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PERMANENTS OF DOUBLY STOCHASTIC KITE MATRICES

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ABSTRACT. Let p, q be integers such that $2 \le p,\ q \le n$, and let $D_{p,q}$ denote the matrix obtained from I_n, the identity matrix of order n, by replacing each of the first p columns by an all 1's vector and by replacing each of the first two rows and each of the last $q-2$ rows by an all 1's vector. In this paper the permanent minimization problem over the face, determined by the matrix $D_{p,q}$, of the polytope of all $n \times n$ doubly stochastic matrices is treated.

1. Introduction

Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices. This set is known to be a convex polytope of dimension $n^2 - 2n + 1$ in the Euclidean n^2-space. For an $n \times n$ matrix $A = [a_{ij}]$, the permanent of A, $\operatorname{per} A$, is defined by

$$\operatorname{per} A = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

where S_n stands for the symmetric group on the set $\{1,\ 2,\ \cdots,\ n\}$. For an $n \times n$ matrix A and for $i, j \in \{1, 2, \cdots, n\}$, let $A(i|j)$ denote the matrix obtained from A by deleting row i and column j. A square $(0,1)$-matrix $D = [d_{ij}]$ is said to have *total support* if per $D(i|j) > 0$ for every (i,j) with $d_{ij} > 0$. For an $n \times n$ $(0,1)$-matrix D with total support, let

$$\Omega(D) = \{X \in \Omega_n | X \le D\},$$

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where $X \leq D$ means that every entry of X is less than or equal to the corresponding entry of D. Then $\Omega(D)$ forms a face of Ω_n and every face of Ω_n is defined in this fashion [2]. After the resolution of the van der Waerden conjecture [4, 5, 16], there have been made many efforts to minimize the permanent function over various faces of Ω_n [3, 6, 8, 9, 10, 11, 12, 13, 14]. Let $\mu(D)$ denote the minimum permanent over $\Omega(D)$. A matrix $A \in \Omega(D)$ is called a *minimizing matrix* over $\Omega(D)$ if $\operatorname{per} A = \mu(D)$. The set of all minimizing matrices over $\Omega(D)$ is denoted by $\mathbf{Min}(D)$. In the literature, the problem of determining $\mu(D)$ and $\mathbf{Min}(D)$ is called the *permanent minimization problem* over $\Omega(D)$. The permanent minimization problem for any $(0,1)$-matrix of order n of which $n-2$ of the rows are all 1's vectors has been studied by Minc[11], and the problem for staircase matrices has been investigated by Hwang[8], where a staircase matrix is a $(0,1)$-matrix of the form

$$\begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1k} \\ D_{21} & D_{22} & \cdots & D_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ D_{k1} & D_{k2} & \cdots & D_{kk} \end{bmatrix}$$

with D_{ij} being a zero matrix if $i < j$, and an all 1's matrix if $i \geq j$. Let

$$C_n = \left[\begin{array}{cc|cccccc} 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 0 & 0 & \cdots & 1 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{array}\right] \tag{1}$$

be of order n, and for an integer p with $2 \leq p \leq n-1$, let $C_{n,p}$ denote the matrix obtained from C_n by replacing each of the first p columns by an all 1's vector. The permanent minimization problem for C_n and $C_{n,p}$ has been done by Song [13, 14]. Let n be a fixed positive integer. For positive integers p, q with $p, q \leq n$, let $D_{p,q}$ denote the matrix obtained from C_n by replacing each of the first p columns and each of the last q rows with an all 1's vector. In order to have this replacement make sense we assume that $p, q \geq 2$. We call $D_{p,q}$ a *kite matrix* of type (p, q). Note that the matrices C_n and $C_{n,p}$ are kite matrices of type $(2, 2)$ and $(p, 2)$ respectively. In this paper we deal with the permanent minimization problem over the faces of Ω_n determined by kite matrices. Note that, if $p + q \geq n + 1$, then $D_{p,q}$ is a staircase matrix, and the permanent minimization problem over $\Omega(D_{p,q})$ reduces to the work in [8]. So, we assume that $p + q \leq n$ in the sequel.

