J. Korean Math. Soc. **35** (1998), No. 2, pp. 399–422

DECIDABILITY AND FINITE DIRECT PRODUCTS

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ABSTRACT. A useful method of proving the finite decidability of an equationally definable class $\mathcal{V}$ of algebras (i.e., *variety*) is to prove the decidability of the class of finite directly indecomposable members of $\mathcal{V}$. The validity of this method relies on the well-known result of Feferman-Vaught: *if a class $\mathcal{K}$ of first-order structures is decidable, then so is the class* $\{\prod_{i<n} \mathbf{A}_i \mid \mathbf{A}_i \in \mathcal{K}\ (i < n),\ n \in \omega\}$. In this paper we show that the converse of this does not necessarily hold.

1. Introduction

For a class $\mathcal{K}$ of algebras of the same type, we let $\mathcal{K}_{\text{fin}}$ be the class of the finite members of $\mathcal{K}$, and let $\mathcal{K}_{\text{DI}}$ be the class of directly indecomposable members of $\mathcal{K}$. Recall that an algebra $\mathbf{A}$ is *directly indecomposable* iff it has at least two elements and whenever $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$, then either $\mathbf{B}$ or $\mathbf{C}$ is the trivial single element algebra.

$\mathcal{K}$ is said to be *decidable* (resp. *finitely decidable*) if there exists an algorithm to determine whether a given first-order sentence in the language of $\mathcal{K}$ has a model (resp. finite model) in $\mathcal{K}$.

The result presented in this paper grew out of an attempt to answer the following question posed by Stanley Burris:[1]

> If $\mathcal{K} = \mathcal{V}_{\text{fin}}$ for some (locally finite) variety $\mathcal{V}$, then is $\mathcal{K}_{\text{DI}}$ necessarily decidable whenever $\mathcal{K}$ is decidable?

One of the useful methods of proving the finite decidability of a variety (i.e., the decidability of the first-order theory of $\mathcal{V}_{\text{fin}}$) is to prove the

Received December 30, 1997.

1991 Mathematics Subject Classification: 03B25, 08A60.

Key words and phrases: decidability, direct products, Feferman-Vaught theorem.

The author wishes to acknowledge the financial support of the Korea Research Foundation made in the program year of 1997.

[1] International Workshop on Decidable Varieties, University of Waterloo, May, 1991.

decidability of the class of finite directly indecomposable members of $\mathcal{V}$ [1]. Whether this method always works (in principle) is equivalent to the Burris' problem.

For a class $\mathcal{D}$ of first-order structures in the same language, we use the notation

$$\mathsf{P}_{\text{fin}}(\mathcal{D}) \stackrel{\text{def}}{=} \{\prod_{i<n} \mathbf{A}_i \mid \mathbf{A}_i \in \mathcal{D}\ (i<n),\ n \in \omega\}.$$

Then the Feferman-Vaught theorem [2] implies

(1) $\mathcal{D}$ is decidable $\Rightarrow$ $\mathsf{P}_{\text{fin}}(\mathcal{D})$ is decidable.

Let us say that a class $\mathcal{K}$ of algebras is *closed under direct factors* iff whenever $\mathbf{B}\times\mathbf{C} \cong \mathbf{A} \in \mathcal{K}$ then $\mathbf{B}, \mathbf{C} \in \mathcal{K}$. If $\mathcal{K}$ is a class of finite algebras closed under finite direct products and direct factors, then $\mathcal{K} = \mathsf{P}_{\text{fin}}(\mathcal{K}_{\text{DI}})$ up to isomorphism since a finite algebra is always isomorphic to a direct product of some directly indecomposable ones. Thus, in this case, (1) implies

(2) $\mathcal{K}_{\text{DI}}$ is decidable $\Rightarrow$ $\mathcal{K}$ is decidable.

In this paper we construct a counterexample $\mathcal{K}$ for the converse of (2). Then $\mathcal{D} \stackrel{\text{def}}{=} \mathcal{K}_{\text{DI}}$ will be a counterexample for the converse of (1).

Throughout we use the following notation and terminology: $\approx$ is the formal symbol for equality. For a set X, $|X|$ means the cardinality of X. By a *stalk* in a direct product $\prod_{i\in I} \mathbf{A}_i$, we mean each factor algebra $\mathbf{A}_i$. If Φ is a finite set of first-order L-formulas where L is some fixed first-order language, then $\bigwedge \Phi$ (resp. $\bigvee \Phi$) is uniquely determined: i.e., the order of conjuncts (resp. disjuncts) is determined by some fixed enumeration of all L-formulas. If Φ is any set of L-formulas, then $\bigwedge_{\text{fin}} \Phi$ (resp. $\bigvee_{\text{fin}} \Phi$) means $\{\bigwedge \Psi \mid \Psi \subseteq \Phi,\ |\Psi| < \omega\}$ (resp. $\{\bigvee \Psi \mid \Psi \subseteq \Phi,\ |\Psi| < \omega\}$). A finite sequence $\langle a_1, \dots, a_n\rangle$ is usually abbreviated as $\bar{a}$. If each entry a_i of this sequence is itself a sequence $a_i : \alpha \to X_i$ with a common domain α and a codomain X_i, then by abuse of notation we may write $\bar{a}(j)$, for each $j \in \alpha$, to mean $\langle a_1(j), \dots, a_n(j)\rangle \in \prod_{i=1}^n X_i$. Sometimes a sequence starts with index 0: e.g., $\bar{x} = \langle x_0, \dots, x_{n-1}\rangle$. When we introduce a formula $\tau(x_1, \dots, x_m)$, τ is assumed to have free variables among $x_1, \dots, x_m$ that are pairwise distinct, unless specified otherwise.

We agree that the product of the empty sequence of natural numbers is 1, and also agree that the direct product of the empty sequence of algebras is the trivial algebra consisting of a single element universe.

2. The Construction

Our language L consists of $\{\mathtt{f}, \mathtt{0}, \mathtt{a}, \mathtt{b}, \mathtt{c}\}$, where $\mathtt{f}$ is a unary function symbol and the rest are constant symbols. For each variable x and for each $n > 0$, let $\mathsf{order}(x) \approx n$ be an abbreviation for the L-formula

$$(\mathtt{f}^n(x) \approx x) \wedge \bigwedge_{0<i<n} \mathtt{f}^i(x) \not\approx x.$$

For an L-structure $\mathbf{A}$ and $u, v \in A$, let $u - v$ mean the least nonnegative integer n such that $f^n(v) = u$ if such an n exists and ω otherwise, and let $|u - v|$ mean the smaller one among $u - v$ and $v - u$.

We let order(u) mean the natural number $n > 0$ such that $\mathbf{A} \models \mathsf{order}(u) \approx n$ if such an n exists and ω otherwise. We may say "order of u" instead of order(u) whenever it is convenient.

DEFINITION 1. We let

$$\mathcal{K} \stackrel{\text{def}}{=} \mathsf{P}_{\text{fin}}(\mathcal{D}),$$

where $\mathcal{D}$ is the class of all L-structures $\mathbf{A} \stackrel{\text{def}}{=} \langle A, f, 0, a, b, c\rangle$ such that

(a) A is finite and its cardinality is $p + 1$ for some prime number p.
(b) f is a bijection.
(c) $f(0) = 0$.
(d) If $|A| = p + 1$ with $p \in \omega$, then order of x is p for all $x \in A - \{0\}$.
(e) If $a \neq 0$, $b \neq 0$ and $c \neq 0$, then $|a - c| = n$ implies $|b - c| = h(n)$ for all $n \in \omega$ where $h : \omega \longrightarrow \omega$ is some fixed recursive function (not depending on $\mathbf{A}$) such that the range of h is nonrecursive.[2]

By a *cycle* in $\mathbf{A} \in \mathcal{K}$, we mean a nonempty subset $C \subseteq A$ such that for all $x, y \in C$ there exists $n \geq 0$ such that $f^n(x) = y$.

For each $k \in \omega$, let τ_k be a fixed L-sentence saying that

$$(a \neq 0) \wedge (b \neq 0) \wedge (c \neq 0) \wedge (|b - c| = k). \tag{3}$$

Throughout, we fix the classes $\mathcal{K}$ and $\mathcal{D}$, and the map $h : \omega \longrightarrow \omega$.

Note that in each member $\mathbf{A}$ of $\mathcal{K}$, f is a bijection satisfying $\big(f(x) = x\big) \Leftrightarrow (x = 0^{\mathbf{A}})$, and the universe A is a disjoint union of finitely many finite cycles. Among these cycles exactly one is a singleton, namely $\{0^{\mathbf{A}}\}$. Also note that $u - v = q \in \omega$ implies $f^q(v) = u$ but not vice versa.

[2]This construction technic is first found in [3].

PROPOSITION 1. *Each member of $\mathcal{D}$ is directly indecomposable in the variety of all L-structures.*

Proof. Suppose that $\mathbf{B} \times \mathbf{C} \cong \mathbf{A} \in \mathcal{D}$ with $|A| = p + 1$, where $\mathbf{B}$ and $\mathbf{C}$ are L-structures and p is a prime number. Then $0^{\mathbf{A}} = \langle 0^{\mathbf{B}}, 0^{\mathbf{C}} \rangle$ by the definition of direct products. Thus $\mathtt{f}^{\mathbf{B}}(0^{\mathbf{B}}) = 0^{\mathbf{B}}$ and $\mathtt{f}^{\mathbf{C}}(0^{\mathbf{C}}) = 0^{\mathbf{C}}$ follow from $\mathtt{f}^{\mathbf{A}}(0^{\mathbf{A}}) = 0^{\mathbf{A}}$ which holds as $\mathbf{A} \in \mathcal{D}$. Assume, for the purpose of getting a contradiction, that $|C| \geq 2$ and $|B| \geq 2$. Pick $c \in C - \{0^{\mathbf{C}}\}$. Then the order of $\langle 0^{\mathbf{B}}, c \rangle$ in $\mathbf{A}$ is the same as the order of c in $\mathbf{C}$, which is at most $|C|$. On the other hand, the order of $\langle 0^{\mathbf{B}}, c \rangle$ must be p because of definition 1.(d). This gives us a contradiction since

$$\text{order}(\langle 0^{\mathbf{B}}, c \rangle) \leq |C| \leq (p+1)/2 < p = \text{order}(\langle 0^{\mathbf{B}}, c \rangle). \qquad \square$$

PROPOSITION 2. *$\mathcal{D}$ is undecidable.*

Proof. For each $k \in \omega$, τ_k has a model in $\mathcal{D}$ iff k is in the range of h. But the range of h is a nonrecursive subset of ω. $\square$

PROPOSITION 3. *τ_k has a model in $\mathcal{K}$ for all $k \in \omega$.*

Proof. Let $k \in \omega$ be given. We will find $\mathbf{A}_0, \mathbf{A}_1 \in \mathcal{D}$ so that $\mathbf{A}_0 \times \mathbf{A}_1 \models \tau_k$. Pick any prime $p > 2 \cdot k$ and let $A_0 \stackrel{\text{def}}{=} A_1 \stackrel{\text{def}}{=} \{0, \dots, p\}$. Let the constant symbols be interpreted in $\mathbf{A}_0$ and $\mathbf{A}_1$ so that $0^{\mathbf{A}_0 \times \mathbf{A}_1} = \langle 0, 0 \rangle$, $\mathtt{a}^{\mathbf{A}_0 \times \mathbf{A}_1} = \langle 0, 1 \rangle$, $\mathtt{b}^{\mathbf{A}_0 \times \mathbf{A}_1} = \langle 1, 0 \rangle$ and $\mathtt{c}^{\mathbf{A}_0 \times \mathbf{A}_1} = \langle k+1, 0 \rangle$. Finally interpret $\mathtt{f}$ so that

$$\mathtt{f}^{\mathbf{A}_0}(n) = \mathtt{f}^{\mathbf{A}_1}(n) = \begin{cases} n+1 & \text{if } 0 < n < p, \\ 0 & \text{if } n = 0, \\ 1 & \text{if } n = p. \end{cases}$$

Then it is clear that $\mathbf{A}_0, \mathbf{A}_1 \in \mathcal{D}$ and $\mathbf{A}_0 \times \mathbf{A}_1 \models \tau_k$. $\square$

Now we state our main result as follows.

