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WEAK TYPE INEQUALITY FOR POISSON TYPE INTEGRAL OPERATORS

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ABSTRACT. A condition for a certain maximal operator to be of weak type (p, p) is studied. This operator unifies various maximal operators cited in the literatures.

1. Introduction

Given a function f in $\mathbb{R}^n$, we define a function $\mathcal{M}f$ in

$$\overline{\mathbb{R}^{n+1}_+} = \{(x,t) : x \in \mathbb{R}^n, t \geq 0\}$$

by setting

$$\mathcal{M}f(x,t) = \sup\{\frac{1}{|Q|}\int_Q |f(y)|\, dy : x \in Q \text{ and sidelength}(Q) \geq t\}.$$

It is well known that this maximal operator $\mathcal{M}$ controls Poisson integral defined by, for $x \in \mathbb{R}^n, t \geq 0$,

$$P(f)(x,t) = \int_{\mathbb{R}^n} f(y)P(x-y,t)dy,$$

where

$$P(x,t) = \frac{c_n t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}$$

is the Poisson kernel.

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Given a positive Borel measure ν in $\overline{\mathbb{R}^{n+1}_+}$, we define a function $\mathcal{N}\nu$ in $\mathbb{R}^n$ by setting

$$\mathcal{N}\nu(x) = \sup_{x \in Q} \frac{\nu(\widetilde{Q})}{|Q|},$$

where the supremum is taken over all cubes Q containing x,

$$\widetilde{Q} = \{(x,t) \in \bar{\mathbb{R}}^{n+1}_+ : x \in Q \text{ and } 0 \le t \le \text{ sidelength}(Q)\}.$$

Also it is well known that

(A) Let $\mathcal{N}(x) \le C < \infty$. For every p with $1 < p < \infty$, there is a constant C_p such that for every f and every μ

$$\left[\int_{\overline{\mathbb{R}^{n+1}_+}} [\mathcal{M}f(x,t)]^p d\nu(x,t)\right]^{\frac{1}{p}} \le C_p \left[\int_{\mathbb{R}^n} |f(x)|^p \mathcal{N}\nu(x) dx\right]^{\frac{1}{p}}.$$

For this, see [6].

(B) For every p with $1 < p < \infty$, $\mathcal{M}$ is bounded from $L^p(\mathbb{R}^n, dx)$ into $L^p(\mathbb{R}^{n+1}_+, d\nu)$ if and only if ν satisfies the Carleson condition

$$\nu(\widetilde{Q}) \le C|Q|$$

for each cube Q in $\mathbb{R}^n$. For this, see [1].

(C) For every p with $1 < p < \infty$, $\mathcal{M}$ is bounded from $L^p(R^n, v(x)dx)$ into $L^p(\mathbb{R}^{n+1}_+, \nu)$ if the following condition is satisfied

$$\sup_{x \in Q} \frac{\nu(\widetilde{Q})}{|Q|} \le cv(x) \ a.e. \ x.$$

For this, see [3].

(D) Recently, Sueiro studied a certain maxiaml operator on spaces of homogeneous type and applied to Poisson-Szegö integral on homogeneous domains.

In this paper we generalized some results in (A), (B), (C), and (D) to spaces of homogeneous type with suitable conditions concerning a family of balls.

2. Preliminaries

DEFINITION 2.1. Let X be a topological space. Assume d is a pseudo-distance on X, i.e., a nonnegative function defined on $X \times X$ satisfying

(1) $d(x,x) = 0$; $d(x,y) > 0$ if $x \neq y$;
(2) $d(x,y) = d(y,x)$;
(3) $d(x,z) \leq K[d(x,y) + d(y,z)]$, where K is some fixed constant.
Assume further that
(a) the balls $B(x,r) = \{y \in X : d(x,y) < r\}$ form a basis of open neighborhoods at $x \in X$ and that μ is a Borel measure on X such that
(b) $0 < \mu(B(x,2r) \leq A\mu(B(x,r)) < \infty$, where A is some fixed constant.
Then we call (X, d, μ) a space of homogeneos type.