2. Preliminaries

In the sequel, let I_n denote the identity matrix of order n and let $J_{m,n}$ denote the $m \times n$ matrix of 1's. The matrix $J_{n,n}$ is denoted by J_n for brevity. An $n \times n$ matrix is called *fully indecomposable* if it does not contain an $s \times (n - s)$ zero submatrix. We start this section with some useful lemmas.

LEMMA 2.1 [7]. *Let $D = [d_{ij}]$ be a fully indecomposable $(0, 1)$-matrix and let $A = [a_{ij}] \in \mathbf{Min}(D)$. Then A is also fully indecomposable, and moreover, for (i, j) with $d_{ij} = 1$, $\operatorname{per} A(i|j) \geq \operatorname{per} A$ with equality if $a_{ij} > 0$.*

LEMMA 2.2 [11]. *Let D and $A = [\,\mathbf{a}_1, \cdots, \mathbf{a}_n]$ be the same as in* Lemma 2.1. *If $\mathbf{d}_1 = \cdots = \mathbf{d}_r$ for some $r \leq n$, then the matrix obtained from A by replacing each of the first r columns by the average of $\mathbf{a}_1, \cdots, \mathbf{a}_r$ is also a matrix in $\mathbf{Min}(D)$. A similar statement holds for rows.*

The following Lemma is a direct consequence of Lemma 2.3 of [10] and we omit the proof.

LEMMA 2.3. *Let $D = [d_{ij}]$ be a $(0, 1)$-matrix with total support of which the first p columns are identical and $d_{11} = \cdots = d_{1,p} =$*

$1, d_{1,p+1} = \cdots = d_{1,n} = 0$. *Then*

$$\mu(D) = \left(\frac{p-1}{p}\right)^{p-1} \mu(D(1|1)).$$

LEMMA 2.4 [13]. *Let C_n be the kite matrix of type $(2,2)$ defined in (1). If $n \geq 6$, then*

$$\mu(C_n) = \frac{2(n-3)(n-4)^{n-4}}{(n-2)^{n-1}}$$

which is attained uniquely at the matrix

$$\frac{1}{n-2}\begin{bmatrix} J_{n-2,2} & (n-4)I_{n-2} \\ O & J_{2,n-2} \end{bmatrix}.$$

3. Minimizing the permanent over $\Omega(D_{p,q})$

We begin with the cases where n is small.

LEMMA 3.1. *Let C_n be the kite matrix of type $(2,2)$.*
(a) If $n = 4$, then

$$\mu(C_4) = (16\alpha^2 - 10\alpha + 3)/14 = 0.10277\cdots \tag{2}$$

where $\alpha = 0.30343\cdots$ is the the unique real root of the polynomial equation

$$28x^3 - 24x^2 + 8x - 1 = 0, \tag{3}$$

and the minimum value is attained uniquely at the matrix

$$\begin{bmatrix} \alpha & \alpha & \beta & 0 \\ \alpha & \alpha & 0 & \beta \\ \gamma & \gamma & \alpha & \alpha \\ \gamma & \gamma & \alpha & \alpha \end{bmatrix}, \tag{4}$$

where $\beta = 1 - 2\alpha$ and $\gamma = (1-2\alpha)/2$.

