J. Korean Math. Soc. **35** (1998), No. 2, pp. 465–490

ON EXISTENCE OF SOLUTIONS OF DEGENERATE WAVE EQUATIONS WITH NONLINEAR DAMPING TERMS

JONG YEOUL PARK AND JEONG JA BAE

ABSTRACT. In this paper, we consider the existence and asymptotic behavior of solutions of the following problem:

$$\begin{aligned}
&u_{tt}(t,x) - (\|\nabla u(t,x)\|_2^2 + \|\nabla v(t,x)\|_2^2)^\gamma \Delta u(t,x) + \delta|u_t(t,x)|^{p-1}u_t(t,x) \\
&\quad = \mu|u(t,x)|^{q-1}u(t,x), \quad x \in \Omega, \quad t \in [0,T], \\
&v_{tt}(t,x) - (\|\nabla u(t,x)\|_2^2 + \|\nabla v(t,x)\|_2^2)^\gamma \Delta v(t,x) + \delta|v_t(t,x)|^{p-1}v_t(t,x) \\
&\quad = \mu|v(t,x)|^{q-1}v(t,x), \quad x \in \Omega, \quad t \in [0,T], \\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \\
&v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \quad x \in \Omega, \\
&u|_{\partial\Omega} = v|_{\partial\Omega} = 0
\end{aligned}$$

where $T > 0$, $q > 1$, $p \geq 1$, $\delta > 0$, $\mu \in R$, $\gamma \geq 1$ and Δ is the Laplacian in R^N.

1. Introduction

Let Ω be a bounded domain in R^N with smooth boundary $\partial\Omega$. In this paper, we consider the existence of solutions of the following problem:

Received February 5, 1998.

1991 Mathematics Subject Classification: 35L70, 35L15, 65M60.

Key words and phrases: existence and uniqueness, asymptotic behavior, degenerate wave equation, Galerkin method.

This is supported by KOSEF 1996 (961-0106-040-2).

(1.1)

$$
\begin{aligned}
&u_{tt}(t,x) - (\|\nabla u(t,x)\|_2^2 + \|\nabla v(t,x)\|_2^2)^\gamma \Delta u(t,x) + \delta |u_t(t,x)|^{p-1} u_t(t,x) \\
&\quad = \mu |u(t,x)|^{q-1} u(t,x), \quad x \in \Omega, \quad t \in [0,T], \\
&v_{tt}(t,x) - (\|\nabla u(t,x)\|_2^2 + \|\nabla v(t,x)\|_2^2)^\gamma \Delta v(t,x) + \delta |v_t(t,x)|^{p-1} v_t(t,x) \\
&\quad = \mu |v(t,x)|^{q-1} v(t,x), \quad x \in \Omega, \quad t \in [0,T], \\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \\
&v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \quad x \in \Omega, \\
&u|_{\partial\Omega} = v|_{\partial\Omega} = 0,
\end{aligned}
$$

where $T > 0$, $q > 1$, $p \geq 1$, $\delta > 0$, $\mu \in R$, $\gamma \geq 1$ and Δ is the Laplacian in R^N. Here

$$
\|\nabla u\|_2^2 = \sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}(t,x)|^2 dx, \quad u_t = \frac{\partial u}{\partial t} \quad \text{and} \quad \Delta u = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}.
$$

Equation (1.1) has its origin in the nonlinear vibrations of an elastic string (cf. R. Narasimha [6]). Many authors have studied the existence and uniqueness of solutions of (1.1) by using various methods.

When $\delta > 0$ and $\mu = 0$, for degenerate case, Nishihara and Yamada [7] have proved the global existence of a unique solution under the assumptions that the initial data $\{u_0, u_1\}$ are sufficiently small and $u_0 \neq 0$. For the problem with linear damping δu_t, there are the works of Brito [1], Ikehata [2], K. Ono [8] and the references therein. In the case of $\gamma = 1$, M. D. Silva Aleves ([9]) has proved the existence of weak solutions of the unilateral problem using the Galerkin method. In the present paper we will study the existence and uniqueness of solutions of unilateral problem (1.1) with $\gamma > 1$ by using Galerkin method and will also investigate its asymptotic behavior.

The contents of this paper are as follows: In section 2, we present the preliminaries and some lemmas. In section 3, we give the statement of the main theorem. In section 4, we deal with a priori estimates for solutions of (1.1) and prove our main Theorem and section 5 deals with the asymptotic behavior of the solutions obtained in section 4.

2. Preliminaries

We first prepare the following well known lemmas which will be needed later.

LEMMA 2.1. (Sobolev-Poincaré [4]) *If either* $1 \le q < +\infty \quad (N = 1,2)$ *or* $1 \le q \le \frac{N+2}{N-2} \quad (N \ge 3)$, *then there is a positive constant* $C(\Omega, q+1)$ *such that*

$$\|u\|_{q+1} \le C(\Omega, q+1)\|\nabla u\|_2 \quad \text{for} \quad u \in H_0^1(\Omega).$$

In other words,

$$C(\Omega, q+1) = \sup\{\frac{\|u\|_{q+1}}{\|\nabla u\|_2} | \quad u \in H_0^1(\Omega), \quad u \neq 0\}$$

is positive and finite.

LEMMA 2.2. (Gagliardo-Nirenberg [4]) *Let* $1 \le r < q \le +\infty$ *and* $p \le q$. *Then the inequality*

$$\|u\|_{W^{k,q}} \le C\|u\|_{W^{m,p}}^{\theta}\|u\|_r^{1-\theta} \quad \text{for} \quad u \in W^{m,p}(\Omega) \cap L^r(\Omega)$$

holds with some $C > 0$ *and*

$$\theta = \left(\frac{k}{N} + \frac{1}{r} - \frac{1}{q}\right)\left(\frac{m}{N} + \frac{1}{r} - \frac{1}{p}\right)^{-1}$$

provided that $0 < \theta \le 1$ (*we assume* $0 < \theta < 1$ *if* $q = +\infty$).

We conclude this section by stating a lemma concerning a difference inequality, which will be used later.

LEMMA 2.3. (Nakao [5]) *Let* $\phi(t)$ *be a nonincreasing and nonnegative function on* $[0,T], T > 1$, *such that*

$$\phi(t)^{1+r} \le k_0(\phi(t) - \phi(t+1)) \quad \text{on} \quad [0,T],$$

where k_0 *is a positive constant and* r *a nonnegative constant. Then we have*

(*i*) *if* $r > 0$, *then* $\phi(t) \le (\phi(0)^{-r} + k_0^{-1}r[t-1]^+)^{-\frac{1}{r}}$,

where $[t-1]^+ = \max\{t-1, 0\}$,

(*ii*) *if* $r = 0$, *then* $\phi(t) \le \phi(0)e^{-k_1[t-1]^+}$ *on* $[0,T]$,

where $k_1 = \log(\frac{k_0}{k_0-1})$.

3. Statement of the result

We consider the following initial value problem:

$$
(3.1)\qquad
\begin{aligned}
&u_{tt}(t) - (\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^\gamma \Delta u(t) + \delta |u_t(t)|^{p-1} u_t(t) \\
&\quad = \mu |u(t)|^{q-1} u(t), \quad t \in [0,T], \\
&v_{tt}(t) - (\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^\gamma \Delta v(t) + \delta |v_t(t)|^{p-1} v_t(t) \\
&\quad = \mu |v(t)|^{q-1} v(t), \quad t \in [0,T], \\
&u(0) = u_0, \quad u_t(0) = u_1, \\
&v(0) = v_0, \quad v_t(0) = v_1, \quad \text{where} \quad \gamma \geq 1.
\end{aligned}
$$

Now we set

$$
\begin{aligned}
J(u,v) &= \frac{1}{2(\gamma+1)}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^{\gamma+1} - \frac{\mu}{q+1}(\|u\|_{q+1}^{q+1} + \|v\|_{q+1}^{q+1}), \\
I(u,v) &= (\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^{\gamma+1} - \mu(\|u\|_{q+1}^{q+1} + \|v\|_{q+1}^{q+1})
\end{aligned}
$$

and define the potential as

$$
W = \{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) | I(u,v) > 0\} \cup \{0\}.
$$

Next, by setting

$$
E(u,v) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|v_t\|^2 + J(u,v),
$$

we can state our main Theorem.