REMARK. Properties (3) and (b) will be referred to as the *triangle inequality* and the *doubling property* respectively.

Note that the condition (b) is equivalent that for every $c > 0$, there exists a constant A_c such that $\mu(B(x,cr)) \leq A_c\mu(B(x,r))$.

DEFINITION 2.2. Assume for each $x \in X$ we are given a set $\Omega_x \subset X \times [0,\infty)$. Let Ω denote the family $\{\Omega_x : x \in X\}$. For each $t \geq 0$ set

$$\Omega_{(x,t)} = \Omega_x \cap \big(X \times [t,\infty)\big)$$

and

$$\mathcal{R}_\alpha(x,t) = \{(y,r) \in X \times [0,\infty) : \Omega_{(y,r)}(t) \cap B(x,\alpha t) \neq \emptyset\ \},$$

where $\Omega_{(y,r)}(t) = \{z \in X : (z,t) \in \Omega_{(y,r)}\}$ is the cross-section of $\Omega_{(y,r)}$ at height t.

DEFINITION 2.3. Assume that we have a family $\{\Omega_x : x \in X\}$. For $f \in L^1_{loc}(X, d\mu)$ and $x \in X$, $t \geq 0$ set

$$\mathcal{M}_\Omega f(x,t) \ = \sup_{(y,s)\in\Omega(x,t)} \frac{1}{\mu(B(y,s))} \int_{B(y,s)} |f| d\mu.$$

DEFINITION 2.4. Let $1 \leq p < \infty$. A pair (ν, w) is said to satisfy the condition $C_p(\Omega)$ if there are constants $C = C(K, A, p, \alpha)$ so that

$$\frac{\nu(\mathcal{R}_\alpha(x,r)}{\mu(B(x,r))} \leq Cw(y), \text{ a.e. on } B(x,r)$$

if $p = 1$ and

$$\frac{\nu(\mathcal{R}_\alpha(x,r))}{\mu(B(x,r))}\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)} w^{-1/(p-1)}d\mu\right)^{p-1} \leq C,$$

if $p > 1$.

DEFINITION 2.5. An operator T defined in $L^p(d\mu)$ is said to be of weak type (p,p) if there is a constant C_p so that

$$\mu\{x : |T(f)(x)| > \lambda\} \leq C_p \frac{\|f\|_p^p}{\lambda^p}$$

for all $\lambda > 0$.

3. Main Theorems

The following lemma is given in [2]. See also [8].

LEMMA 3.1. *Let E be a bounded subset of X i.e. E is contained in some ball. Let $r(x)$ be a positive number for each $x \in E$. Then there is a (finite or infinite) sequence of disjoint balls $B(x_i,r(x_i))$, $x_i \in E$, such that the balls $B(x_i,4Kr(x_i))$ cover E, where K is the constant in the triangle inequality. Furthermore, every $x \in E$ is contained in some ball $B(x_i,4Kr(x_i))$ satisfying $r(x) \leq 2r(x_i)$.*

THEOREM 3.1. *Suppose that if $(y,r_1) \in \Omega_{(x,t)}$ and $r_1 < r_2$ then $(y,r_2) \in \Omega_{(x,t)}$. If $\mathcal{M}_\Omega$ be of weak type (p,p) with respect to the pair (ν, w), $p \geq 1$, then (ν, w) satisfies the condition $C_p(\Omega)$.*

Conversely, if (ν, w) satisfies the condition $C_p(\Omega)$, then $\mathcal{M}_\Omega$ is of weak type (p,p) with respect to the pair (ν, w).

