be nonnegative integers and suppose that $w \in L^2(\mathbb{R}^{d(n_1+n_2)})$. Then there is an operator T_W on $\mathcal{F}_s(L^2(\mathbb{R}^d))$ so that $D_S \subset D(T_W)$ is a core for T_W and*
(a)

$$T_W = \int_{\mathbb{R}^{d(n_1+n_2)}} w(k_1,\cdots,k_{n_1},p_1,\cdots,p_{n_2})(\prod_{i=1}^{n_1} a^*(k_i))(\prod_{i=1}^{n_2} a(p_i))dkdp$$

as quadratic forms on $D_S \times D_S$.
(b)

$$T_W^* = \int_{\mathbb{R}^{d(n_1+n_2)}} \overline{w(k_1,\cdots,k_{n_1},p_1,\cdots,p_{n_2})}(\prod_{i=1}^{n_2} a^*(p_i))(\prod_{i=1}^{n_1} a(k_i))dkdp$$

as quadratic forms on $D_S \times D_S$.
(c) *$(N+1)^{-n_1/2}T_W(N+1)^{-n_2/2}$ is a bounded operator on $\mathcal{F}(L^2)$ and the bound*

$$\|(N+1)^{-n_1/2}T_W(N+1)^{-n_2/2}\| \le C(n_1,n_2)\,\|W\|_{L^2}$$

holds for some constant $C(n_1,n_2)$.
(d) *If $w_n \to w$ in $L^2(\mathbb{R}^{d(n_1+n_2)})$, then $T_{W_n} \to T_W$ strongly on D_S.*
(e) *$\mathcal{F}_0$ is contained in $D(T_W)$, and on vectors in $\mathcal{F}_0$, T_W is given by the explicit formula*

$$(T_W\psi)^{(l-n_2+n_1)} = K(l,n_1,n_2) \times$$
$$S\int_{\mathbb{R}^{dn_2}} w(k_1,\cdots,k_{n_1},p_1,\cdots,p_{n_2})\psi^{(l)}(p_1,\cdots,p_{n_2},k_1,\cdots,k_{n_1+l-n_2})dp$$

and

$$(T_W\psi)^{(n)} = 0 \quad \text{if} \quad n < n_1$$

where S is the symmetrization operator and $K(l, n_1, n_2) = \left(\frac{l!(l+n_1-n_2)!}{((l-n_2)!)^2}\right)^{1/2}$.

The following is a result on kernel operators associated to more general kernels:

THEOREM 2.2 ([3], Proposition 1.2.3 (a)). *Let $w(k,p) \in \mathcal{S}'(\mathbb{R}^{d(n_1+n_2)})$ so that w is a densely defined bilinear form on $\mathcal{S}(\mathbb{R}^{dn_1}) \times \mathcal{S}(\mathbb{R}^{dn_2})$. Denote by $\|w\|$ the norm of the operator w from $L^2(\mathbb{R}^{dn_2})$ to $L^2(\mathbb{R}^{dn_1})$ given by the kernel $w(k,p)$. Let T_W be the bilinear form given by (1.1). Then there exists a constant $C(n_1, n_2)$ such that the bound*

$$\begin{aligned} &\|(N_\tau + 1)^{-n_1/2} T_W (N_\tau + 1)^{-n_2/2}\| \\ \leq \;& C(n_1, n_2) \|(\prod_{i=1}^{n_1} \mu(k_i)^{-\tau/2})\, w\, (\prod_{j=1}^{n_2} \mu(p_j)^{-\tau/2})\| \end{aligned}$$

holds.

3. Kernel operators : Main results

In this section, we reformulate and extend the results in Theorem 2.1 and Theorem 2.2 in the framework of white noise analysis. First let us recall basic facts of white noise analysis. The inner product of $L^2(\mathbb{R}^d; dx)$ is denoted by $(\cdot,\cdot)_2$, the corresponding norm by $|\cdot|_2$. Let $(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}, \mu)$ be the Gaussian measure on $\mathcal{S}'(\mathbb{R}^d)$ with its characteristic function on $\mathcal{S}(\mathbb{R}^d)$ given by

$$\int_{\mathcal{S}'(\mathbb{R}^d)} \exp(i < x, \xi >)\, d\mu(x) = \exp(-\frac{1}{2}|\xi|_2^2), \quad \xi \in \mathcal{S}(\mathbb{R}^d),$$

where $< \cdot, \cdot >$ is the dual pairing. Let $(L^2) = L^2(\mathcal{S}'(\mathbb{R}^d), \mu)$. Recall the definition of $\Gamma(L^2)$ in (2.6). By the chaos decomposition([4, Proposition 2.6]), every $\phi \in (L^2)$ is in one to one correspondence with a sequence $\tilde{\phi} = \{\tilde{\phi}^{(n)}\}$ in $\Gamma(L^2)$, and $\|\phi\|_{(L^2)} = \|\tilde{\phi}\|_{\Gamma(L^2)}$. Thus one may identify (L^2) with $\Gamma(L^2)$.