THEOREM 1. (***The Main Result***) *$\mathcal{K} = \mathsf{P}_{\text{fin}}(\mathcal{D})$ is decidable while $\mathcal{D}$ is not. Thus the converse of* (1) *does not hold.*

We devote the rest of this paper in proving above theorem. The decidability proof is basically done by the elimination of quantifier method.

3. Some Lemmas

In this section we work in a sublanguage L^0 of L, whose symbol set is $\{\mathtt{f}, \mathtt{0}\}$. Let $\mathcal{D}^0$ be a class of L^0-structures obtained from $\mathcal{D}$ by simply forgetting the interpretations of a, b and c: i.e., $\mathcal{D}^0 \stackrel{\text{def}}{=} \{\mathbf{A}|_{L^0} \mid \mathbf{A} \in \mathcal{D}\}$ where $\mathbf{A}|_{L^0}$ is the L^0-reduct of $\mathbf{A}$, or equivalently, $\mathcal{D}^0$ is the class of all L^0-structures $\mathbf{A} \stackrel{\text{def}}{=} \langle A, f, 0\rangle$ in which (a)–(d) of definition 1 holds. Similarly we let $\mathcal{K}^0 = \{\mathbf{A}|_{L^0} \mid \mathbf{A} \in \mathcal{K}\}$. It is clear that $\mathcal{K}^0 = \mathsf{P}_{\text{fin}}(\mathcal{D}^0)$.

DEFINITION 2. For each $\mathbf{A} \in \mathcal{D}^0$, the *height* of $\mathbf{A}$ will mean $|A| - 1$. For each $M > 0$, we let

$$\mathcal{K}^0_{\leq M} \stackrel{\text{def}}{=} \{\mathbf{A} \in \mathcal{K}^0 \mid \text{ each stalk of } \mathbf{A} \text{ has height} \leq M\},$$

$$\mathcal{K}^0_{> M} \stackrel{\text{def}}{=} \{\mathbf{A} \in \mathcal{K}^0 \mid \text{ each stalk of } \mathbf{A} \text{ has height} > M\}.$$

We define $\mathcal{K}^0_{<M}$ and $\mathcal{K}^0_{\geq M}$ in the obvious way.

Observe that for each prime p and for each $n \geq 0$, the number of stalks of $\mathbf{A} \in \mathcal{K}^0$ with height p is n if and only if there are exactly $(p+1)^n - 1$ many elements of A with order p. This is clearly a first-order property, and we will choose a fixed first-order L^0-sentence

$$\mathsf{width}_p \approx n$$

saying this. We also use $\mathsf{width}_p \leq n$ as an abbreviation of the sentence $\mathsf{width}_p \approx 0 \vee \cdots \vee \mathsf{width}_p \approx n$. We define $\mathsf{width}_p < n$, $\mathsf{width}_p \geq n$ and $\mathsf{width}_p > n$ in the obvious way.

We state some useful facts without proof.

LEMMA 1. *Let* $\mathbf{A} \stackrel{\text{def}}{=} \prod_{i \in I} \mathbf{A}_i \in \mathcal{K}^0$ *where each* $\mathbf{A}_i \in \mathcal{D}^0$.

(a) *Given* $a \in \mathbf{A}$, *if we enumerate the elements of the set*

$$\{p \mid p \text{ prime, height of } \mathbf{A}_i \text{ is } p,\ a(i) \neq 0^{\mathbf{A}_i},\ i \in I\}$$

as $p_1, \ldots, p_n$, *then the order of* a *in* $\mathbf{A}$ *is* $\prod_{i=1}^n p_i$. *In particular, order*(a) *is not a multiple of* p^2 *for any prime* p.

(b) *Let* $f = \mathtt{f}^{\mathbf{A}}$ *and let* $u, v \in A$. *Then, for the following three statements* (i), (ii) *and* (iii), *the implications* $(i) \Rightarrow (ii) \Rightarrow (iii)$ *hold.*

(i) $(\exists q \in \omega)\big(f^q(u) = v\big)$

(ii) $(\forall i \in I)\big(u(i) = 0^{\mathbf{A}_i} \Leftrightarrow v(i) = 0^{\mathbf{A}_i}\big)$

(iii) $\mathbf{A} \models \mathsf{order}(u) \approx \mathsf{order}(v)$

Proof. Easy. □

DEFINITION 3. Let $\mathsf{VarCon} \stackrel{\text{def}}{=} \{0, v_0, v_1, v_2 \dots\}$ be the set of all constants and variables in the language L^0. Then we define some sets of L^0-formulas as follows:

$$\begin{aligned}
\Phi_{\text{atom}} &\stackrel{\text{def}}{=} \{\mathtt{f}^q(x) \approx y \mid x, y \in \mathsf{VarCon},\ q \in \omega\},\\
\Phi_{\neg\text{atom}} &\stackrel{\text{def}}{=} \{\neg\varphi \mid \varphi \in \Phi_{\text{atom}}\},\\
\Phi_{\text{order}} &\stackrel{\text{def}}{=} \{\mathsf{order}(x) \approx n \mid x \in \mathsf{VarCon},\ n \in \omega\},\\
\Phi_{\neg\text{order}} &\stackrel{\text{def}}{=} \{\neg\varphi \mid \varphi \in \Phi_{\text{order}}\},\\
\Phi_{\text{width}} &\stackrel{\text{def}}{=} \{\mathsf{width}_p \approx n \mid p \text{ prime},\ n \in \omega\},\\
\Phi_{\neg\text{width}} &\stackrel{\text{def}}{=} \{\neg\varphi \mid \varphi \in \Phi_{\text{width}}\},\\
\Phi_{\text{basic}} &\stackrel{\text{def}}{=} \Phi_{\text{atom}} \cup \Phi_{\neg\text{atom}} \cup \Phi_{\text{order}} \cup \Phi_{\neg\text{order}} \cup \Phi_{\text{width}} \cup \Phi_{\neg\text{width}},\\
\Phi^*_{\text{basic}} &\stackrel{\text{def}}{=} \bigvee_{\text{fin}} \bigwedge_{\text{fin}} \Phi_{\text{basic}}.
\end{aligned}$$

DEFINITION 4. (a) For each formula $\tau \in \Phi^*_{\text{basic}}$, the *level* of τ is a natural number which is defined as follows. First let

$$\Delta_\tau = \{\varphi \in \Phi_{\text{atom}} \cup \Phi_{\neg\text{atom}} \mid \varphi \text{ is a conjunct of a disjunct of } \tau\}$$

and let

$$\text{level}(\tau) = \max\{q \in \omega \mid \mathtt{f}^q(x) \approx y \in \Delta_\tau \text{ or } \mathtt{f}^q(x) \not\approx y \in \Delta_\tau,\ x, y \in \mathsf{VarCon}\} + 1$$

if Δ_τ is nonempty. If $\Delta_\tau = \varnothing$, then just let $\text{level}(\tau) = 0$.

(b) For each $\tau \in \Phi_{\text{basic}}$ and for each $M, n \in \omega$, we will say that τ is *of rank*(M, n) iff for some $x, y \in \{0, v_0, v_1, \dots, v_{n-1}\}$, $k \in \omega$, p prime, one of the following four holds:

(i) $\tau \in \Phi_{\text{atom}} \cup \Phi_{\neg\text{atom}}$ and $\text{level}(\tau) \leq M/(n+1)$.

(ii) $\big(k > 0\big)$ and $\big(\tau = \mathsf{order}(x) \approx k$ or $\tau = \mathsf{order}(x) \not\approx k\big)$ and $\big(p^2$ does not divide k for each prime $p\big)$ and $\big(k = 1$ or each prime divisor of k is $< M\big)$.

(iii) $\big(\tau = \mathsf{width}_p \approx k$ or $\tau = \mathsf{width}_p \not\approx k\big)$ and $p < M$ and $k < (p+1)^n$.

(iv) $\tau = \mathsf{width}_M \approx 0$ or $\tau = \mathsf{width}_M \not\approx 0$.

(c) For each $M, n \in \omega$, we define two finite subsets of L^0-formula as follows:

$$\Phi_{\text{basic}}(M,n) \stackrel{\text{def}}{=} \text{the set of all members of } \Phi_{\text{basic}} \text{ of rank}(M,n),$$

$$\Phi^*_{\text{basic}}(M,n) \stackrel{\text{def}}{=} \bigvee_{\text{fin}} \bigwedge_{\text{fin}} \Phi_{\text{basic}}(M,n).$$

LEMMA 2. *Let M be a prime number and let $n \leq M-1$. If Ψ is a set of L^0-formulas $\alpha(x_1, \dots, x_n) \in \Phi_{\text{atom}} \cup \Phi_{\neg\text{atom}}$ that are of rank(M,n), then for any two members $\mathbf{A}, \mathbf{B} \in \mathcal{K}^0_{\geq M}$, Ψ is satisfiable in $\mathbf{A}$ iff Ψ is satisfiable in $\mathbf{B}$.*

Proof. Let $\mathbf{A}, \mathbf{B} \in \mathcal{K}^0_{\geq M}$ be given, and suppose that $\langle a_1, \dots, a_n\rangle \in A^n$ satisfies $\mathbf{A} \models \alpha(a_1, \dots, a_n)$ for all $\alpha \in \Psi$. We want to find $\langle b_1, \dots, b_n\rangle \in B^n$ such that

$$\mathbf{B} \models \alpha(b_1, \dots, b_n) \text{ for all } \alpha \in \Psi. \tag{4}$$

Let k be the largest integer less than or equal to $M/(n+1)$: that is, if $\mathbf{f}^q(x) \approx y \in \Psi$ or $\mathbf{f}^q(x) \not\approx y \in \Psi$ then $q < k$.

