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INTEGRAL FORMULAS FOR EULER'S CONSTANT

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ABSTRACT. There have been developed many integral representations for Euler's constant some of which are recorded here. We are aiming at showing a (presumably) new integral form of Euler's constant and disproving another integral representation for this constant which were recently proposed by Jean Angelsio, Garches, France, in American Mathematical Monthly. By modifying the Angelsio's incorrectly proposed integral form of Euler's constant, we also provide an integral representation for Euler's constant.

About 260 years ago an important mathematical constant was born by Leonhard Euler, maybe next to π and e, which is called *Euler's* constant γ defined by

$$\gamma := \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \log n\right] \cong 0.577\,215\,664\dots . \tag{1}$$

The value of Euler's constant was given by Mascheroni in 1790 with 32 figures the value of which was turned out to be incorrect in the twentieth place. Maybe, since Mascheroni's error has led to eight additional calculations of this constant, so γ is called the *Euler-Mascheroni*'s constant (see Glaisher [6]).

The true nature of Euler's constant (whether an algebraic or transcendental number) is not known. This is a part of the seventh one among the famous Hilbert's 23 problems.

There have been developed lots of integral representations for γ some of which are recorded here. We are also aiming at showing a (presumably) new integral form of γ and disproving another integral representation for this constant which were recently proposed by Jean Angelsio,

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Garches, France, in American Mathematical Monthly. By modifying the Angelsio's incorrectly proposed integral form of Euler's constant, we also provide an integral representation for Euler's constant.

We first introduce some integral representations for γ (see Seo *et al.* [9]; also Gradshteyn *et al.* [7, pp. 946-947, Entry 8.367]):

$$\gamma = \int_0^1 \frac{1-e^{-t}}{t}\,dt - \int_1^\infty \frac{e^{-t}}{t}\,dt; \tag{2}$$

$$\gamma = \int_0^\infty e^{-t}\left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right) dt, \tag{3}$$

which is due to Gauss;

$$\gamma = \int_0^1 \left(\frac{1}{\log t} + \frac{1}{1-t}\right) dt; \tag{4}$$

$$\gamma = \int_0^1 \left(1 - e^{-t} - e^{-1/t}\right) \frac{dt}{t}; \tag{5}$$

$$\gamma = \int_0^\infty \left(\frac{1}{1+t} - e^{-t}\right) \frac{dt}{t}, \tag{6}$$

which is due to Dirichlet;

$$\gamma = -\int_0^\infty e^{-t} \log t\, dt; \tag{7}$$

$$\gamma = -\int_0^1 \log\left(\log\frac{1}{t}\right) dt; \tag{8}$$

$$\gamma = \frac{1}{2} + 2\int_0^\infty \frac{t}{1+t^2}\,\frac{dt}{e^{2\pi t}-1}, \tag{9}$$

which is called *Poisson's expression* for γ.

We also introduce some integral representations for γ whose integrands contain circular functions (see Seo *et al.* [9]; also Gradshteyn *et al.* [7, pp. 432, Entries 3.781-3]):

$$\gamma = -\int_0^\infty \left(\cos t - \frac{1}{1+t^2}\right)\frac{dt}{t}; \tag{10}$$

$$\gamma = 1 - \int_0^\infty \left(\frac{\sin t}{t} - \frac{1}{1+t}\right)\frac{dt}{t}; \tag{11}$$

$$\gamma = -2\int_0^\infty \left(\cos t - e^{-t^2}\right)\frac{dt}{t}; \tag{12}$$

$$\gamma = \log 2 - \pi \int_0^1 \int_0^{\frac{1}{2}} \tan\frac{\pi t}{2}\left(\frac{\sin \pi t u}{\sin \pi u} - t\right) du dt, \tag{13}$$

which is due to Mikolas (see Campbell [4, p. 216]);

$$\gamma = \frac{1}{2} + 2\int_0^\infty (1+t^2)^{-\frac{1}{2}}\left(e^{2\pi t} - 1\right)^{-1}\sin(\tan^{-1} t)\, dt; \tag{14}$$

$$\gamma = \log 2 - 2\int_0^\infty (1+t^2)^{-\frac{1}{2}}\left(e^{2\pi t} + 1\right)^{-1}\sin(\tan^{-1} t)\, dt, \tag{15}$$

which is due to Jensen (see Bateman [3, p. 33, Equation (13)]).

In 1996, Jean Angelsio, Garches, France, proposed the following integral representation for γ :

$$\lim_{u\to\infty}\int_{1/u}^{u}\left(\frac{\sin x}{x} - \cos x\right)\frac{\ln x}{x}\, dx = 1 - \gamma, \tag{16}$$

the problem number of which was 427 in American Mathematical Monthly and solved in 1997 by Donald A. Darling, Newport Beach, CA and many other persons and Problems Group (U. K.), and NSA Problems Group and the proposer.

In the same year, Jean Angelsio also proposed the following problems in the same journal: Show that

$$\text{(A)} \qquad \gamma = \lim_{u\to\infty} \int_{1/u}^{u} \left(\frac{1}{2} - \cos x\right) \frac{dx}{x}$$

and

$$\text{(B)} \qquad \gamma = \lim_{u\to\infty} \int_{1/u}^{u} \left(\frac{1}{1+x} - \frac{\cos x}{x}\right) \frac{dx}{x}.$$

We will prove (A), and disprove (B), i.e., the proposed formula (B) is incorrect.