(*b*) *If* $n = 5$, *then*

$$\mu(C_5) = (2650\alpha^2 + 147\alpha - 9)/121 = 0.04781\cdots \tag{5}$$

where $\alpha = 0.29513\cdots$ *is the unique real root of the polynomial equation*

$$44x^3 - 16x^2 + 9x - 1 = 0, \tag{6}$$

and the minimum value is attained uniquely at the matrix

$$\begin{bmatrix} \alpha & \alpha & \beta & 0 & 0 \\ \alpha & \alpha & 0 & \beta & 0 \\ \alpha & \alpha & 0 & 0 & \beta \\ \gamma & \gamma & \alpha & \alpha & \alpha \\ \gamma & \gamma & \alpha & \alpha & \alpha \end{bmatrix}, \tag{7}$$

where $\beta = 1 - 2\alpha$ *and* $\gamma = (1 - 3\alpha)/2$.

Proof. (*a*) was proved in [11].
(*b*) It is proved in [12] that

$$\mu(C_5) = (1 - 2\alpha)^2(1 - 5\alpha + 12\alpha^2)/2 \tag{8}$$

where α is the unique real root of the polynomial equation (6) which is attained uniquely at the matrix in (7). The expression (5) is just a simplification of (8) taking account of (6). □

We now discuss our main problem of minimizing the permanent over the face $\Omega(D_{p,q})$ of Ω_n. Recall that the integers p, q are restricted to satisfy $p + q \leq n$ and $p, q \geq 2$.

THEOREM 3.2. *Let* $p + q \leq n$ *and* $p, q \geq 2$. *Then*

$$\mu(D_{p,q}) = \frac{4(p-1)!(q-1)!}{p^{p-1}q^{q-1}} f(p, q)$$

where

$$f(p,q) = \begin{cases} \frac{1}{14}(16\alpha^2 - 10\alpha + 3), & \text{if } p+q=n, \\ \frac{1}{121}(2650\gamma^2 + 147\gamma - 9), & \text{if } p+q=n-1, \\ \frac{2(m-1)(m-2)^{m-2}}{m^{m+1}}, & \text{if } p+q \le n-2, \end{cases}$$

with α *and* γ *being the unique real roots of the polynomial equations* (3) *and* (6) *respectively, and* $m = n - p - q + 2$.

Proof. We prove the theorem by induction on $p+q$. If $p+q=4$, then $p=q=2$ and $D_{p,q} = C_n$. Since

$$\frac{4(p-1)!(q-1)!}{p^{p-1}q^{q-1}} = \frac{4(2-1)!(2-1)!}{2^1 2^1} = 1$$

and $\mu(C_n) = f(2,2)$, our theorem holds for this case, and the induction starts. Suppose that $p+q \ge 5$ and that the theorem holds for $p+q-1$. Without loss of generality we can assume that $p \ge 3$. Let $E = D_{p,q}(1|1)$. Then $E = D_{r,s}$ with $r = p-1$ and $s = q$. By induction hypothsis , we have

$$\mu(E) = \frac{4(r-1)!(s-1)!}{r^{r-1}s^{s-1}} f(r,s).$$

We claim that $f(r,s) = f(p,q)$. This equality is clear for the cases $p+q=n$ or $p+q=n-1$, because $p+q=n$ if and only if $r+s=n-1=$ (order of E) and $p+q=n-1$ if and only if $r+s=n-2=$ (order of E) -1. The equality for the case $p+q \le n-2$ is also evident because $(n-1)-r-s+2 = n-p-q+2 = m$. Since

$$\mu(D_{p,q}) = \left(\frac{p-1}{p}\right)^{p-1} \mu(E),$$

by Lemma 2.3, we finally have

$$\mu(D_{p,q}) = \left(\frac{p-1}{p}\right)^{p-1} \frac{4(p-2)!}{(p-1)^{p-2}} \frac{(q-1)!}{q^{q-1}} f(p,q) = \frac{4(p-1)!(q-1)!}{p^{p-1}q^{q-1}} f(p,q),$$

and the proof is complete. □

THEOREM 3.3. *Let $p+q \leq n$ and $p,q \geq 2$. Then $\Omega(D_{p,q})$ has a unique minimizing matrix A. If $p+q \geq n-1$, then*