THEOREM 3.1. *Let N be a positive integer. Suppose that $\delta > 0$, $\mu > 0$ and $2\gamma < \min\{q-1, (4-N)q + N - 2\}$. Assume that $p < \min\{q, \frac{N+4q-Nq}{2}\}$ is such that*

(*i*) $1 \leq p < +\infty (N = 1, 2)$

(*ii*) $1 \leq p \leq 3, \quad 1 < q \leq 5 (N = 3)$

(*iii*) $1 \leq p \leq \dfrac{N}{N-2}, \quad \dfrac{N}{N-2} \leq q \leq \min\{\dfrac{N+2}{N-2}, \dfrac{N-2}{[N-4]^+}\}(N \geq 4).$

If $(u_0, v_0) \in W \cap (H^2(\Omega) \times H^2(\Omega)), u_1, v_1 \in H_0^1(\Omega)$ *and*

$$\mu[C(\Omega, q+1)]^{q+1}\left(\frac{2(q+1)(\gamma+1)}{q-1-2\gamma}E(u_0, v_0)\right)^{\frac{q-1-2\gamma}{2(\gamma+1)}} < 1,$$

then the problem (3.1) *has solutions* $u = u(t,x)$ *and* $v = v(t,x)$ *satisfying*

$$\begin{aligned} &u, v \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)), \\ &u', v' \in L^\infty(0,T; H_0^1(\Omega)), \\ &u'', v'' \in L^\infty(0,T; L^2(\Omega)). \end{aligned}$$

4. Proof of Theorem 3.1

Throughout this section we always assume that $(u_0, v_0) \in W \cap (H^2(\Omega) \times H^2(\Omega))$ and $u_1, v_1 \in H_0^1(\Omega)$. We employ the Galerkin method to construct a solution. Let $\{\lambda_j\}_{j=1}^\infty$ be a sequence of eigenvalues for $-\Delta w = \lambda w$ in Ω. Let $w_j \in H_0^1(\Omega) \cap H^2(\Omega)$ be the corresponding eigenfunction to λ_j and take $\{w_j\}_{j=1}^\infty$ as a complete orthonormal system in $L^2(\Omega)$. We construct approximate solutions $u_m, v_m \quad (m = 1, 2, \cdots)$ in the form

$$u_m(t) = \sum_{j=1}^m g_{jm}(t) w_j, \quad v_m(t) = \sum_{j=1}^m h_{jm}(t) w_j$$

which are determined by the following ordinary differential equations:

$$\begin{aligned} &(u_m''(t), w) - ((\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma \Delta u_m(t), w) \\ &\quad + \delta |u_m'(t)|^{p-1}(u_m'(t), w) = \mu |u_m(t)|^{q-1}(u_m(t), w), \end{aligned} \tag{4.1}$$

$$\begin{aligned} &(v_m''(t), w) - ((\|\nabla u_m(t)\|^2 + \|\nabla v_m(t)\|^2)^\gamma \Delta v_m(t), w) \\ &\quad + \delta |v_m'(t)|^{p-1}(v_m'(t), w) = \mu |v_m(t)|^{q-1}(v_m(t), w) \end{aligned} \tag{4.2}$$

($' = \frac{\partial}{\partial t}$ and $'' = \frac{\partial^2}{\partial t^2}$) with the initial conditions,

$$(4.3)\qquad \begin{aligned} u_m(0) &= u_{0m} = \sum_{j=1}^{m}(u_0, w_j)w_j \to u_0 \quad \text{in } H_0^1(\Omega)\cap H^2(\Omega), \\ v_m(0) &= v_{0m} = \sum_{j=1}^{m}(v_0, w_j)w_j \to v_0 \quad \text{in } H_0^1(\Omega)\cap H^2(\Omega), \end{aligned}$$

$$(4.4)\qquad \begin{aligned} u_m'(0) &= u_{1m} = \sum_{j=1}^{m}(u_1, w_j)w_j \to u_1 \quad \text{strongly in } H_0^1(\Omega), \\ v_m'(0) &= v_{1m} = \sum_{j=1}^{m}(v_1, w_j)w_j \to v_1 \quad \text{strongly in } H_0^1(\Omega). \end{aligned}$$

Therefore we can solve the system (4.1)-(4.4) by Picard's iteration method. Hence the system (4.1)-(4.4) have a unique solution on some interval $[0, T_m)$ with $0 < T_m \le T$. Note that $u_m(t)$ is in the C^2-class. We shall see that $u_m(t)$ can be extended to $[0, T]$. We can utilize a standard compactness argument for the limiting procedure and it suffices to derive some a priori estimates for u_m. But this procedure allows us to employ the energy method for a smooth solution $u(t)$ to the problem (4.1)-(4.4) (the results should be in fact applied to the approximated solutions).

A Priori Estimates I

Multiplying the equation in (4.1) by $u_m'(t)$ and multiplying the equation in (4.2) by $v_m'(t)$ yield

$$(4.5)\qquad \begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|u_m'(t)\|_2^2 - \frac{\mu}{q+1}\|u_m(t)\|_{q+1}^{q+1}\right) + \delta\|u_m'(t)\|_{p+1}^{p+1} \\ &\quad + \frac{1}{2}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma \frac{d}{dt}\|\nabla u_m(t)\|_2^2 = 0 \end{aligned}$$

and

$$(4.6)\qquad \begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|v_m'(t)\|_2^2 - \frac{\mu}{q+1}\|v_m(t)\|_{q+1}^{q+1}\right) + \delta\|v_m'(t)\|_{p+1}^{p+1} \\ &\quad + \frac{1}{2}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma \frac{d}{dt}\|\nabla v_m(t)\|_2^2 = 0. \end{aligned}$$

Adding (4.5) and (4.6) and then integrating from 0 to t yield the energy identity

$$(4.7)\quad E(u_m(t), v_m(t)) + \delta \int_0^t (\|u_m'(s)\|_{p+1}^{p+1} + \|v_m'(s)\|_{p+1}^{p+1}) ds = E(u_0, v_0)$$

where

$$\begin{aligned} E(u_m(t), v_m(t)) &= \frac{1}{2}\|u_m'(t)\|_2^2 + \frac{1}{2}\|v_m'(t)\|_2^2 \\ &+ \frac{1}{2(\gamma+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1} \\ &- \frac{\mu}{q+1}\|u_m(t)\|_{q+1}^{q+1} - \frac{\mu}{q+1}\|v_m(t)\|_{q+1}^{q+1}. \end{aligned}$$

In particular, $E(u_m(t), v_m(t))$ is nonincreasing on $[0, T]$ and

$$(4.8)\qquad E(u_m(t), v_m(t)) \leq E(u_0, v_0).$$

Now, to obtain a priori estimates, we need the following result.

LEMMA 4.1. *Assume that either*

$$1 \leq q < +\infty (N = 1, 2), \quad \textit{or} \quad 1 \leq q \leq \frac{N+3}{N-2} (N \geq 3).$$

Let $(u_m(t), v_m(t))$ be the solution of (4.1) – (4.4) with $(u_0, v_0) \in W$ and $(u_1, v_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$. If $2\gamma < q - 1$ and

$$(4.9)\qquad \mu[C(\Omega, q+1)]^{q+1} \left(\frac{2(q+1)(\gamma+1)}{q-1-2\gamma} E(u_0, v_0) \right)^{\frac{q-1-2\gamma}{2(\gamma+1)}} < 1,$$

then $(u_m(t), v_m(t)) \in W$ on $[0, T]$, that is,

$$(\|\nabla u_m\|_2^2 + \|\nabla v_m\|_2^2)^{\gamma+1} - \mu(\|u_m\|_{q+1}^{q+1} + \|v_m\|_{q+1}^{q+1}) > 0 \quad on \quad [0, T].$$

Proof. Since $I(u_0, v_0) > 0$, it follows from the continuity of $u_m(t)$ and $v_m(t)$ that

$$I(u(t), v(t)) \geq 0 \quad \text{for some interval near} \quad t = 0. \tag{4.10}$$