Let H be the self-adjoint extension of the differential operator

$$-\Delta + |x|^2 + 1$$

on $L^2(\mathbb{R}^d; dx)$ with domain $\mathcal{S}(\mathbb{R}^d)$. Notice that $H \geq 2$. By $|\cdot|_{2,p}$, $p \in \mathbb{R}$, we denote the norm given by $|H^p \cdot|_2$. More generally, for a function $f^{(l)}$

on $\mathbb{R}^{dl}$, $l \in \mathbb{N}$, we denote by $|f^{(l)}|_{2,p}$, $p \in \mathbb{R}$, the norm $|(H^p)^{\otimes l} f^{(l)}|_2$. Let

$$(\mathcal{S}) \subset (L^2) \simeq \Gamma(L^2) \subset (\mathcal{S})^*$$

be the Gelfand triple [4, Chapter 4.A]. Any $\psi \in (\mathcal{S})$ has a chaos decomposition $\psi = \{\psi^{(l)}; l \in \mathbb{N}\}$, where each $\psi^{(l)}$, $l \in \mathbb{N}$, belongs to the symmetric Schwartz space $\mathcal{S}_s(\mathbb{R}^{dl})$ [4, Proposition 4.3]. Again we identify $(\mathcal{S})$ with the space of such sequences. Notice that $\psi = \{\psi^{(l)}\}$ belongs to $(\mathcal{S})$ if and only if for any $p \geq 0$

$$\|\psi\|_{2,p}^2 = \sum_{l=0}^{\infty} l! |\psi^{(l)}|_{2,p}^2 < \infty. \tag{3.1}$$

See [4, Proposition 4.2]. We remark that the inclusions

$$D_{\mathcal{S}} \subset (\mathcal{S}) \subset (L^2) \simeq \Gamma(L^2) \subset \mathcal{F}(L^2) \tag{3.2}$$

hold and the embeddings are dense.

Let us return to the discussion of kernel operators. Denote the set of continuous linear map from a topological space X to a topological space Y by $\mathcal{L}(X, Y)$. Recall that $W \in \mathcal{L}(L^2(\mathbb{R}^{dn_2}), L^2(\mathbb{R}^{dn_1}))$ is Hilbert-Schmidt operator if and only if there is a kernel function $w \in L^2(\mathbb{R}^{d(n_1+n_2)})$ with

$$(Wf)(k) = \int_{\mathbb{R}^{dn_2}} w(k,p) f(p) dp, \qquad f \in L^2(\mathbb{R}^{dn_2}). \tag{3.3}$$

Moreover the Hilbert-Schmidt norm of W equals to the L^2-norm of the kernel function w [8, Theorem VI.23]. Therefore one can see that Theorem 2.1 is the result for kernel operators associated with Hilbert-Schmidt operators from $L^2(\mathbb{R}^{dn_2})$ to $L^2(\mathbb{R}^{dn_1})$.

In order to describe the main idea in this paper, consider the operators N_τ defined in (2.12). N_τ can be written formally by

$$N_\tau = \int_{\mathbb{R}^{2d}} (k^2+1)^{\tau/4} \delta(k-p)(p^2+1)^{\tau/4} a^*(k) a(p)\, dk dp.$$

Thus, in a sense N_τ $\tau \geq 0$, can be viewed as a kernel operator associated with the kernel $w(k,p) = (k^2+1)^{\tau/4}\delta(k-p)(p^2+1)^{\tau/4}$, which is not in $L^2(\mathbb{R}^{2d})$. However the kernel defines a continuous linear map from $\mathcal{S}(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ via the expression (3.3).