Let $a_0 = 0^{\mathbf{A}}$ and let $U = \{a_0, a_1, \dots, a_n\}$, and define an equivalence relation $\sim$ on the index set $I \stackrel{\text{def}}{=} \{0, 1, \dots, n\}$ of a_i's to be the transitive closure of the binary relation $\{\langle i_1, i_2\rangle \in I^2 \mid |a_{i_1} - a_{i_2}| < k\}$. Let $I_0, \dots, I_{r-1}$ be an enumeration of $I/{\sim}$, and let $s_j = |I_j|$ for each $j < r$. Then let $U_j = \{a_i \mid i \in I_j\}$ for each $j < r$.

Let us reenumerate $a_0, a_1, \dots, a_n$ as

$$a_{(0,1)}, a_{(0,2)}, \dots, a_{(0,s_0)}, a_{(1,1)}, \dots, a_{(1,s_1)}, \dots, a_{(r-1,1)}, \dots, a_{(r-1,s_{r-1})}$$

so that for each $j = 0, \dots, r-1$,

$$U_j = \{a_{(j,1)}, \dots, a_{(j,s_j)}\} \text{ and}$$
$$a_{(j,1)} \preceq a_{(j,2)} \preceq \dots \preceq a_{(j,s_j)},$$

where $\preceq$ is a binary relation on U defined by $u \preceq v \Leftrightarrow f^q(u) = v$ for some $0 \leq q < k$. Without loss of generality we assume that $a_0 = a_{(0,1)}$: that is, $U_0 = \{a_0\}$.

Observe that if we define $\leq$ to be the transitive closure of $\preceq$, then $\leq$ is antisymmetric: that is, $\big(u \leq v \text{ and } v \leq u\big) \Rightarrow u = v$. This is because the universe A of $\mathbf{A} \in \mathcal{K}^0_{\geq M}$ is a disjoint union of finite cycles whose sizes are either 1 or $\geq M \geq k \cdot (n+1)$. The same thing can be said about B, the universe of $\mathbf{B}$. Using this fact we now choose $b_1, \dots, b_n$ that satisfy (4) as follows.

Choose a function $\lambda : \{0, 1, \ldots, n\} \to \omega \times \omega$ so that $a_i = a_{\lambda(i)}$ for all $i = 0, 1, \ldots, n$. First, let $b_{(0,1)} = b_{(0,2)} = \cdots = b_{(0,s_0)} = 0^{\mathbf{B}} \stackrel{\text{def}}{=} b_0$: that is, let $b_i = b_0$ for all i such that $\lambda(i) = (0, t)$ for some $1 \leq t \leq s_0$.

Next, choose any cycle C of $\mathbf{B}$ other than $\{b_0\}$. Let $K > 0$ be the size of C. We must have $K \geq M \geq k \cdot (n+1)$. We are going to choose all b_i's from this cycle C. For convenience, we assume $C = \{1, 2, \ldots, k \cdot (n+1), \ldots, K\}$ and assume $\mathbf{f}^{\mathbf{B}}(n) = n + 1$ for $n < K$ and $\mathbf{f}^{\mathbf{B}}(K) = 1$. There should be no loss of generality in assuming these.

For each $j = 1, \ldots, r-1$, we choose $b_{(j,1)}, \ldots, b_{(j,s_j)}$ from C as follows. For $j = 1$, let

$$b_{(1,t)} = \begin{cases} 1 & \text{if } t = 1, \\ b_{(1,t-1)} + (a_{(1,t)} - a_{(1,t-1)}) & \text{if } 1 < t \leq s_1. \end{cases} \tag{5}$$

For $j > 1$, let

$$b_{(j,t)} = \begin{cases} b_{(j-1,s_{j-1})} + k & \text{if } t = 1, \\ b_{(j,t-1)} + (a_{(j,t)} - a_{(j,t-1)}) & \text{if } 1 < t \leq s_j. \end{cases} \tag{6}$$

We claim that these b_i's satisfy (4).

To prove this claim, suppose that $\alpha(x_1, \ldots, x_n) \in \Psi$ is given. Then $\mathbf{A} \models \alpha(\bar{a})$ by hypotheses. To show $\mathbf{B} \models \alpha(\bar{b})$, we consider two cases.

(CASE1). $\alpha = \mathbf{f}^q(x) \approx y$ for some $x, y \in \{0, x_1, \ldots, x_n\}$ and some $q < k$.

For notational convenience let $x_0 = 0$, and choose $i, i' \in \{0, 1, \ldots, n\}$ so that $x = x_i$ and $y = x_{i'}$. Then $f^q(a_i) = a_{i'}$ should hold as $\mathbf{A} \models \mathbf{f}^q(x_i) \approx x_{i'}$. Thus $\{a_i, a_{i'}\} \subseteq U_j$ for some $j < r$. In other words, $a_i = a_{(j,t)}$ and $a_{i'} = a_{(j,t')}$ for some $t, t' \leq s_j$.

From the construction (5) and (6), it is obvious that $a_{(j,t')} - a_{(j,t)} = q = b_{(j,t')} - b_{(j,t)}$. Thus $f^q(b_i) = b_{i'}$ holds in $\mathbf{B}$, as was desired.

(CASE2). $\alpha = \mathbf{f}^q(x) \not\approx y$ for some $x, y \in \{0, x_1, \ldots, x_n\}$ and some $q < k$.

Choose $i, i' \in \{0, \ldots, n\}$ as in the previous case: i.e., $f^q(a_i) \neq a_{i'}$. Two subcases arise depending on whether $a_i \sim a_{i'}$. In case $a_i \sim a_{i'}$, $b_{(j,t')} - b_{(j,t)} = a_{(j,t')} - a_{(j,t)} \neq q$, and hence $f^q(b_i) \neq b_{i'}$ in $\mathbf{B}$ as was desired. In case $a_i \not\sim a_{i'}$, we are forced to have $b_i = b_{(j,t)}$ and $b_{i'} = b_{(j',t')}$ for some $j \neq j'$. Thus

$b_{i'} - b_i \geq k > q$ from the construction (5) and (6), which implies $\mathbf{B} \models \mathsf{f}^q(b_i) \not\approx b_{i'}$. □

DEFINITION 5. For each $M \geq 2$ and $n \geq 0$, we define a finite (up to isomorphism) subset $\mathcal{K}^0_{\leq M,n}$ of $\mathcal{K}^0$ as follows:

$$\mathcal{K}^0_{\leq M,n} \stackrel{\text{def}}{=} \{\mathbf{A} \in \mathcal{K}^0_{\leq M} \mid \mathbf{A} \models \mathsf{width}_p \leq (p+1)^n \text{ for all prime } p < M, \text{ and } \mathbf{A} \models \mathsf{width}_M \leq 1\}.$$

LEMMA 3. *Let M be a prime and let $n \geq m \geq 0$. Then given $\mathbf{A} \in \mathcal{K}^0$ and $\bar{a} \in A^m$, there exist $\mathbf{A}' \in \mathcal{K}^0_{\leq M,n}$ and $\bar{a}' \in A'^m$ such that*

(*a*) *Whenever $\tau(x_0, \ldots, x_{m-1}) \in \Phi^*_{\text{basic}}(M,n)$, we have*

$$\mathbf{A} \models \tau(\bar{a}) \Leftrightarrow \mathbf{A}' \models \tau(\bar{a}'). \tag{7}$$

(*b*) *Moreover, if $\tau \in \Phi^*_{\text{basic}}(M,n)$ has the property that every conjunct of every disjunct of τ belongs to $\Phi_{\text{atom}} \cup \Phi_{\neg\text{atom}}$, then $\tau(\bar{a}(i))$ holds in $\mathbf{A}_i$ for every $i \in I$ iff $\tau(\bar{a}'(i'))$ holds in $\mathbf{A}'_{i'}$ for every $i' \in I'$, where $\mathbf{A} = \prod_{i \in I} \mathbf{A}_i$ with each $\mathbf{A}_i \in D^0$ and $\mathbf{A}' = \prod_{i' \in I'} \mathbf{A}'_i$ with each $\mathbf{A}'_i \in D^0$.*

(*c*) *Further, if $m < n$, then for every $a'_m \in A'$ there exists $a_m \in A$ such that for all $\tau(\bar{x}, x_m) \in \Phi^*_{\text{basic}}(M,n)$ the biimplication*

$$\mathbf{A} \models \tau(\bar{a}, a_m) \Leftrightarrow \mathbf{A}' \models \tau(\bar{a}', a'_m) \tag{8}$$

holds.

Proof. Let $M, n, m, \mathbf{A}$ and $\bar{a} \in A^m$ be given as in the hypothesis. Let $\mathbf{A} \stackrel{\text{def}}{=} \prod_{i \in I} \mathbf{A}_i$ with each $\mathbf{A}_i \in \mathcal{D}^0$.

Let $\bar{a} \stackrel{\text{def}}{=} \langle a^0, \ldots, a^{m-1} \rangle$ and let the i-th coordinate of a^j be $a^j(i)$ for each $j < m$, $i \in I$. Recall that by $\bar{a}(i)$ we mean $\langle a^0(i), \ldots, a^{m-1}(i) \rangle$. For each prime p, let

$$I(p) = \{i \in I \mid \text{height of } \mathbf{A}_i = p\}.$$

Observe that for each $k \in \omega$, $\mathbf{A} \models \mathsf{width}_p \approx k$ iff $|I(p)| = k$.

Now to prove item (a) of the lemma, we define a subset $I'(p) \subseteq I(p)$, for each prime $p < M$, as follows. If $I(p) \leq (p+1)^n$, then just let $I'(p) = I(p)$. Otherwise choose $I'(p) \subseteq I(p)$ so that $|I'(p)| = (p+1)^n$ and

$$(\forall i \in I(p))(\exists i' \in I'(p))(\bar{a}(i) = \bar{a}(i')). \tag{9}$$

This is possible because for each p, the set $\{\bar{a}(i) \mid i \in I(p)\}$ has cardinality at most $(p+1)^m$ which is $\leq (p+1)^n$.