Let t_{max} be a maximal time (possibly $t_{max} = T_m$) when (4.10) holds on $[0, t_{max})$. Note that

(4.11)
$$\begin{aligned} J(u_m(t), v_m(t)) &= \frac{1}{2(\gamma+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1} \\ &\quad - \frac{\mu}{q+1}(\|u_m(t)\|_{q+1}^{q+1} + \|v_m(t)\|_{q+1}^{q+1}), \\ &= \frac{q-2\gamma-1}{2(\gamma+1)(q+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1} \\ &\quad + \frac{1}{q+1} I(u_m(t), v_m(t)) \\ &\geq \frac{q-2\gamma-1}{2(\gamma+1)(q+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1} \\ &\quad \text{on} \quad [0, t_{max}). \end{aligned}$$

By the energy identity (4.7), (4.8) and (4.11), we have

(4.12)
$$\begin{aligned} (\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1} &\leq \frac{2(\gamma+1)(q+1)}{q-2\gamma-1} J(u_m(t), v_m(t)) \\ &\leq \frac{2(\gamma+1)(q+1)}{q-2\gamma-1} E(u_m(t), v_m(t)) \\ &\leq \frac{2(\gamma+1)(q+1)}{q-2\gamma-1} E(u_0, v_0) \\ &\quad \text{on} \quad [0, t_{max}). \end{aligned}$$

It follows from the Sobolev- Poincaré inequality and (4.11) that

(4.13)
$$\begin{aligned} \mu\|u_m(t)\|_{q+1}^{q+1} &\leq \mu C(\Omega, q+1)^{q+1} \|\nabla u_m(t)\|_2^{q+1} \\ &= \mu C(\Omega, q+1)^{q+1} \|\nabla u_m(t)\|_2^{q-2\gamma-1} \|\nabla u_m(t)\|_2^{2(\gamma+1)} \\ &\leq \mu C(\Omega, q+1)^{q+1} \left(\frac{2(\gamma+1)(q+1)}{q-2\gamma-1} E(u_0, v_0) \right)^{\frac{q-2\gamma-1}{2(\gamma+1)}} \\ &\quad \times \|\nabla u_m(t)\|_2^{2(\gamma+1)} \quad \text{on} \quad [0, t_{max}). \end{aligned}$$

Similarly,

(4.14)
$$\mu\|v_m(t)\|_{q+1}^{q+1} \le \mu C(\Omega, q+1)^{q+1}\left(\frac{2(\gamma+1)(q+1)}{q-2\gamma-1}E(u_0,v_0)\right)^{\frac{q-2\gamma-1}{2(\gamma+1)}} \times \|\nabla v_m(t)\|_2^{2(\gamma+1)} \quad \text{on} \quad [0, t_{max}).$$

Thus from (4.9), (4.13) and (4.14), we obtain

(4.15)
$$\begin{aligned}
&\mu(\|u_m(t)\|_{q+1}^{q+1} + \|v_m(t)\|_{q+1}^{q+1}) \\
&\le \mu C(\Omega, q+1)^{q+1}\left(\frac{2(\gamma+1)(q+1)}{q-2\gamma-1}E(u_0,v_0)\right)^{\frac{q-2\gamma-1}{2(\gamma+1)}} \\
&\quad \times (\|\nabla u_m(t)\|_2^{2(\gamma+1)} + \|\nabla v_m(t)\|_2^{2(\gamma+1)}) \\
&\le \mu C(\Omega, q+1)^{q+1}\left(\frac{2(\gamma+1)(q+1)}{q-2\gamma-1}E(u_0,v_0)\right)^{\frac{q-2\gamma-1}{2(\gamma+1)}} \\
&\quad \times (\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1} \\
&\le (\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1} \quad \text{on} \quad [0, t_{max}).
\end{aligned}$$

Therefore we get $I(u(t), v(t)) > 0$ on $[0, t_{max})$. This implies that we can take $t_{max} = T_m$. This completes the proof of Lemma 4.1. □

Using Lemma 4.1, we can deduce a priori-bounded on u_m and v_m:

(4.16)
$$\begin{aligned}
E(u_m(t), v_m(t)) &= \frac{1}{2}\|u_m'(t)\|_2^2 + \frac{1}{2}\|v_m'(t)\|_2^2 + J(u_m(t), v_m(t)) \\
&= \frac{1}{2}\|u_m'(t)\|_2^2 + \frac{1}{2}\|v_m'(t)\|_2^2 + \frac{1}{q+1}I(u_m(t), v_m(t)) \\
&\quad + \frac{q-1-2\gamma}{2(q+1)(\gamma+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1} \\
&\ge \frac{1}{2}\|u_m'(t)\|_2^2 + \frac{1}{2}\|v_m'(t)\|_2^2 \\
&\quad + \frac{q-1-2\gamma}{2(q+1)(\gamma+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1}.
\end{aligned}$$

Thus from (4.7) and (4.16), we get

(4.17)

$$\frac{1}{2}(\|u'_m(t)\|_2^2 + \|v'_m(t)\|_2^2) + \frac{q-1-2\gamma}{2(q+1)(\gamma+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1}$$
$$+ \delta \int_0^t (\|u'_m(s)\|_{p+1}^{p+1} + \|v'_m(s)\|_{p+1}^{p+1}) ds$$
$$\leq E(u_0, v_0).$$

In fact, the inequality (4.17) shows that Lemma 4.1 holds on $[0,\infty)$, that is,

$$(\|\nabla u_m\|_2^2 + \|\nabla v_m\|_2^2)^{\gamma+1} - \mu(\|u_m\|_{q+1}^{q+1} + \|v_m\|_{q+1}^{q+1}) > 0 \quad \text{on} \quad [0,\infty).$$

A Priori Estimates II

Multiplying the equation (4.1) by $-\Delta u'_m(t)$, multiplying the equation (4.2) by $-\Delta v'_m(t)$ and adding these equations give

(4.18)

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u'_m(t)\|_2^2 + \|\nabla v'_m(t)\|_2^2)$$
$$+ \frac{1}{2}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma}\frac{d}{dt}(\|\Delta u_m(t)\|_2^2 + \|\Delta v_m(t)\|_2^2)$$
$$+ p\delta(|u'_m(t)|^{p-1}\nabla u'_m(t), \nabla u'_m(t)) + p\delta(|v'_m(t)|^{p-1}\nabla v'_m(t), \nabla v'_m(t))$$
$$= \mu(\nabla[|u_m(t)|^{q-1}u_m(t)], \nabla u'_m(t)) + \mu(\nabla[|v_m(t)|^{q-1}v_m(t)], \nabla v'_m(t)).$$

We set

$$H(t) = H(u_m(t), v_m(t))$$
$$= \frac{\|\nabla u'_m(t)\|_2^2 + \|\nabla v'_m(t)\|_2^2}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma}} + \|\Delta u_m(t)\|_2^2 + \|\Delta v_m(t)\|_2^2.$$

From (4.18), we obtain

(4.19)

$$\frac{1}{2}H'(t) + \frac{p\delta(|u'_m(t)|^{p-1}\nabla u'_m(t), \nabla u'_m(t)) + p\delta(|v'_m(t)|^{p-1}\nabla v'_m(t), \nabla v'_m(t))}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma}}$$
$$= -\frac{\gamma(\|\nabla u'_m(t)\|_2^2 + \|\nabla v'_m(t)\|_2^2)((\nabla u_m(t), \nabla u'_m(t)) + (\nabla v_m(t), \nabla v'_m(t)))}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1}}$$
$$+ \frac{\mu(\nabla[|u_m(t)|^{q-1}u_m(t)], \nabla u'_m(t)) + \mu(\nabla[|v_m(t)|^{q-1}v_m(t)], \nabla v'_m(t))}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma}}$$
$$\equiv I_1(t) + I_2(t).$$

Now, we note that from (4.17) we have

$$\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2 \le \left(\frac{2(q+1)(\gamma+1)}{q-1-2\gamma}E(u_0,v_0)\right)^{\frac{1}{\gamma+1}}.$$