$$A = \begin{bmatrix} \frac{1}{p}J_{p-2,p} & O & O \\ \frac{2\alpha}{p}J_{m,p} & (1-2\alpha)I_m & O \\ \frac{2-2k\alpha}{pq}J_{q,p} & \frac{2\alpha}{q}J_{q,m} & \frac{1}{q}J_{q,q-2} \end{bmatrix},$$

where $k=2$ if $p+q=n$, and $k=3$ if $p+q=n-1$, and α is the unique real root of the equation (3) if $p+q=n$, and of the equation (6) if $p+q=n-1$. If $p+q=n-2$, then

$$A = \begin{bmatrix} \frac{1}{p}J_{p-2,p} & O & O \\ \frac{2}{mp}J_{m,p} & \frac{m-2}{m}I_m & O \\ O & \frac{2}{mq}J_{q,m} & \frac{1}{q}J_{q,q-2} \end{bmatrix},$$

where $m=n-p-q+2$.

Proof. Again we use induction on $p+q$. The case $p+q=4$ is done in Lemmas 2.4 and 3.1. So, we let $p+q \geq 5$. We may assume that $p \geq q$ without loss of generality. Then $p \geq 3$. Suppose first that $q=2$. Then $m=n-p$. If $m=2$, then $\Omega(D_{p,2})$ has a unique minimizing matrix by the works in [11]. Since the matrix

$$\begin{bmatrix} \frac{1}{p}J_{p-2,p} & O \\ \frac{2\alpha}{p}J_{2,p} & (1-2\alpha)I_2 \\ \frac{1-2\alpha}{p}J_{2,p} & \alpha J_2 \end{bmatrix},$$

with α being the unique real root of equation (3), has permanent $(16\alpha^2-10\alpha+3)/14$, this matrix is the unique minimizing matrix and our theorem holds for this case. If $m \geq 3$, then the proof reduces to the work in [14]. Finally, suppose that $q \geq 3$. Let $E = D_{p,q}(1|1)$. Then

$E = D_{p-1,q}$. By induction, $\Omega(E)$ has a unique minimizing matrix G. If $p+q \geq n-1$, then $(p-1)+q = (n-1)-1$, and by induction

$$G = \begin{bmatrix} \frac{1}{p-1}J_{p-3,p-1} & O & O \\ \frac{2\alpha}{p-1}J_{k,p-1} & (1-2\alpha)I_k & O \\ \frac{2-2k\alpha}{p-q}J_{q,p-1} & \frac{2\alpha}{q}J_{q,k} & \frac{1}{q}J_{q,q-2} \end{bmatrix},$$

where α and k are the same numbers stated in the theorem. If $p+q \leq n-2$, then $(p-1)+q \leq (n-1)-2$, and by induction

$$G = \begin{bmatrix} \frac{1}{p-1}J_{p-3,p-1} & O & O \\ \frac{2}{mp}J_{m,p-1} & \frac{m-2}{m}I_m & O \\ O & \frac{2}{mq}J_{q,m} & \frac{1}{q}J_{q,q-2} \end{bmatrix},$$

where $m = (n-1)-(p-1)-q+2 = n-p-q+2$. Let $A \in \mathbf{Min}(D_{p,q})$ and let $A_1 = A(\frac{1}{p}J_p \oplus I_{n-p})$. Then by Lemma 2.2, we have $A_1 \in \mathbf{Min}(D_{p,q})$. Let $A_2 = A_1(\frac{p}{p-1}I_p \oplus I_{n-p})$ and let $B = A_2(1|1)$. Then $B \in \Omega(E)$ and

$$\text{per } B = \left(\frac{p}{p-1}\right)^{p-1} \text{per } A_1(1|1) = \left(\frac{p}{p-1}\right)^{p-1} \text{per } A_1 = \left(\frac{p}{p-1}\right)^{p-1} \mu(D_{p,q}).$$