Let W be a continuous linear map from $\mathcal{S}(\mathbb{R}^{dn_2})$ to $L^2(\mathbb{R}^{dn_1})$, i.e., $W \in \mathcal{L}(\mathcal{S}(\mathbb{R}^{dn_2}), L^2(\mathbb{R}^{dn_1}))$. Then there exist constants $p > 0$ and $C > 0$

such that

$$|Wf|_{L^2} \le C|(H^p)^{\otimes n_2} f|_2 \quad f \in \mathcal{S}(\mathbb{R}^{dn_2}). \tag{3.4}$$

See [8, Chapter V]. Let $W \in \mathcal{L}(\mathcal{S}(\mathbb{R}^{dn_2}), L^2(\mathbb{R}^{dn_1}))$ be given. For any $l \ge n_2$, $\mathcal{S}(\mathbb{R}^{dl}) \simeq \mathcal{S}(\mathbb{R}^{dn_2}) \otimes \mathcal{S}(\mathbb{R}^{d(l-n_2)})$, and so $W \otimes \mathbf{1}^{(l-n_2)}$ is a continuous linear map from $\mathcal{S}(\mathbb{R}^{dl})$ to $L^2(\mathbb{R}^{d(l-n_2+n_1)})$, where $\mathbf{1}^{(l-n_2)}$ is the identity map on $L^2(\mathbb{R}^{d(l-n_2)})$.

The following is our main result:

THEOREM 3.1. *Let n_1 and n_2 be non-negative integers and suppose that a continuous linear map W from $\mathcal{S}(\mathbb{R}^{dn_2})$ to $L^2(\mathbb{R}^{dn_1})$ is given. For any $\psi = \{\psi^{(l)}\} \in (\mathcal{S})$, let $T_W\psi$ be defined by*

$$\begin{aligned} (T_W\psi)^{(l-n_2+n_1)} &= K(l,n_1,n_2)S(W \otimes \mathbf{1}^{(l-n_2)})\psi^{(l)}, \quad l \ge n_2, \\ (T_W\psi)^{(n)} &= 0, \quad n < n_1, \end{aligned} \tag{3.5}$$

where $S = S_{l-n_2+n_1}$ is the symmetrization operator and $K(l,n_1,n_2)$ is the constant given in Theorem 2.1 (e). *Then the following results hold:*
(a) *T_W becomes a continuous linear map from $(\mathcal{S})$ to (L^2). Thus T_W is an unbounded linear operator on $\Gamma(L^2)$ (and so $\mathcal{F}(L^2)$) with domain $(\mathcal{S})$.*
(b) *If the adjoint W^* of W defines a continuous linear map from $\mathcal{S}(\mathbb{R}^{dn_1})$ to $L^2(\mathbb{R}^{dn_2})$, then T_{W^*} is also continuous map from $(\mathcal{S})$ to (L^2). In this case $T_{W^*} = (T_W)^*$ on $(\mathcal{S})$, and so T_W is closable.*
(c) *Let p be any positive constant satisfying* (3.4). *Then $T_W(d\Gamma(H^{2p}))^{-n_2/2}$ $(N+1)^{-n_1/2}$ is a bounded operator on $\mathcal{F}(L^2)$), and the bound*

$$\begin{aligned} &\|T_W(d\Gamma(H^{2p}))^{-n_2/2}(N+1)^{-n_1/2}\|_{\mathcal{L}(\mathcal{F}(L^2),\mathcal{F}(L^2))} \\ &\le C(n_1,n_2)\|W(H^{-p})^{\otimes n_2}\|_{\mathcal{L}(L^2(\mathbb{R}^{dn_2}),L^2(\mathbb{R}^{dn_1}))} \end{aligned}$$

holds for some constant $C(n_1,n_2)$. The same result as above holds on (L^2) if one replaces $T_W(d\Gamma(H^{2p}))^{-n_2/2}(N+1)^{-n_1/2}$ by $T_W(d\Gamma(H^{2p}))^{-n_2/2}$ $(N+1)^{-n_1/2}(N+1)^{-(n_1-n_2)/2}$.

REMARK 3.2. (a) If $W \in \mathcal{L}(\mathcal{S}(\mathbb{R}^{dn_2}), L^2(\mathbb{R}^{dn_1}))$ is defined by the kernel $w(k,p)$ via the expression (3.3), then T_W can be expressed as the integral (1.1) as a bilinear form $(\mathcal{S}) \times (\mathcal{S})$.
(b) Theorem 3.1(c) is the result analogous to Theorem 2.2 [3, Proposition 1.2.3 (a)].