Thus we get

$$\begin{aligned}
I_1(t) &\le \gamma \frac{(\|\nabla u_m'(t)\|_2^2 + \|\nabla v_m'(t)\|_2^2}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1}} \\
&\quad \times (\|\nabla u_m(t)\|_2 \|\nabla u_m'(t)\|_2 + \|\nabla v_m(t)\|_2 \|\nabla v_m'(t))\|_2) \\
&\le \frac{\gamma}{2}\frac{\|\nabla u_m'(t)\|_2^2 + \|\nabla v_m'(t)\|_2^2}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma}} \\
&\quad + \frac{\gamma}{2}\frac{(\|\nabla u_m'(t)\|_2^2 + \|\nabla v_m'(t))\|_2^2)^2}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1}} \\
&= \frac{\gamma}{2}\frac{\|\nabla u_m'(t)\|_2^2 + \|\nabla v_m'(t)\|_2^2}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma}} \qquad (4.20) \\
&\quad + \frac{\gamma}{2}\left(\frac{\|\nabla u_m'(t)\|_2^2 + \|\nabla v_m'(t))\|_2^2}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma}}\right)^2 \\
&\quad \times (\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma-1} \\
&\le \frac{\gamma}{2}H(t) + \frac{\gamma}{2}H(t)^2\left(\frac{2(\gamma+1)(q+1)}{q-1-2\gamma}E(u_0,v_0)\right)^{\frac{\gamma-1}{\gamma+1}}.
\end{aligned}$$

Now we shall compute the second term in the right hand side of (4.19). In the case $\frac{N}{N-2} \le q \le \min\{\frac{N+2}{N-2}, \frac{N-2}{[N-4]^+}\}(N \ge 3)$, we also see that

$$\begin{aligned}
&|(\nabla[|u_m(t)|^{q-1}u_m(t)], \nabla u_m'(t))| \\
&\qquad \le q\||u_m(t)|^{q-1}\nabla u_m(t)\|_2 \|\nabla u_m'(t)\|_2 \\
&\qquad \le q\|u_m(t)\|_{(q-1)N}^{q-1} \|\nabla u_m(t)\|_{\frac{2N}{N-2}} \|\nabla u_m'(t)\|_2 \qquad (4.21) \\
&\qquad \le qC\|u_m(t)\|_{(q-1)N}^{q-1} \|\Delta u_m(t)\|_2 \|\nabla u_m'(t)\|_2
\end{aligned}$$

where we have used Hölder's inequality and Sobolev-Poincaré's inequality. We observe from Gagliardo-Nirenberg inequality and Sobolev-Pointcaré's inequality that

$$
\begin{aligned}
\|u_m(t)\|_{(q-1)N}^{q-1} &\le C\|u_m(t)\|_{\frac{2N}{N-2}}^{(q-1)(1-\theta)}\|\Delta u_m(t)\|_2^{(q-1)\theta} \\
&\le C\|\nabla u_m(t)\|_2^{(q-1)(1-\theta)}\|\Delta u_m(t)\|_2^{(q-1)\theta} \\
&\quad \text{with} \quad \theta = \frac{N-2}{2} - \frac{1}{q-1}(<1).
\end{aligned} \tag{4.22}
$$

Thus, (4.19)-(4.22) imply

$$
\begin{aligned}
&|\mu(\nabla[|u_m(t)|^{q-1}u_m(t)], \nabla u_m'(t))| \\
&\le q\mu C\|\nabla u_m(t)\|_2^{(q-1)(1-\theta)}\|\Delta u_m(t)\|_2^{1+(q-1)\theta}\|\nabla u_m'(t)\|_2 \\
&\le \frac{q\mu C}{2}\|\nabla u_m(t)\|_2^{2(q-1)(1-\theta)}\|\Delta u_m(t)\|_2^{2+2(q-1)\theta} \\
&\quad + \frac{q\mu C}{2}\|\nabla u_m'(t)\|_2^2.
\end{aligned} \tag{4.23}
$$

Similarly

$$
\begin{aligned}
&|\mu(\nabla[|v_m(t)|^{q-1}v_m(t)], \nabla v_m'(t))| \\
&\le \frac{q\mu C}{2}\|\nabla v_m(t)\|_2^{2(q-1)(1-\theta)}\|\Delta v_m(t)\|_2^{2+2(q-1)\theta} \\
&\quad + \frac{q\mu C}{2}\|\nabla v_m'(t)\|_2^2.
\end{aligned} \tag{4.24}
$$

From (4.17), (4.19), (4.23) and (4.24), we obtain

(4.25)

$$
\begin{aligned}
I_2(t) &\leq \frac{q\mu C}{2}\frac{\|\nabla u_m(t)\|_2^{2(q-1)(1-\theta)}\|\Delta u_m(t)\|_2^{2+2(q-1)\theta}}{(\|\nabla u_m(t)\|_2^2+\|\nabla v_m(t)\|_2^2)^{\gamma}} \\
&\quad + \frac{q\mu C}{2}\frac{\|\nabla v_m(t)\|_2^{2(q-1)(1-\theta)}\|\Delta v_m(t)\|_2^{2+2(q-1)\theta}}{(\|\nabla u_m(t)\|_2^2+\|\nabla v_m(t)\|_2^2)^{\gamma}} \\
&\quad + \frac{q\mu C}{2}\frac{\|\nabla u_m'(t)\|_2^2+\|\nabla v_m'(t)\|_2^2}{(\|\nabla u_m(t)\|_2^2+\|\nabla v_m(t)\|_2^2)^{\gamma}} \\
&\leq \frac{q\mu C}{2}(\|\nabla u_m(t)\|_2^2+\|\nabla v_m(t)\|_2^2)^{(q-1)(1-\theta)-\gamma} \\
&\quad \times (\|\Delta u_m(t)\|_2^{2+2(q-1)\theta}+\|\Delta v_m(t)\|_2^{2+2(q-1)\theta}) \\
&\quad + \frac{q\mu C}{2}\frac{\|\nabla u_m'(t)\|_2^2+\|\nabla v_m'(t)\|_2^2}{(\|\nabla u_m(t)\|_2^2+\|\nabla v_m(t)\|_2^2)^{\gamma}} \\
&\leq \frac{q\mu C}{2}\left(\frac{2(q+1)(\gamma+1)}{q-1-2\gamma}E(u_0,v_0)\right)^{\frac{(q-1)(1-\theta)-\gamma}{\gamma+1}} \\
&\quad \times (\|\Delta u_m(t)\|_2^2+\|\Delta v_m(t)\|_2^2)^{1+(q-1)\theta} \\
&\quad + \frac{q\mu C}{2}\frac{\|\nabla u_m'(t)\|_2^2+\|\nabla v_m'(t)\|_2^2}{(\|\nabla u_m(t)\|_2^2+\|\nabla v_m(t)\|_2^2)^{\gamma}} \\
&\leq \frac{q\mu C}{2}\left(\frac{2(q+1)(\gamma+1)}{q-1-2\gamma}E(u_0,v_0)\right)^{\frac{(q-1)(1-\theta)-\gamma}{\gamma+1}} H(t)^{1+(q-1)\theta} \\
&\quad + \frac{q\mu C}{2}H(t).
\end{aligned}
$$

Thus (4.19), (4.20) and (4.25) imply

(4.26)

$$
\begin{aligned}
\frac{1}{2}H'(t) &\leq \frac{\gamma}{2}H(t)+\frac{\gamma}{2}H(t)^2\left(\frac{2(\gamma+1)(q+1)}{q-1-2\gamma}E(u_0,v_0)\right)^{\frac{\gamma-1}{\gamma+1}} \\
&\quad + \frac{q\mu C}{2}\left(\frac{2(q+1)(\gamma+1)}{q-1-2\gamma}E(u_0,v_0)\right)^{\frac{(q-1)(1-\theta)-\gamma}{\gamma+1}} H(t)^{1+(q-1)\theta} \\
&\quad + \frac{q\mu C}{2}H(t) \\
&\leq C_1(H(t)+H(t)^{1+(q-1)\theta}+H(t)^2)
\end{aligned}
$$

for some constant $C_1>0$

where we have used

$$\frac{p\delta(|u'_m(t)|^{p-1}\nabla u'_m(t), \nabla u'_m(t)) + p\delta(|v'_m(t)|^{p-1}\nabla v'_m(t), \nabla v'_m(t))}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma} \geq 0.$$