Let $I_{<M} = \bigcup\{I(p) \mid p < M,\ p \text{ prime}\}$, and let $I'_{<M} = \bigcup\{I'(p) \mid p < M,\ p \text{ prime}\}$. Let $\mathbf{A}_{<M} = \prod_{i \in I_{<M}} \mathbf{A}_i$ and let $\mathbf{A}'_{<M} = \prod_{i \in I'_{<M}} \mathbf{A}_i$. For each $j < m$, let $a^j_{<M}$ and $a'^j_{<M}$ be the restriction of a^j to $I_{<M}$ and $I'_{<M}$ respectively. Then we claim that, whenever $\tau(\bar{x}) \in \Phi^*_{\text{basic}}(M,n)$, we have

$$\mathbf{A}_{<M} \models \tau(\bar{a}_{<M}) \ \Leftrightarrow\ \mathbf{A}'_{<M} \models \tau(\bar{a}'_{<M}). \tag{10}$$

The proof of this claim is fairly straightforward—it goes as follows. First note that we only have to show (10) for $\tau \in \Phi_{\text{basic}}(M,n) \cap (\Phi_{\text{atom}} \cup \Phi_{\text{order}} \cup \Phi_{\text{width}})$.

Suppose that τ is atomic. Then $\Rightarrow$ of (10) is obvious, for $\bar{a}'_{<M}$ is simply a restriction map of $\bar{a}_{<M}$. To show $\Leftarrow$, we need (9): if the equation τ fails at a stalk $\mathbf{A}_i$ for some prime $p < M$ and $i \in I(p)$, then τ must fail at $\mathbf{A}_{i'}$ where $i' \in I(p)$ is chosen according to (9).

Next suppose that $\tau \in \Phi_{\text{order}}$. We can easily see (10) holds by lemma 1.(a).

Finally suppose that $\tau \stackrel{\text{def}}{=} \mathsf{width}_p \approx k \in \Phi_{\text{width}}$. In this case, (10) reduces to

$$|I(p)| \approx k \ \Leftrightarrow\ |I'(p)| \approx k, \tag{11}$$

where $k < (p+1)^n$. If $I(p) < (p+1)^n$, then $I(p) = I'(p)$ and hence (11) hold immediately. If $I(p) \geq (p+1)^n$, then $I'(p) = (p+1)^n$ and thus both sides of (11) fail: i.e., the biimplication holds. This completes the proof of the claim.

Now, if $I_{<M} = I$ (i.e., $\mathbf{A}_{<M} = \mathbf{A}$), then we will be done by letting $\mathbf{A}' = \mathbf{A}'_{<M}$ and $\bar{a}' = \bar{a}'_{<M}$. So we will assume that $I_{\geq M} \stackrel{\text{def}}{=} I - I_{<M} \neq \varnothing$, and then construct $\mathbf{A}'$ and $\bar{a}'$ that satisfy (7).

First, let Ψ be the set of formulas $\alpha(x_0, \ldots, x_{m-1}) \in \Phi_{\text{basic}}(M,n) \cap (\Phi_{\text{atom}} \cup \Phi_{\neg\text{atom}})$ such that $\mathbf{A}_{\geq M} \models \alpha(\bar{a}_{\geq M})$ where $\mathbf{A}_{\geq M}$ is the restriction of $\mathbf{A}$ on $I_{\geq M}$ and $\bar{a}_{\geq M}$ is the restriction of $\bar{a}$ on $I_{\geq M}$. Then pick any $\mathbf{B}$ from $\mathcal{K}^0_{\leq M,n} \cap \mathcal{K}^0_{\geq M}$ and pick $\bar{b} \in \mathbf{B}^m$ so that $\mathbf{B} \models \alpha(\bar{b})$ for all $\alpha \in \Psi$. This is possible by lemma 2. Note that $\mathbf{B}$ consists of a single stalk of height M.

Then, let $\mathbf{A}'$ be the direct product of $\mathbf{A}_{<M} \times \mathbf{B}$ and let $a'^j = a^j_{<M} \cup b^j$ for each $j < m$. We want to show $\mathbf{A} \models \tau(\bar{a}) \Leftrightarrow \mathbf{A}' \models \tau(\bar{a}')$ for $\tau \in \Phi_{\text{basic}}(M,n) \cap (\Phi_{\text{atom}} \cup \Phi_{\text{order}} \cup \Phi_{\text{width}})$.

First, $\tau \in \Phi_{\text{atom}}$ case is obvious, since $\tau, \neg\tau \in \Psi$ in this case.

Next we consider the case $\tau \stackrel{\text{def}}{=} \mathsf{order}(x) \approx k \in \Phi_{\text{order}}$ where $x \in \{0, x_0, \ldots, x_{m-1}\}$ and k is some natural number such that $k = 1$, or each prime divisor of k is $< M$ and k does not have any prime square factor. Because of this condition imposed on k, if $\langle \mathbf{A}, \bar{a} \rangle \models \mathsf{order}(x_j) \approx k$, then

$$(12) \qquad a^j(i) \text{ is } 0 \text{ for all } i \in I_{\geq M},$$

should hold: that is, $a^j_{\geq M}$ is the constant 0-map for each $j < m$. If we look at the proof of lemma 2 then we can clearly see that b^j is the constant 0-map for each $j < m$. (Actually each b^j is a map on a singleton domain. But this does not matter.) For the converse, the fact that "$\langle \mathbf{A}', \bar{a}' \rangle \models \mathsf{order}(x) \approx k$ implies each $a^j_{\geq M}$ is the constant 0-map" is again easily seen in the proof of lemma 2.

Therefore, under the current supposition that $\tau \in \Phi_{\text{order}}$, (7) reduces to (10), which has been proved already.

Finally we consider the case $\tau \stackrel{\text{def}}{=} \mathsf{width}_p \approx k$. Again, by (10) we only have to consider the case when $p = M$. But this case is trivial since $k = 0$ is required from the definition of $\text{rank}(M, n)$ in 4.(*b*). This completes the proof of item (*a*) of this lemma.

Item (*b*) is more or less obvious from the construction of $\langle \mathbf{A}', \bar{a}' \rangle$, and hence the proof will be omitted.

Now it remains to prove item (*c*): i.e., the existence of $a_m \in A$ for the biimplication (8). But the process of obtaining $a_m \in A$ from $a'_m \in A'$ is really a reverse of what are described above already. We will skip the straightforward proof. □

Above lemma says that "*modulo* $\Phi^*_{\text{basic}}(M, n)$" every member of $\mathcal{K}^0$ is elementarily equivalent to some member of $\mathcal{K}^0_{\leq M,n}$, which is only a finite set (up to isomorphism) of finite algebras. The next lemma tries to remove the "modulo" condition.

LEMMA 4. (Elimination of quantifiers) *There exists a recursive map from the set of all L^0-formulas into Φ^*_{basic}, written $\tau(x_1, \ldots, x_n) \mapsto \tau^*(x_1, \ldots, x_n)$, such that*

$$(13) \qquad \mathcal{K}^0 \models \tau \leftrightarrow \tau^*.$$

Proof. We will describe an effective procedure that produces τ^* from τ. First, we write τ in prenex normal form, and let $\tau_0(y_1, \ldots, y_m, x_1, \ldots, x_n)$

be the quantifier-free part of τ in its prenex normal form: that is,

$$\tau = Q_m y_m Q_{m-1} y_{m-1} \cdots Q_1 y_1 \tau_0(y_1, \dots, y_m, x_1, \dots, x_n),$$

where each Q_i $(0 < i \leq m)$ is either $\forall$ or $\exists$. For each $0 < i \leq m$, let

$$\tau_i = Q_i y_i \cdots Q_1 y_1 \tau_0(y_1, \dots, y_m, x_1, \dots, x_n).$$

We will obtain, for each $i = 0, \dots, m$, a formula $\tau_i^* \in \Phi^*_{\text{basic}}(M, n+m)$ such that $\mathcal{K}^0 \models \tau_i \leftrightarrow \tau_i^*$, where M is the least prime $> 2^m(n+m+1) \cdot \text{level}(\tau_0)$. Then, at $i = m$, we will be done by letting $\tau^* = \tau_m^*$.

To be precise, we will construct a finite sequence $\langle \tau_i^* \mid 0 \leq i \leq m \rangle$ of formulas in Φ^*_{basic} with the following properties for every $i \leq m$:

(a) $\mathcal{K}^0 \models \tau_i^* \leftrightarrow \tau_i$,
(b) $\tau_i^* \in \Phi^*_{\text{basic}}(M, n+m)$,
(c) $\text{level}(\tau_i^*) \leq 2^i \cdot \text{level}(\tau_0)$,
(d) $\{\text{free variables of } \tau_i^*\} \subseteq \{y_{i+1}, \dots, y_m, x_1, \dots, x_m\}$.

The construction of this sequence $\langle \tau_i^* \mid 0 \leq i \leq m \rangle$ is done by induction on $i = 0, \dots, m$.

First we define τ_0^* as follows: for each $\mathbf{B} \in \mathcal{K}^0_{\leq M, n+m}$ and for each $\bar{b} \in B^{n+m}$, we let $\psi_{\mathbf{B},\bar{b}}$ be the conjunction of all formulas $\alpha(y_1, \dots, y_m, x_1, \dots, x_n)$ such that $\mathbf{B} \models \alpha(\bar{b})$ and moreover one of the following two holds:

(i) $\alpha \in \Phi_{\text{atom}} \cup \Phi_{\neg\text{atom}}$ and $\text{level}(\alpha) \leq \text{level}(\tau_0)$,
(ii) $\alpha \in \Phi_{\text{order}} \cup \Phi_{\neg\text{order}} \cup \Phi_{\text{width}} \cup \Phi_{\neg\text{width}}$ and $\alpha \in \Phi_{\text{basic}}(M, n+m)$.

Note that

$$\mathbf{B} \models \psi_{\mathbf{B},\bar{b}}(\bar{b}) \tag{14}$$

holds obviously.

Henceforth we will assume without loss of generality that $\mathcal{K}^0_{\leq M,n}$ is a finite set (of finite algebras) for all M and n. We claim that

$$\tau_0^* \stackrel{\text{def}}{=} \bigvee \{\psi_{\mathbf{B},\bar{b}} \mid \mathbf{B} \models \tau_0(\bar{b}),\ \mathbf{B} \in \mathcal{K}^0_{\leq M, n+m}\ \bar{b} \in B^{n+m}\} \tag{15}$$

satisfies the 4 properties (a) – (d) as desired.[3]

[3]Of course it should have been easier if we just took τ_0^* to be the disjunctive normal form for τ_0. Nevertheless we stick to this construction because certain properties of τ_0^* obtained in this way will be used later, in the proof of theorem 2. We adopt the convention that the disjunction of the empty set of formulas is defined to be the contradiction $0 \not\approx 0$.