Thus, by Lemma 2.3, we have that per $B = \mu(E)$ and $B \in \mathbf{Min}(E)$, and hence that $B = G$. Therefore A has the form

$$A = \begin{bmatrix} A_{11} & O & O \\ A_{21} & (1-2\alpha)I_k & O \\ A_{31} & \frac{2\alpha}{q}J_{q,k} & \frac{1}{q}J_{q,q-2} \end{bmatrix},$$

if $p+q \geq n-1$, or

$$A = \begin{bmatrix} A_{11} & O & O \\ A_{21} & \frac{m-2}{m} I_m & O \\ Om & \frac{2}{mq} J_{q,m} & \frac{1}{q} J_{q,q-2} \end{bmatrix},$$

if $p + q \leq n - 2$. A similar argument applied to $D_{p,q}(n|n)$ assures that $A_{11} = (1/p)J_{p-2,p}$, $A_{21} = (2\alpha/p)J_{k,p}$ if $p + q \geq n - 1$, and $A_{21} = (2/mp)J_{m,p}$ if $p + q \leq n - 2$. Then $A_{31} = ((2 - 2k\alpha)/pq)J_{q,p}$ automatically, and the proof is complete. □

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References

[1] R. A. Brualdi, *An interesting face of the polytope of doubly stochastic matrices*, Linear and Multilinear Algebra **17** (1985), 5-18.

[2] R. A. Brualdi and P. M. Gibson, *The convex polyhedron of doubly stochastic matrices: I. Applications of the permanent function*, J. Combin. Theory (A) **22** (1977), 194-230.

[3] R. A. Brualdi and B. L. Shader, *Minimum permanents on special faces of the polytope of doubly stochastic matrices*, Linear Algebra Appl. **201** (1994), 103-111.

[4] G. P Egoryĉev, *Reŝnie problemy van-der-Vardena dla permanentov*, Dokl Akad. Nauk SSR **258** (1981), 1041-1044.

[5] D. I. Falikman, *Dokazatel'stvo gipotezy van der Vardena o permanente dvaẑdy stochasticeskoî matricy*, Mat. Zam. **29** (1981), 931-938.

[6] Ismor Fischer and Suk-Geun Hwang, *Certain nonbarycentric cohesive matrices*, Linear Algebra Appl. **239** (1996), 185-200.

[7] T. H. Foregger, *On the minimum value of the permanent of a nearly decomposable doubly stochastic matrix*, Linear Algebra Appl. **32** (1980), 75-85.

[8] Suk-Geun Hwang, *Minimum permanent on faces of staircase type of the polytope of doubly stochastic matrices*, Linear and Multilinear Algebra **18** (1985), 271-306.

[9] ———, *Minimizing the permanent over some faces of the polytope of doubly stochastic matrices*, Linear Algebra Appl. **219** (1995), 195-205.

[10] Suk-Geun Hwang and Sun-Jeong Shin, *A faces of the polytope of doubly stochastic matrices associated with certain matrix expansions*, Linear Algebra Appl. **253** (1997), 125-140.

[11] H. Minc, *Minimum permanents of doubly stochastic matrices with prescribed zero entries*, Linear and Multilinear Algebra **15** (1984), 225-243.
[12] Seok-Zun Song, *Minimum permanents on certain faces of matrics containing an identity submatrix*, Linear Algebra Appl. **108** (1988), 263-280.
[13] ———, *Minimum permanents on certain doubly stochastic matrices*, Linear Algebra Appl. **143** (1991), 49-56.
[14] ———, *Minimum permanents on certain doubly stochastic matrices II*, Linear Algebra Appl. **161** (1992), 265-277.
[15] R. Sinkhorn and P. Knopp, *Concerning nonnegative matrices and doubly stochastic matrices*, Pacific J. Math **21** (1967), 343-348.
[16] B. L. Van der Waerden, *Aufgabe 45*, Jahresber. Deutch. Math.-Verein **35** (1926), 117.

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