Integrating the inequality (4.26) from 0 to t gives

$$\frac{1}{2}H(t) \leq \frac{1}{2}H(0) + C_1 \int_0^t (H(s) + H(s)^{1+(q-1)\theta} + H(s)^2)ds.$$

Here, we set $g(s) = s + s^{1+(q-1)\theta} + s^2$ on $s \geq 0$. Then we have

$$\frac{1}{2}H(t) \leq \frac{1}{2}H(0) + C_1 \int_0^t g(H(s))ds.$$

Note that $g(s)$ is continuous and nondecreasing on $s \geq 0$. By applying Bihari-Langenhop's inequality (cf. [2]), we get

$$H(t) \leq M_1 \quad \text{for some constant} \quad M_1 > 0.$$

Hence we get

$$\|\Delta u_m(t)\|_2^2 + \|\Delta v_m(t)\|_2^2 \leq M_2 \quad \text{for some constant} \quad M_2 > 0. \tag{4.27}$$

A Priori Estimates III

Finally, by multiplying the equation (4.1) by $u''_m(t)$, we have

$$\begin{aligned}\|u''_m(t)\|_2^2 \leq{}& (\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma \|\Delta u_m(t)\|_2 \|u''_m(t)\|_2 \\ &+ |\delta(|u'_m(t)|^{p-1}u'_m(t), u''_m(t))| + |\mu(|u_m(t)|^{q-1}u_m(t), u''_m(t))|.\end{aligned}$$

Note that

$$\begin{aligned}&\delta(|u'_m(t)|^{p-1}u'_m(t)), u''_m(t)) \\ &\leq \delta \int_\Omega |u'_m(t)|^p |u''_m(t)| dx \\ &\leq \delta \Big(\int_\Omega |u'_m(t)|^{2p} dx\Big)^{\frac{1}{2}} \Big(\int_\Omega |u''_m(t)|^2 dx\Big)^{\frac{1}{2}} \\ &= \delta \|u'_m(t)\|_{2p}^p \|u''_m(t)\|_2\end{aligned}$$

and similarly

$$\mu|u_m(t)|^{q-1}(u_m(t)), u_m''(t)) \le \mu\|u_m(t)\|_{2q}^q \|u_m''(t)\|_2.$$

Now, it follows from (4.17) and the Gagliardo-Nirenberg inequality that

$$\begin{aligned}\|u_m'(t)\|_{2p}^p &\le C_2\|\nabla u_m'(t)\|_2^{p\theta_1}\|u_m'(t)\|_2^{p(1-\theta_1)}\\ &\le C_3\|\nabla u_m'(t)\|_2^{p\theta_1}, \quad \text{with} \quad \theta_1 = \frac{(p-1)N}{2p}\\ \|u_m(t)\|_{2q}^q &\le C_4\|\nabla u_m(t)\|_2^{q\theta_2}\|u_m(t)\|_2^{q(1-\theta_2)}\\ &\le C_5\|\nabla u_m(t)\|_2^{q\theta_2} \quad \text{with} \quad \theta_2 = \frac{(q-1)N}{2q}.\end{aligned}$$

Thus, we get

$$\begin{aligned}\|u_m''(t)\|_2 \le (\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma \|\Delta u_m(t)\|_2\\ + C_2\|\nabla u_m'(t)\|_2^{p\theta_1} + C_4\|\nabla u_m(t)\|_2^{q\theta_2}\\ \le M_3 \quad \text{for some constant} \quad M_3 > 0.\end{aligned} \tag{4.28}$$

By applying a similar method as the one for u_m, we get

$$\|v_m''(t)\|_2 \le M_4 \quad \text{for some constant} \quad M_4 > 0. \tag{4.29}$$

Limiting process

By the above estimates (4.17), (4.27), (4.28) and (4.29), $\{u_m\}, \{v_m\}$ have subsequences still denoted by $\{u_m\}, \{v_m\}$ such that

$$u_m \to u, \quad v_m \to v \quad \text{in} \quad L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)) \quad \text{weak}^*, \tag{4.30}$$

$$u_m' \to u', \quad v_m' \to v' \quad \text{in} \quad L^\infty(0,T; H_0^1(\Omega)) \quad \text{weak}^*, \tag{4.31}$$

$$u_m'' \to u'', \quad v_m'' \to v'' \quad \text{in} \quad L^\infty(0,T; L^2(\Omega)) \quad \text{weak}^*, \tag{4.32}$$

(4.33) $u'_m \to u', \quad v'_m \to v' \quad \text{in} \quad L^{p+1}(0,T;L^{p+1}(\Omega)) \quad \text{weak},$

(4.34)

$$(\|\nabla u_m\|_2^2 + \|\nabla v_m\|_2^2)^\gamma \Delta u_m \to \xi_1 \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \quad \text{weak*},$$
$$(\|\nabla u_m\|_2^2 + \|\nabla v_m\|_2^2)^\gamma \Delta v_m \to \xi_2 \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \quad \text{weak*},$$

(4.35)

$$|u'_m|^{p-1}u'_m \to \phi_1, \quad |v'_m|^{p-1}v'_m \to \phi_2 \quad \text{in} \quad L^{\frac{p+1}{p}}(0,T\times\Omega) \quad \text{weak},$$

(4.36)

$$|u_m|^{q-1}u_m \to \psi_1, \quad |v_m|^{q-1}v_m \to \psi_2 \quad \text{in} \quad L^{\frac{q+1}{q}}(0,T\times\Omega) \quad \text{weak}.$$

It follows from a classical compactness argument (cf. Lions [3]) that

$$|u'_m|^{p-1}u'_m \to |u'|^{p-1}u', \quad |v'_m|^{p-1}v'_m \to |v'|^{p-1}v' \quad \text{in} \quad L^{\frac{p+1}{p}}(0,T\times\Omega) \quad \text{weak},$$
$$|u_m|^{q-1}u_m \to |u|^{q-1}u, \quad |v_m|^{q-1}v_m \to |v|^{q-1}v \quad \text{in} \quad L^{\frac{q+1}{q}}(0,T\times\Omega) \quad \text{weak}.$$

We shall show $\xi_1 = (\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^\gamma \Delta u$, $\xi_2 = (\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^\gamma \Delta v$. For any $w \in C_0(0,\infty;L^2(\Omega))$, the mean value theorem imply

$$\int_0^T ((\|\nabla u_m\|_2^2 + \|\nabla v_m\|_2^2)^\gamma - (\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^\gamma)(\Delta u_m, w)dt$$
$$\le C\int_0^T (\|\nabla u_m - \nabla u\|_2 + \|\nabla v_m - \nabla v\|_2)dt \to 0 \quad \text{as} \quad m\to\infty.$$

Thus we have

$$\begin{aligned}
&\int_0^T (\xi_1 - (\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^\gamma)\Delta u, w)dt \\
&= \int_0^T (\xi_1 - (\|\nabla u_m\|_2^2 + \|\nabla v_m\|_2^2)^\gamma \Delta u_m, w)dt \\
&\quad + \int_0^T (\|\nabla u_m\|_2^2 + \|\nabla v_m\|_2^2)^\gamma)(\Delta u_m - \Delta u, w)dt \\
&\quad + \int_0^T ((\|\nabla u_m\|_2^2 + \|\nabla v_m\|_2^2)^\gamma - (\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^\gamma)(\Delta u, w)dt \\
&\qquad \to 0 \quad \text{as} \quad m\to\infty.
\end{aligned}$$

and hence we conclude $\xi_1 = (\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^\gamma \Delta u$.

Similarly, we have $\xi_2 = (\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^\gamma \Delta v$.