Among these four properties, $(b), (c)$ and (d) are more or less obvious. So we will be done by showing (a): that is,

$$\mathcal{K}^0 \models \tau_0^*(\bar{x}) \leftrightarrow \tau_0(\bar{x}). \tag{16}$$

To prove $\leftarrow$ of (16), suppose $\mathbf{A} \models \tau_0(\bar{a})$ where $\mathbf{A} \in \mathcal{K}^0$ and $\bar{a} \in A^{n+m}$. Then $\psi_{\mathbf{A},\bar{a}}$ must be a disjunct of τ^*, and hence $\mathbf{A} \models \tau^*(\bar{a})$ as desired.

To prove the converse $\rightarrow$, let $\mathbf{A} \in \mathcal{K}^0$, $\bar{a} \in A^{n+m}$, and suppose $\mathbf{A} \models \psi_{\mathbf{B},\bar{b}}(\bar{a})$ for some $\mathbf{B} \in \mathcal{K}^0$ and $\bar{b} \in B^{n+m}$ such that $\mathbf{B} \models \tau_0(\bar{b})$. Assume without loss of generality that τ_0 is in disjunctive normal form and let φ be a disjunct of τ_0 such that $\mathbf{B} \models \varphi(\bar{b})$. We may further assume that this disjunct φ is a conjunction of formulas in $\Phi_{\text{atom}} \cup \Phi_{\neg\text{atom}}$. Then since $\mathbf{B} \models \psi_{\mathbf{B},\bar{b}}(\bar{b})$ and $\mathbf{B} \models \varphi(\bar{b})$, every conjunct of φ must also be a conjunct of $\psi_{\mathbf{B},\bar{b}}$. Therefore, from $\mathbf{A} \models \psi_{\mathbf{B},\bar{b}}(\bar{a})$, we get $\mathbf{A} \models \varphi(\bar{a})$ and consequently $\mathbf{A} \models \tau_0(\bar{a})$ as desired. This completes the proof for the initial stage $i = 0$ of our induction.

Before we proceed induction, observe that we can assume without loss of generality that the quantifiers Q_i are all $\exists$'s because Φ^*_{basic} is closed under $\neg$ modulo propositional equivalence, and the level and rank of a formula in Φ^*_{basic} are invariant under $\neg$ and the propositional rearrangements involved.

Suppose that we have obtained τ^*_{i-1} with all the desired properties. To improve readability, we rename the variable y_{m-i+1} as x_{n+i} for each $1 \leq i \leq m$. Then we may write

$$\tau_i(x_1, \dots, x_s) = \exists x_{s+1} \tau_{i-1}(x_1, \dots, x_s, x_{s+1}),$$

where $s = n + m - i$. (Thus $x_{s+1} = y_i$.) Let $\bar{x} = \langle x_1, \dots, x_s \rangle$, $\bar{b} = \langle b_1, \dots, b_s \rangle$ etc. For each $\mathbf{B} \in \mathcal{K}^0_{\leq M, n+m}$ and $\bar{b} \in B^s$, we define $\psi_{\mathbf{B},\bar{b}}$ as the conjunction of $\alpha(\bar{x})$'s as before except that for $\alpha \in \Phi_{\text{atom}} \cup \Phi_{\neg\text{atom}}$ we now require

$$\text{level}(\alpha) \leq 2 \cdot \text{level}(\tau^*_{i-1}).$$

We claim that

(17)
$$\tau_i^* \stackrel{\text{def}}{=} \bigvee \{\psi_{\mathbf{B},\bar{b}}(\bar{x}) \mid \mathbf{B} \models \tau_{i-1}(\bar{b}, b_{s+1}),\ \mathbf{B} \in \mathcal{K}^0_{\leq M, n+m},\ \bar{b} \in B^s,\ b_{s+1} \in B\}$$

satisfies the 4 properties $(a) - (d)$ as desired. Among these four properties, $(b), (c)$ and (d) are more or less obvious. So we will be done by showing (a): that is,

$$\mathcal{K}^0 \models \tau_i^*(\bar{x}) \leftrightarrow \exists x_{s+1}\tau_{i-1}(\bar{x}, x_{s+1}). \tag{18}$$

To prove $\leftarrow$ of (18), suppose that $\mathbf{A} \models \exists x_{s+1}\tau_{i-1}(\bar{a}, x_{s+1})$, where $\mathbf{A} \in \mathcal{K}^0$ and $\bar{a} \in A^s$. Then choose $a_{s+1} \in A$ so that

$$\mathbf{A} \models \tau_{i-1}(\bar{a}, a_{s+1}). \tag{19}$$

By induction hypothesis (a) for $i-1$, (19) reduces to

$$\mathbf{A} \models \tau_{i-1}^*(\bar{a}, a_{s+1}). \tag{20}$$

Now by lemma 3.(a), we choose $\mathbf{A}' \in \mathcal{K}^0_{\leq M,n+m}$ and $\bar{a}' \in A'^s$, $a'_{s+1} \in A'$ so that $\mathbf{A} \models \alpha(\bar{a}', a'_{s+1})$ for all $\alpha(\bar{x}, x_{s+1}) \in \Phi^*_{\text{basic}}(M, n+m)$. Then (20) implies

$$\mathbf{A}' \models \tau_{i-1}^*(\bar{a}', a'_{s+1}), \tag{21}$$

as $\tau_{i-1}^* \in \Phi^*_{\text{basic}}(M, n+m)$ by induction hypothesis (b) for $i-1$. Again by the equivalence of τ_{i-1}^* and τ_{i-1} over $\mathcal{K}^0$, (21) reduces to

$$\mathbf{A}' \models \tau_{i-1}(\bar{a}', a'_{s+1}),$$

which means that $\psi_{\mathbf{A}',\bar{a}'}$ is a disjunct of τ_i^* by definition (17). But $\mathbf{A}' \models \psi_{\mathbf{A}',\bar{a}'}(\bar{a}')$ as we noted earlier in (14). Consequently we get

$$\mathbf{A}' \models \tau_i^*(\bar{a}'). \tag{22}$$

From (22) and our choice of $\langle \mathbf{A}', \bar{a}' \rangle$ and the fact that $\tau_i^*(\bar{x}) \in \Phi^*_{\text{basic}}(M, n+m)$, we get $\mathbf{A} \models \tau_i^*(\bar{a})$ as desired.

Proving $\rightarrow$ of (18) needs more work. Suppose that $\mathbf{A} \models \tau_i^*(\bar{a})$ where $\mathbf{A} \in \mathcal{K}^0$ and $\bar{a} \in A^s$. We want to show $\mathbf{A} \models \tau_i(\bar{a})$, or equivalently

$$\mathbf{A} \models \exists x_{s+1}\tau_{i-1}(\bar{a}, x_{s+1}). \tag{23}$$

First choose $\mathbf{A}' \in \mathcal{K}^0_{\leq M,n+m}$ and $\bar{a}' \in A'^s$ corresponding to $\mathbf{A}$ and $\bar{a} \in A^s$ as in lemma 3.(a). Then, among other things, we have $\mathbf{A}' \models \tau_i^*(\bar{a}')$ as $\mathbf{A} \models \tau_i^*(\bar{a})$ by supposition. Then by the definition (17) of τ_i^*, there exist $\mathbf{B} \in \mathcal{K}^0_{\leq M,n+m}$ and $\bar{b} \in B^s$, $b_{s+1} \in B$ such that $\mathbf{A}' \models \psi_{\mathbf{B},\bar{b}}(\bar{a}')$ and

$$\mathbf{B} \models \tau_{i-1}(\bar{b}, b_{s+1}),$$

or equivalently

$$\mathbf{B} \models \tau_{i-1}^*(\bar{b}, b_{s+1}). \tag{24}$$

Note that $\mathbf{B}$ is isomorphic to $\mathbf{A}'$ since the formula $\psi_{\mathbf{B},\bar{b}}$, which holds in both $\mathbf{A}'$ and $\mathbf{B}$, contains all the information on the number of stalks of height $\leq M$. The detail of this argument is shown in lemma 5.

We want to show that there exists $a'_{s+1} \in A'$ such that

(25) $$\mathbf{A}' \models \tau^*_{i-1}(\bar{a}', a'_{s+1}).$$

Assuming this we can choose $a_{s+1} \in A$ that corresponds to $a'_{s+1} \in A'$ by lemma 3.(c). Then (25) would imply $\mathbf{A} \models \tau^*_{i-1}(\bar{a}, a_{s+1})$, which in turn implies $\mathbf{A} \models \tau_{i-1}(\bar{a}, a_{s+1})$ by induction hypothesis, and consequently (23) should follow.

We will complete the proof of this lemma by finding $a'_{s+1} \in A'$ that satisfies (25). First note that we have the following two satisfaction relations:

(26) $$\mathbf{A}' \models \psi_{\mathbf{B},\bar{b}}(\bar{a}') \quad \text{and} \quad \mathbf{B} \models \psi_{\mathbf{B},\bar{b}}(\bar{b}),$$

which imply that $\langle \mathbf{A}', \bar{a}' \rangle$ and $\langle \mathbf{B}, \bar{b} \rangle$ look almost the same—these two need not be isomorphic but they are similar enough that (25) follows from (24) for some suitable $a'_{s+1} \in A'$. The point to observe is that while $\tau^*_{i-1}(\bar{x}, x_{s+1})$ has one more free variable than $\psi_{\mathbf{B},\bar{b}}(\bar{x})$, the latter has higher level for atomic and negated atomic conjuncts.

Let Q be the level of τ^*_{i-1}. Then the level of $\psi_{\mathbf{B},\bar{b}}$ would be $2Q$. Let $b_0 = 0^{\mathbf{B}}$ and let $U_B = \{b_0, b_1, \ldots, b_s\}$. Let $a'_0 = 0^{\mathbf{A}'}$ and let $U_{A'} = \{a'_0, a'_1, \ldots, a'_s\}$. A cycle C in $\mathbf{B}$ will be said to be U_B-*dense* iff it has the property

$$(\forall c \in C)(\exists b \in U_B)(|c - b| < Q).$$

In the other case C will be said to be U_B-*sparse*. Similarly we define the notion of $U_{A'}$-*dense* and $U_{A'}$-*sparse* for cycles in $\mathbf{A}'$. So, in a U_B-cycle C, there exists $c \in C$ such that $|c - b| \geq Q$ for all $b \in U_B$ if and only if C is U_B-sparse. We can say a similar thing for $U_{A'}$-cycles.

For each $k > 0$, the number of cycles with size k in $\mathbf{B}$ is the same as the number of cycles with size k in $\mathbf{A}'$. This is because $\mathbf{B}$ and $\mathbf{A}'$ are isomorphic to each other. In fact, from (26):

> CLAIM: The number of U_B-dense (resp. U_B-sparse) cycles with some fixed size k in $\mathbf{B}$ is the same as the number of $U_{A'}$-dense (resp. $U_{A'}$-sparse) cycles with size k in $\mathbf{A}'$.