Proof. Suppose that $\mathcal{M}_\Omega$ be of weak type (p,p) with respect to the pair (ν, w). Let $f \geq 0$ without loss of generality. Let $(x_o,t) \in \mathcal{R}_\alpha(x,r)$. Then $\Omega_{(x_o,t)}(r) \cap B(x,\alpha r) \neq \emptyset$. So we can choose $y \in \Omega_{(x_o,t)}(r) \cap B(x,\alpha r)$. By the triangle inequality,

$$B(x,r) \subset B(y,K(\alpha+1)r) \subset B(x,(K^2\alpha+K\alpha+K^2)r). \tag{1}$$

Thus

$$f_{B(y,K(\alpha+1)r)} \leq \mathcal{M}_\Omega(f\chi_{B(y,K(\alpha+1)r)})(x_o,t),$$

where

$$f_B = \frac{1}{\mu(B)}\int_B f d\mu.$$

Choose λ so that $0 < \lambda < f_{B(y,K(\alpha+1)r)}$. If we denote

$$E_\lambda = \{\mathcal{M}_\Omega(f\chi_{B(y,K(\alpha+1)r)}) > \lambda\},$$

then $\mathcal{R}_\alpha(x,r) \subset E_\lambda$, and so

$$\nu(\mathcal{R}_\alpha(x,r)) \leq \frac{C}{\lambda^p}\int_{B(y,K(\alpha+1)r)} f^p w d\mu.$$

Hence if we replace f by $f\chi_B(x,r)$, then

$$f^p_{B(x,r)}\nu(\mathcal{R}_\alpha(x,r)) \leq C\int_{B(x,r)} f^p w d\mu. \tag{2}$$

In particualr, if $S \subset B(x,r)$, then by the doubling property of μ we have

$$\left(\frac{1}{\mu(B(x,r))}\int_S f d\mu\right)^p \nu(\mathcal{R}_\alpha(x,r)) \leq C\int_S f^p w d\mu. \tag{3}$$

If $f = 1$, then (3) can be written as

$$\left(\frac{\mu(S)}{\mu(B(x,r))}\right)^p \nu(\mathcal{R}_\alpha(x,r)) \leq C' w(S), \tag{4}$$

where $w(S) = \int_S w d\mu$.

If $p = 1$, then

$$\frac{\nu(\mathcal{R}_\alpha(x,r))}{\mu(B(x,r))} \le C\frac{1}{\mu(S)}\int_S w d\mu \tag{5}$$

holds for every measurable set $S \subset B(y,r)$. Thus

$$\frac{\nu(\mathcal{R}_\alpha(x,r))}{\mu(B(x,r))} \le C\text{ess.inf}_{y\in B(x,r)} w(y) \le Cw(y) \tag{6}$$

a.e. $y \in B(x,r)$.

Next let $p > 1$. Choose $f = w^{-1/(p-1)}$ so that $f = f^p w$ on $B(x,r)$. Then from (3), we have

$$\left(\frac{1}{\mu(B(x,r))}\int_S w^{-1/(p-1)} d\mu\right)^p \nu(\mathcal{R}_\alpha(x,r)) \le C\int_S w^{-1/(p-1)} d\mu.$$

for all measurable set $S \subset B(x,r)$.

Thus

$$\left(\frac{1}{\mu(B(x,r))}\int_S w^{-1/(p-1)} d\mu\right)^{p-1} \frac{\nu(\mathcal{R}_\alpha(x,r))}{\mu(B(x,r))} \le C(K,A,\alpha,p).$$

and so by letting $S \uparrow B(x,r)$, we obtain

(7)

$$\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)} w^{-1/(p-1)} d\mu\right)^{p-1} \frac{\nu(\mathcal{R}_\alpha(x,r))}{\mu(B(x,r))} \le C(K,A,\alpha,p).$$

To prove the converse, suppose that (ν, w) satisfies the condition $C_p(\Omega)$, $p \ge 1$. We follow the idea of Sueiro[8]. For each $\lambda > 0$, define

$$E_\lambda = \{(x,t) \in X \times [0,\infty) : \mathcal{M}_\Omega f(x,t) > \lambda\}$$

and

$$E'_\lambda = \{x \in X : \sup_{r>0} \frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f| d\mu > \lambda\}$$