On the other hand, using Aubin-Lions's compactness lemma, we can extract from $\{u_m\}$ and $\{v_m\}$ subsequences still denoted by $\{u_m\}$ and $\{v_m\}$, respectively, such that for each $t \in [0,T]$

$$(4.37) \qquad u_m(t) \to u(t), \quad v_m(t) \to v(t) \quad \text{strongly in} \quad H_0^1(\Omega).$$

By letting $m \to \infty$ in (4.1) and (4.2), we can find that u and v satisfy the equations:

(4.38)

$$(u''(t), w) - ((\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^\gamma \Delta u(t), w) \\ + \delta |u'(t)|^{p-1}(u'(t), w) = \mu |u(t)|^{q-1}(u(t), w) \quad \text{for all} \quad w \in H_0^1(\Omega),$$

(4.39)

$$(v''(t), w) - ((\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^\gamma \Delta v(t), w) \\ + \delta |v'(t)|^{p-1}(v'(t), w) = \mu |v(t)|^{q-1}(v(t), w) \quad \text{for all} \quad w \in H_0^1(\Omega).$$

Now, the above result (4.37) imply

$$(4.40) \qquad u_m(0) = u_{0m} \to u_0 \quad \text{strongly in} \quad H_0^1(\Omega).$$

Thus, from (4.3) and (4.40), $u(0) = u_0$. Also, from (4.31) we obtain

$$(4.41) \quad (u'_m(0) - u'(0), w) \to 0 \quad \text{as} \quad m \to \infty \quad \text{for each} \quad w \in H_0^1(\Omega).$$

Thus, (4.4) and (4.41) imply $u'(0) = u_1$. Similarly, we obtain $v(0) = v_0$ and $v'(0) = v_1$. This completes the proof of Theorem 3.1.

5. Asymptotic behavior of solutions

THEOREM 5.1. *Let $u(t), v(t)$ and q be the same as in* Theorem 3.1. *Assume that either $1 \le p < +\infty$ $(N = 1, 2)$ or $1 \le p \le \frac{N}{N-2}$ $(N \ge 3)$ holds. If $p > \frac{1}{2\gamma+1}$, then we have the decay estimate*

$$E(u(t), v(t)) \le C_1(1+t)^{-\frac{2(\gamma+1)}{(2\gamma+1)p-1}} \quad \text{on} \quad [0, +\infty)$$

where C_1 is a positive constant depending on $\|\nabla u_0\|_2$ and $\|u_1\|_2$.

To prove Theorem 5.1, we need the following Lemma.

LEMMA 5.2. *Let $u(t)$ and q be the same as in* Lemma 4.1. *Then there is a certain number η_0 with $0 < \eta_0 < 1$ such that*

$$\mu(\|u(t)\|_{q+1}^{q+1}+\|v(t)\|_{q+1}^{q+1}) \leq (1-\eta_0)(\|\nabla u(t)\|_2^2+\|\nabla v(t)\|_2^2)^{\gamma+1} \quad on \quad [0,\infty)$$

where

$$\eta_0 \equiv 1 - \mu C(\Omega, q+1)^{q+1}\left(\frac{2(q+1)(\gamma+1)}{q-1-2\gamma}E(u_0, v_0)\right)^{\frac{q-1-2\gamma}{2(\gamma+1)}}.$$

Proof. It follows from the Sobolev-Poincaré inequality and (4.17) that

$$\begin{aligned}\mu\|u(t)\|_{q+1}^{q+1} &\leq \mu C(\Omega, q+1)^{q+1}\|\nabla u(t)\|_2^{q+1} \\ &= \mu C(\Omega, q+1)^{q+1}\|\nabla u(t)\|_2^{q-1-2\gamma}\|\nabla u(t)\|_2^{2(\gamma+1)} \\ &\leq \mu C(\Omega, q+1)^{q+1}\left(\frac{2(q+1)(\gamma+1)}{q-1-2\gamma}E(u_0, v_0)\right)^{\frac{q-1-2\gamma}{2(\gamma+1)}} \\ &\quad \times \|\nabla u(t)\|_2^{2(\gamma+1)}\end{aligned}$$

and

$$\begin{aligned}\mu\|v(t)\|_{q+1}^{q+1} &\leq \mu C(\Omega, q+1)^{q+1}\left(\frac{2(q+1)(\gamma+1)}{q-1-2\gamma}E(u_0, v_0)\right)^{\frac{q-1-2\gamma}{2(\gamma+1)}} \\ &\quad \times \|\nabla v(t)\|_2^{2(\gamma+1)}.\end{aligned}$$

Thus we get

$$\begin{aligned}&\mu\|u(t)\|_{q+1}^{q+1} + \mu\|v(t)\|_{q+1}^{q+1} \\ &\quad \leq \mu C(\Omega, q+1)^{q+1}\left(\frac{2(q+1)(\gamma+1)}{q-1-2\gamma}E(u_0, v_0)\right)^{\frac{q-1-2\gamma}{2(\gamma+1)}} \\ &\qquad \times (\|\nabla u(t)\|_2^{2(\gamma+1)} + \|\nabla v(t)\|_2^{2(\gamma+1)}) \\ &\quad \leq \mu C(\Omega, q+1)^{q+1}\left(\frac{2(q+1)(\gamma+1)}{q-1-2\gamma}E(u_0, v_0)\right)^{\frac{q-1-2\gamma}{2(\gamma+1)}} \\ &\qquad \times (\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^{\gamma+1} \\ &\quad \equiv (1-\eta_0)(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^{\gamma+1} \quad \text{on} \quad [0,\infty).\end{aligned}$$

This completes the proof of Lemma 5.2. □

Proof of Theorem 5.1. For simplicity of notation, let us denote $E(u(t),v(t))$ by $E(t)$ and $E(u_0,v_0)$ by $E(0)$. Let $u(t)$ and $v(t)$ be solutions of the following problems:

$$
\begin{aligned}
&u''(t) - (\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^\gamma \Delta u(t) + \delta|u'(t)|^{p-1}u'(t) \\
&= \mu|u(t)|^{q-1}u(t),
\end{aligned} \tag{5.1}
$$

$$
\begin{aligned}
&v''(t) - (\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^\gamma \Delta v(t) + \delta|v'(t)|^{p-1}v'(t) \\
&= \mu|v(t)|^{q-1}v(t),
\end{aligned} \tag{5.2}
$$

$$
\begin{aligned}
u(0) = u_0, \quad u'(0) = u_1, \\
v(0) = v_0, \quad v'(0) = v_1.
\end{aligned} \tag{5.3}
$$

By multiplying the equation (5.1) by $u'(t)$, multiplying the equation (5.2) by $v'(t)$ and adding these two equations and then integrating over $[t, t+1] \times \Omega$, we get

$$
\begin{aligned}
\delta \int_t^{t+1} (\|u'(s)\|_{p+1}^{p+1} + \|v'(s)\|_{p+1}^{p+1}) ds &= E(t) - E(t+1) \\
&\equiv \delta F(t)^{p+1},
\end{aligned} \tag{5.4}
$$

where

$$
\begin{aligned}
E(t) = \frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{2}\|v'(t)\|_2^2 + \frac{1}{2(\gamma+1)}(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^{\gamma+1} \\
- \frac{\mu}{q+1}(|u(t)|^{q-1}u(t) + |v(t)|^{q-1}v(t)).
\end{aligned}
$$

It follows from Hölder's inequality and (5.4) that

(5.5)

$$
\begin{aligned}
\int_t^{t+1} \|u'(s)\|_2^2 ds &= \int_t^{t+1} \int_\Omega |u'(s)|^2 dx ds \\
&\le m(\Omega)^{\frac{p-1}{p+1}} \int_t^{t+1} \left(\int_\Omega |u'(s)|^{p+1} dx \right)^{\frac{2}{p+1}} ds \\
&\le m(\Omega)^{\frac{p-1}{p+1}} \int_t^{t+1} \|u'(s)\|_{p+1}^2 ds \\
&\le m(\Omega)^{\frac{p-1}{p+1}} \left(\int_t^{t+1} \|u'(s)\|_{p+1}^{p+1} ds \right)^{\frac{2}{p+1}} \left(\int_t^{t+1} ds \right)^{\frac{p-1}{p+1}} \\
&\le m(\Omega)^{\frac{p-1}{p+1}} F(t)^2.
\end{aligned}
$$