The proof of this claim is given at the end of this lemma.

Now the construction of a'_{s+1} is done in cases.

(CASE1). $|b_{s+1} - b_j| \geq Q$ for all $j \leq s$.

In this case b_{s+1} lies in a U_B-sparse cycle, say C. Then, by the CLAIM, there exists a $U_{A'}$-sparse cycle $C' \subseteq A'$ with $|C'| = |C|$. Take any $a'_{s+1} \in C'$ so that $|a'_{s+1} - a'_j| \geq Q$ for all $j \leq s$. Then we want to show that (25) holds for this a'_{s+1}: i.e., $\mathbf{A}' \models \tau^*_{i-1}(\bar{a}', a'_{s+1})$.

Recall that τ^*_{i-1} is a disjunction of conjunctions of formulas in Φ_{basic}. So (24) implies $\mathbf{B} \models \varphi(\bar{b}, b_{s+1})$ for some disjunct φ of τ^*_{i-1}. It suffices to show that $\mathbf{A}' \models \varphi(\bar{a}', a'_{s+1})$. Let $\sigma \in \Phi_{\text{basic}}$ be any conjunct of φ.

If σ is of the form $\mathsf{width}_p \approx k$ or $\mathsf{width}_p \not\approx k$, then it is trivial to see that $\mathbf{A}' \models \sigma$ because σ has no free variables, $\mathbf{B} \models \sigma$ and $\mathbf{B}$ is isomorphic to $\mathbf{A}'$.

Suppose that σ is of the form $\mathsf{order}(x_j) \approx k$ or $\mathsf{order}(x_j) \not\approx k$. For $j = 0, \dots, s$, σ is already in $\psi_{\mathbf{B},\bar{b}}$, and hence $\mathbf{A}' \models \sigma(\bar{a}', a'_{s+1})$ follows from (26). For $j = s+1$ just note that we have chosen a'_{s+1} so that it has the same order as b_{s+1}.

If σ is of the form $\mathbf{f}^q(x_j) \approx x_{j'}$ with $q < Q$, then without loss of generality we may assume that $j = s+1$ or $j' = s+1$, for otherwise σ holds in $\langle \mathbf{B}, \bar{b}, b_{s+1} \rangle$ iff it is a conjunct of $\psi_{\mathbf{B},\bar{b}}$ iff it holds in $\langle \mathbf{A}', \bar{a}', a'_{s+1} \rangle$. Moreover we can exclude the possibility of $j = s+1 = j'$, for otherwise $(\mathbf{f}^{\mathbf{B}})^q(b_{s+1}) = b_{s+1}$ would imply $q = 0$ or q is a multiple of $|C| = |C'|$, and in either case $\sigma = \mathbf{f}^q(x_{s+1}) \approx x_{s+1}$ must hold in $\langle \mathbf{A}', \bar{a}', a'_{s+1} \rangle$.

Under this assumption σ must fail in $\langle \mathbf{B}, \bar{b}, b_{s+1} \rangle$ because we are in (CASE1). Hence this case will not happen.

Finally if σ is of the form $\mathbf{f}^q(x_j) \not\approx x_{j'}$ with $q < Q$, again we may assume without loss of generality that $j = s+1$ or $j' = s+1$ but $j \neq j'$. Then σ must hold in $\langle \mathbf{A}', \bar{a}', a'_{s+1} \rangle$ by our choice of a'_{s+1}.

(CASE2). $|b_{s+1} - b_j| < Q$ for some $j \leq s$.

In this case $b_{s+1} - b_j = q$ or $b_j - b_{s+1} = q$ for some $0 \leq q < Q$. We will only consider the first case

$$b_{s+1} - b_j = q$$

because the second case can be handled by a similar argument.

Let C be the cycle in $\mathbf{B}$ such that $b_j \in C$. Then $\mathsf{order}(x_j) \approx |C|$ must be a conjunct of $\psi_{\mathbf{B},\bar{b}}$, and hence $|C'| = |C|$ where C'

is the cycle in $\mathbf{A}'$ such that $a'_j \in C'$, provided that $|C|$ satisfies the condition that one of $\mathsf{order}(x_j) \approx |C|$ or $\mathsf{order}(x_j) \not\approx |C|$ appears as a conjunct of $\psi_{\mathbf{B},\bar{b}}$. If $|C|$ does not satisfy such a condition, than $|C'|$ does not either. (For the "condition", see definition 4.(b).(ii).)

We let $a'_{s+1} = (\mathbf{f}^{\mathbf{A}'})^q(a'_j)$: that is, we choose $a'_{s+1} \in A'$ so that

$$a'_{s+1} - a'_j = q \tag{27}$$

and want to show that (25) holds for this a'_{s+1}.

As before, let $\sigma \in \Phi_{\text{basic}}$ be any conjunct of φ where φ is a disjunct of τ^*_{i-1} such that $\mathbf{B}' \models \varphi(\bar{b}, b_{s+1})$.

The case when $\sigma \in \Phi_{\text{width}} \cup \Phi_{\neg\text{width}}$ is trivial.

Next case $\sigma \in \Phi_{\text{order}} \cup \Phi_{\neg\text{order}}$ is trivial too because if $\sigma = \mathsf{order}(x_j) \approx k$ or $\sigma = \mathsf{order}(x_j) \not\approx k$, then

$$\text{order}(a'_{s+1}) = \text{order}(a'_j) = \text{order}(b_j) = \text{order}(b_{s+1}).$$

Finally if $\sigma \in \Phi_{\text{atom}} \cup \Phi_{\neg\text{atom}}$, then various cases exist. Among these cases, we will only consider the case when $\sigma = \mathbf{f}^{q'}(x_{s+1}) \approx x_{j'}$ with $j' \neq s+1$, because it is easy and tedious to check all the remaining cases.

In this case the formula $\mathbf{f}^{q+q'}(x_j) \approx x_{j'}$ must be a conjunct of $\psi_{\mathbf{B},\bar{b}}$ because

$$\Big((\mathbf{f}^{\mathbf{B}})^q(b_j) = b_{s+1} \text{ and } (\mathbf{f}^{\mathbf{B}})^{q'}(b_{s+1}) = b_{j'}\Big) \;\Rightarrow\; (\mathbf{f}^{\mathbf{B}})^{q+q'}(b_j) = b_{j'}$$

and $q + q' < 2Q$. Thus $(\mathbf{f}^{\mathbf{A}'})^{q+q'}(a'_j) = a'_{j'}$.

From this and the equality (27), we see that $(\mathbf{f}^{\mathbf{A}'})^{q'}(a'_{s+1}) = a'_{j'}$ must hold: i.e., σ holds in $\langle \mathbf{A}', \bar{a}', a'_{s+1}\rangle$ as was desired.

Now that we have finished the construction of a'_{s+1}, we can complete the proof of this lemma by verifying the CLAIM.

Suppose that C is a U_B-dense cycle with size k in $\mathbf{B}$ and let $J = \{j \leq s \mid b_j \in C\}$. Note that J is nonempty since C is U_B-dense. Let $|J| = r > 0$. Given $j \in J$, the set $\{(\mathbf{f}^{\mathbf{B}})^q(b_j) \mid 0 < q < 2Q\}$ has a nonempty intersection with $\{b_j \mid j \in J\}$, for otherwise there would be no $b \in U_B$ such that $|(\mathbf{f}^{\mathbf{B}})^Q(b_j) - b| < Q$, which contradicts our supposition that C is U_B-dense. Similarly the set $\{(\mathbf{f}^{\mathbf{B}})^{-q}(b_j) \mid 0 < q < 2Q\}$ has a nonempty intersection with $\{b_j \mid j \in J\}$.

Thus we can enumerate the members of J as $j_1, \dots, j_r$ and choose nonnegative integers $q_1, \dots, q_r < 2Q$ so that all the r-equations

$$\mathtt{f}^{q_1}(x_{j_1}) \approx x_{j_2}, \mathtt{f}^{q_2}(x_{j_2}) \approx x_{j_3}, \dots, \mathtt{f}^{q_r}(x_{j_r}) \approx x_{j_1} \tag{28}$$

hold in $\langle \mathbf{B}, \bar{b} \rangle$. Hence all these equations are conjuncts of $\psi_{\mathbf{B},\bar{b}}$, and consequently hold in $\langle \mathbf{A}', \bar{a}' \rangle$.

Moreover all inequations of the form $\mathtt{f}^q(x_j) \not\approx x_{j'}$, where $j \neq j' \in J$, $0 < q < 2Q$ and $\mathtt{f}^q(x_j) \approx x_{j'}$ is not mentioned in (28), should hold in $\langle \mathbf{B}, \bar{b} \rangle$, and also in $\langle \mathbf{A}', \bar{a}' \rangle$.

Therefore all a_j's with $j \in J$ must belong to the same $U_{A'}$-dense cycle, say C', and moreover it is clear that $|C'| = \sum_{\ell=1}^{r} q_\ell = |C|$.

If there is another U_B-dense cycle C_1 in $\mathbf{B}$, then let $J_1 = \{j \leq s \mid b_j \in C_1\}$. J_1 must be nonempty and disjoint from J. Consider the equations and inequations related to J_1 as before, and so on .... It is straightforward to continue this line of argument to show that there is a 1-1 correspondence between the set of U_B-dense cycles with some fixed size k in $\mathbf{B}$ and the set of U_B-dense cycles with size k in $\mathbf{A}'$. This completes the proof of the CLAIM. □

LEMMA 5. *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{K}^0_{\leq M,n}$. *If, for all* $\alpha \in \Phi_{\text{basic}}(M,n) \cap (\Phi_{\text{width}} \cup \Phi_{\neg\text{width}})$

$$\mathbf{A} \models \alpha \Leftrightarrow \mathbf{B} \models \alpha$$

holds, then $\mathbf{A}$ *is isomorphic to* $\mathbf{B}$.

Proof. Let $\mathbf{A} = \prod_{i \in I_A} \mathbf{A}_i$ where each $\mathbf{A}_i \in \mathcal{D}^0$. Then $I_A = \bigcup \{I_A(p) \mid p \leq M,\ p \text{ prime}\}$ where $I_A(p) \stackrel{\text{def}}{=} \{i \in I_A \mid \mathbf{A}_i \text{ has height } p\}$ for each prime $p \leq M$. Similarly we let $\mathbf{B} = \prod_{i \in I_B} \mathbf{B}_i$ and $I_B = \bigcup \{I_B(p) \mid p \leq M,\ p \text{ prime}\}$. First we want to show that $|I_A(p)| = |I_B(p)|$ for each prime $p \leq M$.