Also for each $x \in E'_\lambda$, if we put

$$r(x) = \sup\{r > 0 : \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| d\mu > \lambda\},$$

then $r(x) > 0$ and

$$\frac{1}{\mu(B(x,r(x)))} \int_{B(x,r(x))} |f| d\mu \geq \lambda.$$

Assume for a moment that E'_λ is bounded. Then by the covering lemma, there exists a sequence of balls $\{B(x_i, r(x_i))\}$ so that $E'_\lambda \subset \cup_i B(x_i, 4Kr_i)$. Now we want to verify

$$E_\lambda \subset \cup_i \mathcal{R}_\alpha(x_i, 4Kr_i/\alpha_i) \tag{8}$$

In fact, if $(x,t) \in E_\lambda$, then

$$\frac{1}{\mu(B(y,r))} \int_{B(y,r)} |f| d\mu > \lambda$$

for some $(y,r) \in \Omega_{(x,t)}$. Thus $y \in E'_\lambda$ and $t \leq r \leq r(y)$. By the last part of the Covering lemma, $y \in B(x_i, 4Kr_i)$ for some i such that $r(y) \leq 2r_i$. Here we may assume that $\alpha < 2K$. Consequently, $t \leq r \leq r(y) \leq 2r_i < (4K/\alpha)r_i$ and so $(y, 4Kr_i/\alpha) \in \Omega_{(x,t)}$. But $y \in B(x_i, \alpha(4K/\alpha)r_i)$, so we get

$$y \in \Omega_{(x,t)}(4Kr_i/\alpha) \cap B(x_i, \alpha(4K/\alpha)r_i),$$

and hence $(x,t) \in \mathcal{R}_\alpha(x_i, 4Kr_i/\alpha)$. Thus (8) holds.

Since $4K/\alpha > 1$, we have

$$B(x_i, r_i) \subset B(x_i, 4Kr_i/\alpha). \tag{9}$$

Now let $p = 1$. Then by the doubling property and (9), we have

$$\begin{aligned}
\nu(E_\lambda) &\leq \sum_i \nu(\mathcal{R}_\alpha(x_i, 4Kr_i/\alpha)) \\
&\leq \sum_i \mu(B(x_i, 4Kr_i/\alpha)) \text{ess.inf}_{y \in B(x_i, 4Kr_i/\alpha)} w(y) \\
&\leq C \sum_i \mu(B(x_i, r_i)) \text{ess.inf}_{y \in B(x_i, 4Kr_i/\alpha)} w(y) \\
&\leq \frac{C}{\lambda} \sum_i \int_{B(x_i, r_i)} |f| w d\mu.
\end{aligned}$$

Next, let $p > 1$. By Hölder's inequality we have
(10)

$$\begin{aligned}\mu(B(x_i, r_i)) &\leq \frac{1}{\lambda}\int_{B(x_i,r_i)} f d\mu \\ &\leq \frac{1}{\lambda}\left(\int_{B(x_i,r_i)} f^p w d\mu\right)^{1/p}\left(\int_{B(x_i,r_i)} w^{-1/(p-1)} d\mu\right)^{(p-1)/p}\end{aligned}$$

Therefore by (9) and (10), we obtain

$$\begin{aligned}\nu(E_\lambda) &\leq \sum_i \nu(\mathcal{R}_\alpha(x_i, 4Kr_i/\alpha)) \\ &\leq \left[\mu(B(x_i, 4Kr_i/\alpha))\right]^p \left[\int_{B(x_i, 4cKr_i/\alpha)} w^{-1/(p-1)} d\mu\right]^{1-p} \\ &\leq \left[\mu(B(x_i, r_i))\right]^p \left[\int_{B(x_i,r_i)} w^{-1/(p-1)} d\mu\right]^{1-p} \\ &\leq \frac{C}{\lambda^p}\int_X f^p w d\mu.\end{aligned}$$

If E'_λ is not bounded, fix $a \in X$ and $R > 0$. Now we replace E'_λ by $E'_\lambda \cap B(a, R)$ to apply the previous argument. Letting $R \to \infty$, then the weak type estimate is obtained as before. This completes the proof. □