Similarly, we obtain

$$\int_t^{t+1} \|v'(s)\|_2^2 ds \le m(\Omega)^{\frac{p-1}{p+1}} F(t)^2. \tag{5.6}$$

Applying the mean value theorem to the left hand sides of (5.5) and (5.6), there exist two points $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u'(t_i)\|_2 \le 2m(\Omega)^{\frac{p-1}{2(p+1)}} F(t) \quad i = 1,2 \tag{5.7}$$

and

$$\|v'(t_i)\|_2 \le 2m(\Omega)^{\frac{p-1}{2(p+1)}} F(t) \quad i = 1,2. \tag{5.8}$$

Next, multiplying (5.1) by $u(t)$, multiplying (5.2) by $v(t)$, adding these two equations and then integrating over $[t_1, t_2] \times \Omega$ give (cf.(5.7), (5.8))
(5.9)

$$\begin{aligned}
&\int_{t_1}^{t_2} I(u(s), v(s)) ds \\
&= \int_{t_1}^{t_2} ((\|\nabla u(s)\|_2^2 + \|\nabla v(s)\|_2^2)^{\gamma+1} - \mu\|u(s)\|_{q+1}^{q+1} - \mu\|v(s)\|_{q+1}^{q+1}) ds \\
&\le \sum_{i=1}^{2} (\|u'(t_i)\|_2 \|u(t_i)\|_2 + \|v'(t_i)\|_2 \|v(t_i)\|_2) + \int_{t_1}^{t_2} (\|u'(s)\|_2^2 + \|v'(s)\|_2^2) ds \\
&\quad + \delta \left| \int_{t_1}^{t_2} |u'(s)|^{p-1}(u'(s), u(s)) ds \right| + \delta \left| \int_{t_1}^{t_2} |v'(s)|^{p-1}(v'(s), v(s)) ds \right| \\
&\le 4m(\Omega)^{\frac{p-1}{2(p+1)}} F(t) (\max_{t_1 \le s \le t_2} \|u(s)\|_2 + \max_{t_1 \le s \le t_2} \|v(s)\|_2) + 2m(\Omega)^{\frac{p-1}{(p+1)}} F(t)^2 \\
&\quad + \delta \int_{t_1}^{t_2} \int_\Omega |u'(s)|^p |u(s)| dx ds + \delta \int_{t_1}^{t_2} \int_\Omega |v'(s)|^p |v(s)| dx ds \\
&\le 8m(\Omega)^{\frac{p-1}{2(p+1)}} F(t) \max_{t_1 \le s \le t_2} E(s)^{\frac{1}{2(\gamma+1)}} + 2m(\Omega)^{\frac{p-1}{(p+1)}} F(t)^2 \\
&\quad + \delta \int_{t_1}^{t_2} \int_\Omega |u'(s)|^p |u(s)| dx ds + \delta \int_{t_1}^{t_2} \int_\Omega |v'(s)|^p |v(s)| dx ds.
\end{aligned}$$

Here we note that

$$
(5.10)\quad
\begin{aligned}
&\delta\int_{t_1}^{t_2}\int_\Omega |u'(s)|^p|u(s)|dxds\\
&\le \delta\int_{t_1}^{t_2}\left(\int_\Omega |u'(s)|^{p+1}dx\right)^{\frac{p}{p+1}}\left(\int_\Omega |u(s)|^{p+1}dx\right)^{\frac{1}{p+1}}ds\\
&= \delta\int_{t_1}^{t_2}\|u'(s)\|_{p+1}^p\|u(s)\|_{p+1}ds\\
&\le \delta C(\Omega,p+1)\int_{t_1}^{t_2}\|u'(s)\|_{p+1}^p\|\nabla u(s)\|_2ds
\end{aligned}
$$

where we have used Hölder's inequality and Sobolev-Poincaré's inequality.

Since $I(u(t),v(t))\ge 0$ on $[0,\infty)$, we see that

$$
(5.11)\quad
\begin{aligned}
E(t) &\ge J(u(t),v(t))\\
&= \frac{1}{q+1}I(u(t),v(t)) + \frac{q-1-2\gamma}{2(q+1)(\gamma+1)}(\|\nabla u(t)\|_2^2+\|\nabla v(t)\|_2^2)^{\gamma+1}\\
&\ge \frac{q-1-2\gamma}{2(q+1)(\gamma+1)}(\|\nabla u(t)\|_2^2+\|\nabla v(t)\|_2^2)^{\gamma+1}.
\end{aligned}
$$

From (5.4), (5.10) and (5.11), we get

$$
(5.12)\quad
\begin{aligned}
&\delta\int_{t_1}^{t_2}\int_\Omega |u'(s)|^p|u(s)|dxds\\
&\le \delta C(\Omega,p+1)\left(\int_{t_1}^{t_2}\|u'(s)\|_{p+1}^{p+1}ds\right)^{\frac{p}{p+1}}\left(\int_{t_1}^{t_2}ds\right)^{\frac{1}{p+1}}\\
&\qquad\times\left(\frac{2(q+1)(\gamma+1)}{q-1-2\gamma}\right)^{\frac{1}{2(\gamma+1)}}\sup_{t_1\le s\le t_2}E(s)^{\frac{1}{2(\gamma+1)}}\\
&\le \delta C(\Omega,p+1)\left(\frac{2(q+1)(\gamma+1)}{q-1-2\gamma}\right)^{\frac{1}{2(\gamma+1)}}F(t)^p\sup_{t_1\le s\le t_2}E(s)^{\frac{1}{2(\gamma+1)}}.
\end{aligned}
$$

Similarly, we have

(5.13)
$$\begin{aligned}&\delta \int_{t_1}^{t_2} \int_{\Omega} |v'(s)|^p |v(s)| dx ds \\ &\leq \delta C(\Omega, p+1) \left(\frac{2(q+1)(\gamma+1)}{q-1-2\gamma} \right)^{\frac{1}{2(\gamma+1)}} F(t)^p \sup_{t_1 \leq s \leq t_2} E(s)^{\frac{1}{2(\gamma+1)}}.\end{aligned}$$

From (5.9), (5.12) and (5.13), we have

(5.14)
$$\begin{aligned}&\int_{t_1}^{t_2} I(u(s), v(s)) ds \\ &\leq 8m(\Omega)^{\frac{p-1}{2(p+1)}} F(t) \max_{t_1 \leq s \leq t_2} E(s)^{\frac{1}{2(\gamma+1)}} + 2m(\Omega)^{\frac{p-1}{p+1}} F(t)^2 \\ &\quad + 2\delta C(\Omega, p+1) \left(\frac{2(q+1)(\gamma+1)}{q-1-2\gamma} \right)^{\frac{1}{2(\gamma+1)}} F(t)^p \sup_{t_1 \leq s \leq t_2} E(s)^{\frac{1}{2(\gamma+1)}}.\end{aligned}$$

On the other hand, from Lemma 5.2 and the definition of $I(u(t), v(t))$, we have

$$\eta_0 (\|\nabla u(t)\|_2^2 + \|\nabla u(t)\|_2^2)^{\gamma+1} \leq I(u(t), v(t)). \tag{5.15}$$

From (5.5), (5.6) and (5.15), we see that

$$\begin{aligned}&\int_{t_1}^{t_2} E(s) ds \\ &= \frac{1}{2} \int_{t_1}^{t_2} (\|u'(s)\|_2^2 + \|v'(s)\|_2^2) ds + \int_{t_1}^{t_2} J(u(s), v(s)) ds \\ &= \frac{1}{2} \int_{t_1}^{t_2} (\|u'(s)\|_2^2 + \|v'(s)\|_2^2) ds + \frac{1}{q+1} \int_{t_1}^{t_2} I(u(s), v(s)) ds \\ &\quad + \frac{q-1-2\gamma}{2(q+1)(\gamma+1)} \int_{t_1}^{t_2} (\|\nabla u(s)\|_2^2 + \|\nabla v(s)\|_2^2)^{\gamma+1} ds \\ &\leq m(\Omega)^{\frac{p-1}{p+1}} F(t)^2 \\ &\quad + \left(\frac{1}{q+1} + \frac{q-1-2\gamma}{2\eta_0 (q+1)(\gamma+1)} \right) \int_{t_1}^{t_2} I(u(s), v(s)) ds.\end{aligned} \tag{5.16}$$