Proof of Theorem 3.1. (a) For any $\psi = \{\psi^{(l)}\} \in (\mathcal{S})$, it follows from (3.5) and (3.4) that

$$
\begin{aligned}
&|(T_W\psi)^{(l-n_2+n_1)}|_2 \le K(l,n_1,n_2)|(W\otimes \mathbf{1}^{(l-n_2)})\psi^{(l)}|_2 \\
(3.6)\qquad &\le K(l,n_1,n_2)\|W(H^{-p})^{\otimes n_2}\|\;|((H^p)^{\otimes n_2}\otimes \mathbf{1}^{(l-n_2)})\psi^{(l)}|_2.
\end{aligned}
$$

Notice that $K(l,n_1,n_2)^2 \le l^{n_2}(l+n_1)^{n_1}$. Since $H \ge 2$, we have that

$$|((H^p)^{\otimes n_2}\otimes \mathbf{1}^{(l-n_2)})\psi^{(l)}|_2 \le |(H^p)^{\otimes l}\psi^{(l)}|_2.$$

Thus it follows from (3.6) that

$$
\begin{aligned}
\|T_W\psi\|^2_{(L^2)} &\le C\sum_{l\ge n_2}(l-n_2+n_1)!\; l^{n_2}(l+n_1)^{n_1}|\psi^{(l)}|^2_{2,p} \\
&\le \tilde{C}\|\psi\|^2_{2,\tilde{p}}
\end{aligned}
$$

for some $\tilde{p} \ge p$. This proves the part (a).

(b) For given $l \ge n_2$, let $f \in L^2(\mathbb{R}^{dl})$ and $g \in \mathcal{S}(\mathbb{R}^{d(l-n_2+n_1)})$. Using the fact $K(l,n_1,n_2) = K(l-n_2+n_1,n_2,n_1)$ and $S^2 = S$, we have

$$
\begin{aligned}
&K(l,n_1,n_2)(S(W\otimes \mathbf{1}^{(l-n_2)})Sf,g)_2 \\
(3.7)\qquad &= K(l-n_2+n_1,n_2,n_1)\,(Sf,S(W^*\otimes \mathbf{1}^{(l-n_2)})Sg)_2.
\end{aligned}
$$

Thus, it follows from the above relation that $(T_W)^* = T_{W^*}$ on $(\mathcal{S})$. This proves the part (b).

(c) Since each $\psi^{(l)}$ belongs to $\mathcal{S}_s(\mathbb{R}^{dl})$, we have

$$
\begin{aligned}
|((H^p)^{\otimes n_2}\otimes \mathbf{1}^{(l-n_2)})\psi^{(l)}|_2^2 &= (\psi^{(l)},((H^{2p})^{\otimes n_2}\otimes \mathbf{1}^{(l-n_2)})\psi^{(l)})_2 \\
(3.8)\qquad &= (\psi^{(l)},S((H^{2p})^{\otimes n_2}\otimes \mathbf{1}^{(l-n_2)})\psi^{(l)})_2.
\end{aligned}
$$

For any $j \in \{1,2,\cdots,l\}$, denote by H_j the operator H acting on $L^2(\mathbb{R}^d,dx_j)$, i.e., $L^2(\mathbb{R}^d)$- space with x_j-variable. For any $A = \{i_1,i_2,\cdots,i_{n_2}\} \subset \{1,2,\cdots,l\}$, $i_j < i_k$ if $j<k$, we write for a sake of brevity that

$$(H^{2p})^{\otimes A} = \mathbf{1}\otimes\cdots\otimes H^{2p}_{i_1}\otimes\mathbf{1}\otimes\cdots\otimes H^{2p}_{i_2}\otimes\mathbf{1}\otimes\cdots\otimes H^{2p}_{i_{n_2}}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}.$$

Then it can be checked that

$$
\begin{aligned}
S((H^{2p})^{\otimes n_2}\otimes \mathbf{1}^{(l-n_2)})S &= \frac{(l-n_2)!\,n_2!}{l!}\sum_{A:|A|=n_2}(H^{2p})^{\otimes A} \\
(3.9)\qquad &\le \frac{(l-n_2)!}{l!}(d\Gamma(H^{2p}))^{n_2},
\end{aligned}
$$

where the sum is taken over the subsets of $\{1, 2, \cdots, l\}$ with n_2-elements. By substituting (3.9) into (3.8), it follows from (3.6) that

$$\begin{aligned} & |(T_W\psi)^{(l-n_2+n_1)}|_2 \\ \leq \;& C\left(\frac{(l-n_2+n_1)!}{(l-n_2)!}\right)^{1/2} \|W(H^{-p})^{\otimes n_2}\| \; |(d\Gamma(H^{2p})^{n_2/2}\psi^{(l)}|_2 \\ (3.10) \quad \leq \;& C(n_1,n_2)\|W(H^{-p})^{\otimes n_2}\| \; |(N+1)^{n_1/2}(d\Gamma(H^{2p}))^{n_2/2}\psi^{(l)}|_2. \end{aligned}$$