REMARK 1. Let $\mathcal{M}$ is the original maximal operator given in the introduction. Following Ruiz[4], a measure ν on $\mathbb{R}^{n+1}_+$ satisfies the condtion $C_p(w)$ if, for $1 \leq p < \infty$,

$$\sup_Q \frac{\nu(\widetilde{Q})}{|Q|}\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}\right)^{(p-1)} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and, for $p = 1$

$$\sup_Q \frac{\nu(\widetilde{Q})}{|Q|} \leq Cw(x), \text{ a.e.}$$

With this in mind, we have then the following:

COROLLARY 3.1 (RUIZ[4]). *Let $1 \le p < \infty$, then $\nu \in C_p(w)$ if and only if the maximal operator M is of weak type (p,p) with respect to the pair (ν, w).*

THOREM 3.2. *Assume that Ω satisfies the conditions (1) of theorem 3.1. If (ν, w) satisfies the condition $C_1(\Omega)$, then $\mathcal{M}_\Omega$ is bounded from $L^p(X, wd\mu)$ into $L^p(X \times [0,\infty), d\nu)$.*

Proof. It suffices to show that $\mathcal{M}_\Omega$ is of strong type (∞,∞). Suppose that $w(x) \neq 0$ for all x and $\lambda > \|f\|_{L^\infty(wd\mu)}$. (There is no loss of generality in doing so for if $w(x) = 0$ for some x, then $\nu(\mathcal{R}_\alpha(y,r)) = 0$ for any $y \in X$ and $r > 0$. This says that $\nu(X \times [0,\infty)$, so there is nothing to prove.) We have then $\int_{\{|f|>\lambda\}} wd\mu = 0$ and consequently $\mu\{|f| > 0\} = 0$. Hence $|f(x)| \le \lambda$, a.e., from which we have $\mathcal{M}_\Omega \le \lambda$ and so $\|f\|_{L^\infty(wd\mu)}$. This proves that $\mathcal{M}_\Omega$ is of strong type (∞,∞). □

REMARK 3.1. If $d\nu = ud\mu \otimes d\delta_o$, where $d\delta_o$ is the Dirac measure concentrated on $t = 0$, then the $C_p(\Omega)$-condition is reduced to the condition obtained by Wenjie[9]. In fact, $C_p(\Omega)$- condition can be simplifies as

$$\frac{1}{\mu(B(x,r))^p} \int_{B(x,r)} ud\mu \left(\int_{B(x,r)} w^{-1/(p-1)} d\mu \right)^{p-1} \le C,$$

if $p > 1$, and

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} ud\mu \le Cw(y), \text{ a.e. on } y \in B(x,r),$$

if $p = 1$. In fact, our condition is slightly improved than that of Wenjie.

References

[1] L. Carleson, *Interpolation by bounded analytic functions and the corona problem*, Ann. Math. **76** (1962), 547-559.

[2] R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certain espaces homogènes*, Lecture Notes in Math. **242** (1971), Springer-Verlag, Berlin.

[3] C. Fefferman C. and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. **93** (1971), 107-115.

[4] F. J. Ruiz, *A Unified approach to Carleson measures and A_p weights*, Pacific J. Math. **117** (1985).
[5] F. J. Ruiz and J. L. Torrea, *A Unified approach to Carleson measures and A_p weights II*, Pacific J. Math. **120** (1985).
[6] J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topic*, Elsevier Science, North-Holland, 1985.
[7] E. M. Stein, *Harmonic Analysis: Real variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, N. J., 1993.
[8] J. Sueiro, *On maximal functions and Poisson-Szegö integrals*, Trans. Amer. Math. Soc. **298** (1986), 653-669.
[9] P. Wenjie, *Weighted Norm Inequaities for Certain Maximal Operator with Approach Region*, Lecture Notes in Math. **1494** (1988), Springer Verlag, Berlin, 169-175.

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