From (5.14) and (5.16), we get

(5.17)

$$\begin{aligned}\int_{t_1}^{t_2} E(s)ds \\ &\leq C_1\Big(F(t)\sup_{t\leq s\leq t+1} E(s)^{\frac{1}{2(\gamma+1)}} + F(t)^2 + F(t)^p \sup_{t\leq s\leq t+1} E(s)^{\frac{1}{2(\gamma+1)}}\Big) \\ &\leq C_2(E(t)^{\frac{1}{2(\gamma+1)}}F(t) + F(t)^2 + E(t)^{\frac{1}{2(\gamma+1)}}F(t)^p),\end{aligned}$$

where

$$\begin{aligned}C_1 = \max\{&m(\Omega)^{\frac{p-1}{p+1}}, 8m(\Omega)^{\frac{p-1}{2(p+1)}}\left(\frac{1}{q+1}+\frac{q-1-2\gamma}{2\eta_0(q+1)(\gamma+1)}\right), \\ &2m(\Omega)^{\frac{p-1}{p+1}}\left(\frac{1}{q+1}+\frac{q-1-2\gamma}{2\eta_0(q+1)(\gamma+1)}\right), \\ &2\delta C(\Omega,p+1)\left(\frac{2(q+1)(\gamma+1)}{q-1-2\gamma}\right)^{\frac{1}{2(\gamma+1)}} \\ &\quad\times\left(\frac{1}{q+1}+\frac{q-1-2\gamma}{2\eta_0(q+1)(\gamma+1)}\right)\}\end{aligned}$$

and C_2 is a constant.

Again multiplying (5.1) by $u'(t)$, multiplying (5.2) by $v'(t)$, adding these two equations and integrating over $[t,t_2]\times\Omega$ give

$$E(t) = E(t_2) + \delta\int_t^{t_2}(\|u'(s)\|_{p+1}^{p+1} + \|v'(s)\|_{p+1}^{p+1})ds. \tag{5.18}$$

Since $t_2 - t_1 \geq \frac{1}{2}$, we get

$$\begin{aligned}\int_{t_1}^{t_2} E(s)ds &\geq \int_{t_1}^{t_2} E(t_2)ds \\ &= (t_2-t_1)E(t_2) \\ &\geq \frac{1}{2}E(t_2),\end{aligned}$$

that is,

$$E(t_2) \leq 2\int_{t_1}^{t_2} E(s)ds. \tag{5.19}$$

From (5.4), (5.17), (5.18) and (5.19), we have

$$\begin{aligned}
E(t) &= E(t_2) + \delta \int_t^{t_2} (\|u'(s)\|_{p+1}^{p+1} + \|v'(s)\|_{p+1}^{p+1})ds \\
&\leq 2\int_{t_1}^{t_2} E(s)ds + \delta \int_t^{t_2} (\|u'(s)\|_{p+1}^{p+1} + \|v'(s)\|_{p+1}^{p+1})ds \\
&\leq 2C_2(E(t)^{\frac{1}{2(\gamma+1)}} F(t) + F(t)^2 + E(t)^{\frac{1}{2(\gamma+1)}} F(t)^p) + \delta F(t)^{p+1} \\
&\leq C_3(E(t)^{\frac{1}{2(\gamma+1)}} F(t) + F(t)^2 + E(t)^{\frac{1}{2(\gamma+1)}} F(t)^p + F(t)^{p+1})
\end{aligned}$$

for some constant $C_3 > 0$.
Hence, we obtain

$$E(t) \leq C_4(F(t)^{\frac{2(\gamma+1)}{2\gamma+1}} + F(t)^2 + F(t)^{\frac{2(\gamma+1)p}{2\gamma+1}} + F(t)^{p+1}) \tag{5.20}$$

for some constant $C_4 > 0$.
Note that since $E(t)$ is decreasing and $E(t) \geq 0$ on $[0, \infty)$,

$$\begin{aligned}
\delta F(t)^{p+1} &= E(t) - E(t+1) \\
&\leq E(0).
\end{aligned}$$

Thus, we have

$$F(t) \leq (\frac{1}{\delta}E(0))^{\frac{1}{p+1}}. \tag{5.21}$$

It follows from (5.20) and (5.21) that

$$\begin{aligned}
E(t) &\leq C_4(1 + F(t)^{\frac{2\gamma}{2\gamma+1}} + F(t)^{\frac{2(\gamma+1)(p-1)}{2\gamma+1}} + F(t)^{\frac{(2\gamma+1)p-1}{2\gamma+1}})F(t)^{\frac{2(\gamma+1)}{2\gamma+1}} \\
&\leq C_5\Bigg(1 + (\frac{1}{\delta}E(0))^{\frac{2\gamma}{(2\gamma+1)(p+1)}} + (\frac{1}{\delta}E(0))^{\frac{2(\gamma+1)(p-1)}{(2\gamma+1)(p+1)}} \\
&\qquad + (\frac{1}{\delta}E(0))^{\frac{(2\gamma+1)p-1}{(2\gamma+1)(p+1)}}\Bigg) \times F(t)^{\frac{2(\gamma+1)}{2\gamma+1}} \\
&\equiv C_6(E(0))F(t)^{\frac{2(\gamma+1)}{2\gamma+1}} \quad \text{with} \quad \lim_{E(0)\to 0} C_6(E(0)) = C_7 > 0.
\end{aligned}$$

Thus we get

$$E(t)^{1+\frac{(2\gamma+1)p-1}{2(\gamma+1)}} \le C_6(E(0))^{\frac{(2\gamma+1)(p+1)}{2(\gamma+1)}} F(t)^{p+1} \le \frac{1}{\delta} C_6(E(0))^{\frac{(2\gamma+1)(p+1)}{2(\gamma+1)}} (E(t) - E(t+1)). \tag{5.22}$$

Setting

$$C(E(0)) \equiv \delta C_6(E(0))^{-\frac{(2\gamma+1)(p+1)}{2(\gamma+1)}}$$

and applying Nakao's inequality (cf. Lemma 2.3) to (5.22) yield

$$E(t) \le \left(E(0)^{-\frac{(2\gamma+1)p-1}{2(\gamma+1)}} + \frac{(2\gamma+1)p-1}{2(\gamma+1)} C(E(0))[t-1]^+ \right)^{-\frac{2(\gamma+1)}{(2\gamma+1)p-1}}.$$

This completes the proof of Theorem 5.1. □

References

[1] E. H. Brito, *Nonlinear initial boundary value problems*, Nonlinear analysis, Theory, Methods, Applications, **11** (1987), no. 1, 125-137.

[2] R. Ikehata, *On the existence of global solutions for some nonlinear hyperbolic equations with Neumann conditions*, TRU mathematics **24** (1988), 1-17.

[3] J. L. Lions, *Quelques méthode de Résolution des probléme aux limites Non-linéaire*, Dunod Gauthier-Villars, Paris, 1969.

[4] T. Matsuyama, R. Ikehata, *On global solutions and energy decay for the wave equations of Kirchhoff type with nonlinear damping terms*, J. Mathematical analysis and applications **204** (1996), 729-753.

[5] M. Nakao,, *Asymptotic stability of the bounded or almost periodic solutions of the wave equations with nonlinear damping terms*, J. Mathematical analysis and applications **58** (1977), 336-343.

[6] K. Narasimha, *Nonlinear vibration of an elastic string*, J. Sound Vib. **8** (1968), 134-146.

[7] K. Nishihara, Y. Yamada, *On global solutions of some degenerate quasilinear hyperbolic equation with dissipative damping terms*, Funkcialaj Ekvacioj **33** (1990), 151-159.

[8] K. Ono, *Global existence, decay and blowup of solutions for some mildly degenerate nonlinear Kirchhoff strings*, Journal of differential equations **137** (1997), 273-301.

[9] M. D. Silva Alves, *Variational inequality for a nonlinear model of the oscillations of beams*, Nonlinear analysis, Theory, Methods, Applications, **28** (1997), no. 6, 1101-1108.

Department of Mathematics
Pusan National University
Pusan 609-735, Korea