Thus the first part of (c) follows from the above estimate. And

$$\|T_W\psi\|^2_{\Gamma(L^2)} = \sum_{l\geq n_2}(l-n_2+n_1)! \, |(T_W\psi)^{(l-n_2+n_1)}|_2^2.$$

The second part of (c) follows from (3.10) and the above equality. This completes the proof of the theorem. □

Next, let us consider optimal conditions on kernels so that the corresponding kernel operators are densely defined operators on Fock spaces $\mathcal{F}(L^2(\mathbb{R}^d))$ and $\Gamma(L^2(\mathbb{R}^d))$. For any dense subset D in $L^2(\mathbb{R}^d)$, denote by $D_s^{(n)}$ the n-fold symmetric algebraic tensor product of D. For a given dense subset $D \subset L^2(\mathbb{R}^d)$, let

$$\mathcal{F}_0(D) = \{f = (f^{(n)}) \in \mathcal{F}_0 : \; f^{(n)} \in D_s^{(n)}, \; n \in \mathbb{N}\}. \tag{3.11}$$

Denote by $\mathcal{E}(D)$ the algebra generated by the exponential vectors $e(f)$, $f \in D$, where for $f \in D$

$$e(f) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{\otimes n}. \tag{3.12}$$

Then $\mathcal{F}_0(D)$ and $\mathcal{E}(D)$ are dense in $\mathcal{F}(L^2(\mathbb{R}^d))$ and also in $\Gamma(L^2(\mathbb{R}^d))$.

THEOREM 3.3. *Let n_1 and n_2 be nonnegative integers. For a dense subset $D \subset L^2(\mathbb{R}^d)$, let $D^{(n_2)}$ be the n_2-fold algebraic tensor product of D. Suppose W be an unbounded operator from $L^2(\mathbb{R}^{dn_2})$ to $L^2(\mathbb{R}^{nd_1})$ with domain $D^{(n_2)}$. For any $\psi = \{\psi^{(l)}\} \in \mathcal{F}_0(D)$, define $T_W\psi$ by the expression (3.5). Then the following results holds:*
(a) T_W is well-defined on $\mathcal{F}_0(D)$. Thus T_W becomes an unbounded operator on $\mathcal{F}(L^2(\mathbb{R}^d))$ and also on $\Gamma(L^2(\mathbb{R}^d))$ with domain $\mathcal{F}_0(D)$.
(b) T_W is well-defined on $\mathcal{E}(D)$. Thus T_W becomes an unbounded operator on $\mathcal{F}(L^2(\mathbb{R}^d))$ and also on $\Gamma(L^2(\mathbb{R}^d))$ with domain $\mathcal{E}(D)$.
(c) Suppose that W^* is well-defined on $D^{(n_1)}$, i.e., $D^{(n_1)} \subset D(W^*)$. Then $T_{W^*} \subset (T_W)^*$. Thus T_W is closable.

Proof. (a) Notice that for $\psi^{(l)} = S(f_1 f_2 \cdots f_l)$, $l \geq n_2$

$$S(W \otimes \mathbf{1}^{(l-n_2)})\psi^{(l)} = \sum_{\pi \in S(l)} W(f_{\pi(1)} \cdots f_{\pi(n_2)}) f_{\pi(n_2+1)} \cdots f_{\pi(l)},$$

where $S(l)$ is the permutation group of $\{1, \cdots l\}$. Since each $W(f_{\pi(1)} \cdots f_{\pi(n_2)})$ belongs to $L^2(\mathbb{R}^{dn_1})$, we see that $S(W \otimes \mathbf{1}^{(l-n_2)})\psi^{(l)} \in L^2(\mathbb{R}^{d(l-n_2+n_1)})$. Thus (a) follows from (3.5).