Let $|I_A(p)| = k$. We consider two cases, $p < M$ and $p = M$. For the case $p < M$, if $k < (p+1)^n$, then the sentence $\alpha \stackrel{\text{def}}{=} \mathsf{width}_p \approx k$ should hold in $\mathbf{A}$. So it holds in $\mathbf{B}$ too. Thus $|I_B(p)| = k = |I_A(p)|$ as desired. If $k = (p+1)^n$, then for each $m = 1, \dots, (p+1)^n - 1$, $\mathbf{A} \models \mathsf{width}_p \not\approx m$, and the same is true for $\mathbf{B}$. Thus $|I_B(p)| = (p+1)^n = |I_A(p)|$.

For the case $p = M$, we know, from the definition of $\mathcal{K}^0_{\leq M,n}$, that $I_A(M)$ is either a singleton set or an emptyset, and the same is true for $I_B(M)$. So by letting $\alpha \stackrel{\text{def}}{=} \mathsf{width}_M \approx 0$ and/or $\alpha \stackrel{\text{def}}{=} \mathsf{width}_M \not\approx 0$ appropriately, we easily see that $|I_A(p)| = |I_B(p)|$.

Now without loss of generality we let

$$I_A = I_B \stackrel{\text{def}}{=} I,$$

$$A_i = B_i = \{0, 1, \dots, p_i\}, \text{ where } p_i = \text{ height of } \mathbf{A}_i \text{ for each } i \in I,$$

$$0^{\mathbf{A}_i} = 0^{\mathbf{B}_i} = 0,$$

$$\mathtt{f}^{\mathbf{A}_i}(0) = \mathtt{f}^{\mathbf{B}_i}(0) = 0,$$

$$\mathtt{f}^{\mathbf{A}_i}(j) = \mathtt{f}^{\mathbf{B}_i}(j) = \begin{cases} j+1, & \text{if } 1 \le j < p_i \\ 1, & \text{if } j = p_i \end{cases} \text{ for each } i \in I.$$

Then it is clear that the map $\lambda : A \to B$ given by

$$(\lambda(a))(i) = a(i) \text{ for all } a \in A,\ i \in I$$

is an isomorphism from $\mathbf{A}$ onto $\mathbf{B}$. □

4. Proof of the Main Theorem

Now it is easy to see that $\mathcal{K}^0$ is decidable: given an L^0-sentence τ, first obtain τ^* that is equivalent to τ over $\mathcal{K}^0$ as in lemma 4. Then τ has a model in $\mathcal{K}^0$ iff τ^* has a model in $\mathcal{K}^0$. In the process of obtaining τ^* we get integers M and n such that $\tau^* \in \Phi^*_{\text{basic}}(M, n)$. Then by lemma 3.$(a)$, τ^* has a model in $\mathcal{K}^0$ iff τ^* has a model in $\mathcal{K}^0_{\le M,n}$. But $\mathcal{K}^0_{\le M,n}$ is only a finite set of finite algebras in a finite language, and thus decidable. We have shown that $\mathcal{K}^0$ is decidable.

Before we prove the decidability of $\mathcal{K}$, defined at definition 1, we first let $\mathcal{K}^1$ be $\mathcal{K}^0$ in the expanded signature $\{\mathtt{f}, 0, \mathtt{a}, \mathtt{b}, \mathtt{c}\} = L$. (It is obvious that $\mathcal{K}^1 \supseteq \mathcal{K}$. In fact $\mathcal{K}^1$ is identical with $\mathcal{K}$ except that the stalks of each member of $\mathcal{K}^1$ does not have to satisfy the condition (e) of definition 1.) Then $\mathcal{K}^1$ is also decidable because an L-sentence $\varphi(a, b, c)$ has a model in $\mathcal{K}^1$ iff an L^0-sentence $\exists x \exists y \exists z \varphi(x, y, z)$ has a model in $\mathcal{K}^0$, which was shown to be decidable just before.

As a matter of fact we need a little more than just the decidability of $\mathcal{K}^1$. We need all the definitions and lemmas in the previous section except lemma 5, for $\mathcal{K}^1$ instead of $\mathcal{K}^0$. But this is straightforward to verify once we change some definitions as follows.

First, in definition 4.(b).(i), which is shown below,

$$\tau \in \Phi_{\text{atom}} \cup \Phi_{\neg\text{atom}} \text{ and level}(\tau) \le M/(n+1),$$

$M/(n+1)$ should be changed to $M/(n+4)$ due to the fact that L has four constant symbols while L^0 has only one.

Second, in definition 4.(b).(iii), which is shown below,

$$\left(\tau = \mathsf{width}_p \approx k \text{ or } \tau = \mathsf{width}_p \not\approx k\right) \text{ and } p < M \text{ and } k < (p+1)^n,$$

$k < (p+1)^n$ should be changed to $k < (p+1)^{n+3}$. The reason for this change will become clear shortly.

Third, in 5, which defines $\mathcal{K}^0_{\leq M,n}$, $\mathbf{A} \models \mathsf{width}_p \leq (p+1)^n$ should be changed to $\mathbf{A} \models \mathsf{width}_p \leq (p+1)^{n+3}$. Also the corresponding proof in lemma 3, around line (9), $(p+1)^n$ should be changed to $(p+1)^{n+3}$ and $(p+1)^m$ to $(p+1)^{m+3}$. To understand the necessity of these changes, observe that for members of $\mathcal{K}^0$, each stalk $\mathbf{A}_i$ with a designated m-tuple is determined up to isomorphism by $\langle a^0(i), \ldots, a^{m-1}(i)\rangle$, which may have a configuration among $(p+1)^m$ possible ones (where p is the height of $\mathbf{A}_i$), while for members of $\mathcal{K}^1$, each stalk with a designated m-tuple is determined up to isomorphism by $\langle a^0(i), \ldots, a^{m-1}(i), \mathsf{a}^{\mathbf{A}_i}, \mathsf{b}^{\mathbf{A}_i}, \mathsf{c}^{\mathbf{A}_i}\rangle$ which may have a configuration among $(p+1)^{m+3}$ possible ones.

Incidentally lemma 5 may not fully hold for $\mathcal{K}^1$ because those basic formulas mentioned there do not give enough information for a definitive interpretation of the three constant symbols a, b and c.

Now we are ready to prove theorem 1, our main result, which we state again and prove below:

THEOREM 2. (***The Main Result***) *$\mathcal{K} = \mathsf{P}_{\mathrm{fin}}(\mathcal{D})$ is decidable while $\mathcal{D}$ is not. Thus the converse of* (1) *does not hold.*

Proof. Given an L-sentence τ, obtain an L-sentence $\tau^* \in \Phi^*_{\mathrm{basic}}$ that is equivalent over $\mathcal{K}^1 \supseteq \mathcal{K}$. It is enough to describe an effective procedure to determine whether

$$(\star) \qquad \tau^* \text{ has a model in } \mathcal{K}.$$

Note that a model $\mathbf{A}$ of τ^* in $\mathcal{K}^1$ is in $\mathcal{K}$ if and only if for every $k \in \omega$ the following condition is satisfied.

cond_k : every stalk of $\mathbf{A}$ satisfies the sentence σ_k where

$$\sigma_k \stackrel{\mathrm{def}}{=} \left(\mathsf{a} \not\approx 0 \wedge \mathsf{b} \not\approx 0 \wedge \mathsf{c} \not\approx 0 \wedge |\mathsf{a} - \mathsf{c}| \approx k\right) \rightarrow |\mathsf{b} - \mathsf{c}| = h(k).$$

Let us look at the sentence τ^*. Since it has no free variable, it must be a disjunction of formulas of the form $\psi_{\mathbf{B}}$, where $\mathbf{B} \in \mathcal{K}^1_{\leq M,n}$ for some integers M and n. Further, these integers M and n are effectively computed as we have seen in the proof of lemma 4. Since τ^* has a model in $\mathcal{K}$ iff one of its (finitely many) disjuncts has a model in $\mathcal{K}$, we can assume without loss of generality that τ^* has only one disjunct: that is, $\tau^* = \psi_{\mathbf{B}}$.

In carrying out our proof, we will eliminate various situations for which we can effectively determine whether ($\star$) holds. In the end all possible situations will be eliminated and our proof will be completed.

Let us consider the case when $\mathtt{f}^K(\mathtt{a}) \approx \mathtt{c}$ is a conjunct of τ^* where K is some nonnegative integer. In this case, in every stalk of every model of τ^* in $\mathcal{K}^1$, we must have $|a - c| \leq K$. Thus we only have to check cond_k for $k \leq K$ for models $\mathbf{A}$ of τ^* in $\mathcal{K}^1$ in order to determine whether $\mathbf{A} \in \mathcal{K}$. So whether ($\star$) holds can be effectively checked as follows: choose natural numbers M_1 and n_1 so that τ^* as well as all σ_k's (for $k \leq K$) belong to $\Phi^*_{\text{basic}}(M_1, n_1)$. Let us enumerate all the models of τ^* in $\mathcal{K}^1_{\leq M_1,n_1}$ as $\mathbf{A}_1, \dots, \mathbf{A}_r$. For each $i = 1, \dots, r$, check whether cond_k is satisfied in $\mathbf{A}_i$ for $k = 1, \dots, K$. From lemma 3.(b), it is clear that τ^* has a model in $\mathcal{K}$ iff there exists $1 \leq i \leq r$ such that $\mathbf{A}_i$ satisfies cond_k for all $k \leq K$, and the latter condition can be effectively checked because $\mathcal{K}^1_{\leq M_1,n_1}$ is only a finite set of finite algebras in a finite language.

So we will assume that a formula of the form $\mathtt{f}^q(\mathtt{a}) \approx \mathtt{c}$ does not occur as a conjunct of τ^*. Similarly we can assume that τ^* has no conjunct of the form $\mathtt{f}^q(\mathtt{c}) \approx \mathtt{a}$. Further, we can exclude the situation when τ^* has two conjuncts, one is of the form $\mathtt{f}^q(\mathtt{a}) \approx \mathtt{b}$ or $\mathtt{f}^q(\mathtt{b}) \approx \mathtt{a}$ and the other is of the form $\mathtt{f}^{q'}(\mathtt{c}) \approx \mathtt{b}$ or $\mathtt{f}^{q'}(\mathtt{b}) \approx \mathtt{c}$, for otherwise $|a - c| \leq K \stackrel{\text{def}}{=} q + q'$ in every stalk of every model of τ^* in $\mathcal{K}^1$. Finally, if τ^* has a conjunct of the form $\mathtt{f}^q(0) \approx \mathtt{a}$, $\mathtt{f}^q(0) \approx \mathtt{b}$, $\mathtt{f}^q(0) \approx \mathtt{c}$, $\mathtt{f}^q(\mathtt{a}) \approx 0$, $\mathtt{f}^q(\mathtt{b}) \approx 0$ or $\mathtt{f}^q(\mathtt{c}) \approx 0$, then all σ_k's are vacuously satisfied in every stalk of every model of τ^* in $\mathcal{K}^1$, and hence every model of τ^* in $\mathcal{K}^1$ is a witness for ($\star$). Thus checking whether ($\star$) holds can be replaced by checking whether τ^* has a model in $\mathcal{K}^1_{\leq M,n}$, which can be done effectively. So we will assume that any equation relating 0 and one of a, b or c does not appear as a conjunct in τ^*.