PROOF OF (A). Using (10), we have

$$\begin{aligned}
&\int_0^\infty \left(\frac{1}{2} - \cos t\right) \frac{dt}{t} \\
&\quad = \int_0^\infty \left(-\cos t + \frac{1}{1+t^2} - \frac{1}{1+t^2} + \frac{1}{2}\right) \frac{dt}{t} \\
&\quad = \int_0^\infty \left(-\cos t + \frac{1}{1+t^2}\right) \frac{dt}{t} + \int_0^\infty \left(\frac{1}{2} - \frac{1}{1+t^2}\right) \frac{dt}{t} \\
&\quad = \gamma + \int_0^\infty \left(\frac{1}{2} - \frac{1}{1+t^2}\right) \frac{dt}{t}.
\end{aligned}$$

Now it is sufficient to show that

$$\begin{aligned}
(0 =) \int_0^\infty \left(\frac{1}{2} - \frac{1}{1+t^2}\right) \frac{dt}{t}
&= \lim_{u\to\infty} \int_{1/u}^{u} \left(\frac{1}{2} - \frac{1}{1+t^2}\right) \frac{dt}{t} \\
&= \lim_{u\to\infty} \left[-\log t + \log(1+t^2)\right] \Big|_{1/u}^{u} \\
&= \lim_{u\to\infty} \left[-2\log u + 2\log u\right] = 0.
\end{aligned}$$

□

DISPROOF OF (B). Assume that the proposed identity (B) is true. Using (11), we then have

$$\begin{aligned}
\gamma &= \int_0^\infty \left(\frac{1}{1+t} - \frac{\cos t}{t}\right)\frac{dt}{t} = \lim_{\epsilon\downarrow 0}\int_\epsilon^\infty \left(\frac{1}{1+t} - \frac{\cos t}{t}\right)\frac{dt}{t} \\
&= \lim_{\epsilon\downarrow 0}\int_\epsilon^\infty \left(\frac{1}{1+t} - \frac{\sin t}{t} + \frac{\sin t}{t} - \frac{\cos t}{t}\right)\frac{dt}{t} \\
&= \lim_{\epsilon\downarrow 0}\left[\int_\epsilon^\infty \left(\frac{1}{1+t} - \frac{\sin t}{t}\right)\frac{dt}{t} + \int_\epsilon^\infty \left(\frac{\sin t}{t} - \frac{\cos t}{t}\right)\frac{dt}{t}\right] \\
&= \gamma - 1 + \int_0^\infty \left(\frac{\sin t}{t} - \frac{\cos t}{t}\right)\frac{dt}{t},
\end{aligned}$$

from which we have the following integral formula:

$$\text{(C)} \qquad \int_0^\infty \frac{\cos x - \sin x}{x^2}\,dx = -1.$$

However it is easy to prove the divergence of the integral (C) by recalling a well-known theorem (see Kaplan [8, pp. 374-375]): □

COROLLARY. *Let $f(x)$ be continuous for $a \le x < \infty$; let $f(x)$ decrease as x increases and let $\lim_{x\to\infty} f(x) = 0$. Then the integrals*

$$\int_a^\infty f(x)\sin x\,dx \qquad \text{and} \qquad \int_a^\infty f(x)\cos x\,dx$$

converge.

Indeed,

$$\int_0^\infty \frac{\cos x - \sin x}{x^2}\,dx = \left(\int_0^{\frac{\pi}{8}} + \int_{\frac{\pi}{8}}^\infty\right)\frac{\cos x - \sin x}{x^2}\,dx,$$

in which the latter integral converges by Corollary, but

$$\int_0^{\frac{\pi}{8}} \frac{\cos x - \sin x}{x^2}\,dx \ge b\int_0^{\frac{\pi}{8}} \frac{1}{x^2}\,dx = \infty,$$

where

$$b = \cos\frac{\pi}{8} - \sin\frac{\pi}{8} = \frac{1}{2\sqrt{2}}\left[(\sqrt{2}-1)\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}}\right].$$

As a matter of fact, we obtain an integral formula for γ in replacing $\cos x/x$ by $\cos x$ in (B):

$$\gamma = \int_0^\infty \left(\frac{1}{1+x} - \cos x\right)\frac{dx}{x}. \tag{17}$$

PROOF OF (17). Using (11), we have

$$\begin{aligned}\gamma &= \int_0^\infty \left(\frac{1}{1+x} - \cos x\right)\frac{dx}{x} = \lim_{\epsilon\downarrow 0}\int_\epsilon^\infty \left(\frac{1}{1+x} - \cos x\right)\frac{dx}{x}\\ &= \lim_{\epsilon\downarrow 0}\left[\int_\epsilon^\infty \left(\frac{1}{1+x} - \frac{\sin x}{x}\right)\frac{dx}{x} + \int_\epsilon^\infty \left(\frac{\sin x}{x} - \cos x\right)\frac{dx}{x}\right]\\ &= \gamma - 1 + \int_0^\infty \frac{\sin x - x\cos x}{x^2}\,dx = \gamma,\end{aligned}$$

in which the last equality follows by recalling the integral formula (see Gradshteyn *et al.* [7, p. 433, Entry 3.784(4)]):

$$\int_0^\infty \frac{\sin x - x\cos x}{x^2}\,dx = 1.$$

□

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