(b) A direct computation shows that for $g \in D$

$$T_W(e(g)) = \sum_{l \geq n_2} \frac{K(l, n_1, n_2)}{l!} S(W(g^{\otimes n_2}) \cdot g^{\otimes (l-n_2)}),$$

and

$$\begin{aligned} \|T_W(e(g))\|^2_{\Gamma(L^2(\mathbb{R}^d))} &\leq \sum_{l \geq n_2} (l - n_2 + n_1)! \frac{K(l, n_1, n_2)^2}{(l!)^2} |W g^{\otimes n_2}|_2^2 \, |g|_2^{2(l-n_2)} \\ &< \infty. \end{aligned}$$

Notice that $\|\psi\|_{\mathcal{F}(L^2)} \leq \|\psi\|_{\Gamma(L^2)}$. This proves the part (b).

(c) This follows the method similar to that used in the proof of the part (b) of Theorem 3.1. □

In the rest of the paper, we consider the class of kernel operators associated with bounded operators from $L^2(\mathbb{R}^{dn_2})$ to $L^2(\mathbb{R}^{dn_1})$. In this case we have the results analogous to Theorem 2.1.

THEOREM 3.4. *Let n_1 and n_2 be non-negative integers and let $W \in \mathcal{L}\ (L^2(\mathbb{R}^{dn_2}),\ L^2(\mathbb{R}^{dn_1}))$. Let T_W be an unbounded operator on $\mathcal{F}(L^2)$ defined as in (3.5). Then the following results hold:*
(a) T_W is well-defined on $\mathcal{F}_0$, and leaves $\mathcal{F}_0$ invariant, i.e., $T_W \mathcal{F}_0 \subset \mathcal{F}_0$.
(b) $(T_W)^* = T_{W^*}$ on $\mathcal{F}_0$, and so T_W is closable.
(c) If $W_n \to W$ in $\mathcal{L}(L^2(\mathbb{R}^{dn_2}), L^2(\mathbb{R}^{dn_1}))$, then $T_{W_n} \to T_W$ strongly on $\mathcal{F}_0$.
(d) $T_W(N+1)^{-(n_1+n_2)/2}$ is a bounded operator on $\mathcal{F}(L^2)$, and there exists a constant $C(n_1, n_2)$ such that the bound

$$\|T_W(N + 1)^{-(n_1+n_2)/2}\| \leq C(n_1, n_2)\|W\|$$

holds.

Proof. (a) Since $S(W \otimes \mathbf{1}^{(l-n_2)})\psi^{(l)} \in \mathcal{F}^{(l-n_2+n_1)}(L^2)$, the part (a) follows from (3.5).

(b)For any $f \in L^2(\mathbb{R}^{dl})$ and $g \in L^2(\mathbb{R}^{d(l-n_2+n_1)}$, the equality (3.7) holds. This implies the part (b).

(c) It follows from (3.5) that for any $\psi = \{\psi^{(l)}\} \in \mathcal{F}_0$

$$|(T_{W_n}\psi)^{(l-n_2+n_1)} - (T_W\psi)^{(l-n_2+n_1)}|_2 \le K(l, n_1, n_2)\|W_n - W\| \, |\psi^{(l)}|_2.$$

Thus (c) follows from the above bound.

(d) If one replace H by $\mathbf{1}$ and then replace $d\Gamma(H^{2p})$ by $(N+1)$ in Theorem 3.1 (c), the part (d) follows from Theorem 3.1 (c) as a corollary. □

ACKNOWLEDGEMENTS. This work was supported in part by Basic Research Institute Program (BSRI 98-1421), Korean Ministry of Education, and KOSEF (95-0701-04-01-3).

References

[1] O. Bratelli and D. W. Robinson, *Operator Algebra and Quantum Statistical Mechanics II*, Springer, 1981.

[2] J. Glimm and A. Jaff, *Quantum Physics, Functional integral point of view*, Springer, New York, 1981.

[3] ———, *Some quantum field theory models,* in Statistical Mechanics and Quantum Field Theory (Les Houches 1970), ed. by C. deWitt and R. Stora, Gordon and Breach, New York, 1971.

[4] T. Hida, H. H. Kuo, J. Potthoff and L. Streit, *White Noise : An Infinite Dimensional Calculus*, Kluwer Academic, 1993.

[5] P. A. Meyer, *Quantum Probability for Probabilists*, LNM 1538, Springer-Verlag, New York, 1993, pp 87-96.

[6] N. Obata, *White Noise Calculus and Fock space*, Springer-Verlag, Berlin, 1994.

[7] K. R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus*, Birkhäuser Verlag, Berlin, 1992.

[8] M. Reed and B. Simon, *Methods of Mathematical Physics* vol. I - II Academic press, New York, 1972.

Department of Mathematics
Yonsei University
Seoul 120-749, Korea
E-mail: ympark@bubble.yonsei.ac.kr