Now let us approach from a different direction. Let $\mathbf{A}$ be any member of $\mathcal{K}^1$ and let $\mathbf{A}' \in \mathcal{K}^1_{\leq M,n}$ correspond to $\mathbf{A}$ as in lemma 3.(a). Observe that if $\mathbf{A}_{<M}$ satisfies cond_k for all $k \in \omega$, then so is $\mathbf{A}'_{<M}$ because each stalk of $\mathbf{A}'_{<M}$ is actually a stalk of $\mathbf{A}_{<M}$. (Incidentally the converse holds

too. But we do not need this fact in our proof.) Thus

$$\mathbf{A} \in \mathcal{K} \Rightarrow \mathbf{A}'_{<M} \in \mathcal{K}. \tag{29}$$

If the formula $\mathsf{width}_M \approx 0$ appears as a conjunct in τ^*, then for any model $\mathbf{A}$ of τ^* in $\mathcal{K}$, $\mathbf{A}' = \mathbf{A}'_{<M}$ since $\mathbf{A}'$ is a model of τ^*. So $\mathbf{A}'_{<M}$ is a model of τ^* which in in $\mathcal{K}$. Thus checking whether $(\star)$ holds can be replaced by checking whether τ^* has a model in $\mathcal{K}^1_{\leq M,n}$. Hence we will assume that $\mathsf{width}_M \approx 0$ does not appear as a conjunct in τ^*, or equivalently, $\mathsf{width}_M \not\approx 0$ appears as a conjunct in τ^*.

This time, let us consider the case when a formula of the form $\mathsf{order(a)} \approx K$ is a conjunct of τ^*. $K = 1$ case is trivially eliminated because that would imply $a = 0$ in every stalk of every model of τ^* in $\mathcal{K}^1$ and hence σ_k's would be all vacuously satisfied there. So from the assumption $\tau^* \in \Phi^*_{\text{basic}}(M, n)$, we know that each prime divisor of K is $< M$. Then in every model $\mathbf{A}$ of τ^* in $\mathcal{K}^1$, the constant symbol a must be interpreted as 0 in every stalk of $\mathbf{A}$ with height $\geq M$. Thus σ_k's for these stalks are all vacuously satisfied. So we can exclude this situation again. Similarly we can assume that formulas of the form $\mathsf{order(b)} \approx K$ or $\mathsf{order(c)} \approx K$ cannot occur as a conjunct in τ^*.

Next consider the set $\mathcal{C} \stackrel{\text{def}}{=} \{\mathbf{C}_{<M} \mid \mathbf{C}_{<M} \in \mathcal{K},\ \mathbf{C} \in \mathcal{K}^1_{\leq M,n}\}$. This set can be effectively computed because determining whether $\mathbf{C}_{<M} \in \mathcal{K}$ can be done by checking cond_k (for $\mathbf{C}_{<M}$) only for $k < M$. It is clear, by (29), that

$$\mathcal{C} \supseteq \{\mathbf{A}'_{<M} \mid \mathbf{A} \in \mathcal{K}\} \supseteq \{\mathbf{A}'_{<M} \mid \mathbf{A} \models \tau^*,\ \mathbf{A} \in \mathcal{K}\}.$$

Thus if $\mathcal{C} = \varnothing$, then so is the set of all models of τ^* in $\mathcal{K}$, and hence we may conclude that $(\star)$ does not hold. So we will assume that $\mathcal{C}$ is nonempty.

Suppose that $\mathbf{A}$ is a model of τ^* in $\mathcal{K}$. Then $\mathbf{A}'_{<M}$ should satisfy some of the conjuncts of τ^*. First, all equations should be satisfied. Second, conjuncts of the form $\mathsf{width}_p \approx k$ or $\mathsf{width}_p \not\approx k$ with $p < M$ should be satisfied.

Let $\mathcal{C}^\star$ be the subset of $\mathcal{C}$ consisting of those members of $\mathcal{C}$ satisfying all the conjuncts of τ^* mentioned above. If $\mathcal{C}^\star$ is empty, then we can conclude that $(\star)$ does not hold. So let us assume that $\mathcal{C}^\star \neq \varnothing$.

Now we are going to show that for any $\mathbf{C} \in \mathcal{C}^\star$, there exists a model $\mathbf{A}$ of τ^* in $\mathcal{K}$ such that $\mathbf{A}_{<M} = \mathbf{C}$: i.e., $(\star)$ holds. This will complete the proof.

Let $\mathbf{C} \in \mathcal{C}^\star$ be given. We consider the following three cases.

(CASE1). τ^* has no conjunct which is an equation of the form $\mathtt{f}^q(\mathsf{a}) \approx \mathsf{b}$, $\mathtt{f}^q(\mathsf{b}) \approx \mathsf{a}$, $\mathtt{f}^q(\mathsf{c}) \approx \mathsf{b}$, $\mathtt{f}^q(\mathsf{b}) \approx \mathsf{c}$, $\mathtt{f}^q(\mathsf{a}) \approx \mathsf{c}$ or $\mathtt{f}^q(\mathsf{c}) \approx \mathsf{a}$.

We will construct a member $\mathbf{A}$ of $\mathcal{K}$ by adjoining two stalks $\mathbf{A}_1$ and $\mathbf{A}_2$ of height M to $\mathbf{C}$. Without loss of generality the universe of each of these stalks is $\{0, 1, \ldots, M\}$. In all $\mathbf{A}_i$'s the constant symbol $\mathsf{0}$ and the function symbol $\mathtt{f}$ are interpreted as in the proof of lemma 5. Then use the following interpretations for the remaining symbols: $\mathsf{a}^{\mathbf{A}_1} = 0$, $\mathsf{a}^{\mathbf{A}_2} = 1$, $\mathsf{b}^{\mathbf{A}_1} = 1$, $\mathsf{b}^{\mathbf{A}_1} = 0$, $\mathsf{c}^{\mathbf{A}_1} = \mathsf{c}^{\mathbf{A}_2} = 1$.

Now we will verify that $\mathbf{A}$ is a model of τ^* in $\mathcal{K}$. Let φ be a conjunct of τ^*. Note that $\varphi \notin \Phi_{\mathrm{atom}}$ from our assumptions. If $\varphi \in \Phi_{\neg\mathrm{atom}}$, then it holds in one of the 2 new stalks, and hence it should hold in $\mathbf{A}$. If $\varphi \in \Phi_{\mathrm{order}}$, then φ must be $\mathsf{order}(\mathsf{0}) = 1$ because we have excluded all other possibilities already, and this φ certainly holds in $\mathbf{A}$. Finally suppose that $\varphi = \mathsf{width}_p \approx k$ or $\varphi = \mathsf{width}_p \not\approx k$. For $p < M$, φ should hold in $\mathbf{A}$ because it holds in $\mathbf{C} \in \mathcal{C}^*$. For $p = M$, φ must be $\mathsf{width}_p \not\approx 0$ and this certainly holds in $\mathbf{A}$. Since every conjunct of τ^* holds in $\mathbf{A}$, we conclude that $\mathbf{A}$ is a model of τ^*.

Now to show that $\mathbf{A}$ is in $\mathcal{K}$, we have to show that cond_k holds in $\mathbf{A}$ for every $k \leq M$. But σ_k holds in every stalk of $\mathbf{A}$ with height $< M$ because $\mathbf{C} \in \mathcal{K}$ by assumption. For the 2 new stalks $\mathbf{A}_1$ and $\mathbf{A}_2$, all σ_k's hold vacuously.

(CASE2). τ^* has a conjunct which is an equation of the form $\mathtt{f}^q(\mathsf{a}) \approx \mathsf{b}$ or $\mathtt{f}^q(\mathsf{b}) \approx \mathsf{a}$.

First of all, note that τ^* cannot have an equation of the form $\mathtt{f}^q(\mathsf{c}) \approx \mathsf{b}$ or $\mathtt{f}^q(\mathsf{b}) \approx \mathsf{c}$ because we have eliminated this situation earlier.

Also note that it cannot be the case that τ^* has a conjunct $\mathtt{f}^q(\mathsf{a}) \approx b$ and also has another conjunct $\mathtt{f}^{q'}(\mathsf{b}) \approx \mathsf{a}$. If it were the case, then we would be forced to have $M \leq q + q'$ which contradicts the assumption on the rank of τ^*. By a similar reason τ^* cannot have another conjunct $\mathtt{f}^{q'}(\mathsf{a}) \approx \mathsf{b}$. Without loss of generality we assume that the unique equation which appears as a conjunct in τ^* is $\mathtt{f}^q(\mathsf{a}) \approx \mathsf{b}$.

We adjoin 2 stalks $\mathbf{A}_1$ and $\mathbf{A}_2$ to $\mathbf{C}$ to obtain $\mathbf{A}$. The universes and the interpretations of $\mathsf{0}$ and $\mathtt{f}$ are as before. Then use the following interpretations for the remaining symbols.

$a^{A_1} = 1$, $a^{A_2} = 0$, $b^{A_1} = 1 + q$, $b^{A_2} = 0$, $c^{A_1} = 0$ and $c^{A_2} = 1$. It is straightforward to check that this **A** is indeed a model of τ^* in $\mathcal{K}$.

(CASE3). τ^* has a conjunct which is an equation of the form $f^q(c) \approx b$ or $f^q(b) \approx c$.

This case can be handled by an argument similar to that used in the second case.

Therefore in all possible cases we see that there exists a model **A** of τ^* in $\mathcal{K}$ such that $\mathbf{A}_{<M} = \mathbf{C}$, as was desired. □

5. Discussion

We have made a small progress (in the negative direction) in solving the Burris' problem presented in the beginning of this paper. The result of this paper might be somehow extended to give a negative answer to a weaker version of the Burris' problem:

> If $\mathcal{K}$ is a class of finite algebras of the same type closed under homomorphic images, subalgebras and finite direct products (i.e., pseudovariety), then is $\mathcal{K}_{\mathrm{DI}}$ necessarily decidable whenever $\mathcal{K}$ is ?

But the original Burris' problem is of a different breed—it may very well be the case that the answer to the problem is really "yes" and in this case a totally different approach should be necessary.

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