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TAKENS-BOGDANOV ANALYSIS FOR AN EPIDEMIOLOGICAL MODEL

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ABSTRACT. In this paper, we analyze a system for an epidemiological model which has a double zero eigenvalue at certain parameter values.

1. Introduction

In this paper we introduce a system for an epidemiological model which is studied by Feng [3]. At certain parameter values, this system has a pair of zero eigenvalues and one nonzero eigenvalue after linearizing about the origin. By using the center manifold reduction ([2, 9]) and normal form calculation ([4, 8]), we provide a system which is easier to analyze than the original system. We analyze the system near the singular point by using a perturbation method. We also show that the system has additional structure which is an invariant straight line that represents the stable manifold of the unfolded fixed point. We show that no local unfolding of this situation can produce a Hopf bifurcation. In addition, we compare this center manifold-normal form analysis and the analysis done by Feng [3].

2. The epidemiological model

Feng [3] introduced an epidemiological model that takes the following

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form:

$$\begin{aligned} &\frac{d}{dt}S = \Lambda - \mu S - \sigma S\frac{I}{A} \\ &\frac{d}{dt}I = -(\mu+\gamma)I + \sigma S\frac{I}{A} \\ &\frac{d}{dt}Q = -(\mu+\xi)Q + \gamma I \\ &\frac{d}{dt}R = -\mu R + \xi Q, \end{aligned} \tag{2.1}$$

where S represents individuals that are susceptible to disease, I represents infected non-isolated individuals, Q represents isolated individuals and R represents recovered and immune individuals. Also Λ is the rate at which individuals are born into the population, μ is the per capita mortality rate, σ is the per capita infection rate of an average susceptible individual provided that everybody else is infected, γ is the rate at which individuals leave the infected class and ξ is the rate at which individuals leave the isolated class: they are all positive constants. Denote the size of school population by $N = S + I + Q + R$ and active (i.e., non-isolated individuals) by $A = S + I + R$, also assume that sick children undergo some kind of isolation and they do not infect anybody. Further assume that the disease is nonlethal. Note that $A = N - Q$ and $S = A - I - R$. We can eliminate S from (2.1). Scale time such that $\sigma = 1$ by introducing a new, dimensionless, time $\tau = \sigma t$. Then the system (2.1) becomes

$$\begin{aligned} I' &= -(\nu+\theta)I + (1 - \frac{I+R}{N-Q})I \\ Q' &= -(\nu+\zeta)Q + \theta I \\ R' &= -\nu R + \zeta Q, \end{aligned} \tag{2.2}$$

where

$$\nu = \frac{\mu}{\sigma}, \quad \theta = \frac{\gamma}{\sigma}, \quad \zeta = \frac{\xi}{\sigma}.$$

Introduce the fractions

$$u = \frac{S}{A}, \quad y = \frac{I}{A}, \quad q = \frac{Q}{A}, \quad z = \frac{R}{A}, \tag{2.3}$$

and note that

$$A' = -\frac{d}{dt}Q = (\nu + \zeta)Q - \theta I,$$

and

$$\Lambda - \mu S = \mu(I + Q + R).$$

By differentiating (2.3) and using (2.1), we have

$$\begin{aligned} u' &= \nu(y + q + z) - uy + u(\theta y - (\nu + \zeta)q) \\ y' &= -(\nu + \theta)y + uy + y(\theta y - (\nu + \zeta)q) \\ q' &= (1 + q)(\theta y - (\nu + \zeta)q) \\ z' &= \zeta q - \nu z + z(\theta y - (\nu + \zeta)q). \end{aligned} \tag{2.4}$$

The relation $A = S + I + R$ implies that $u + y + z = 1$, so we can eliminate the equation for u' in (2.4) and obtain

$$\begin{aligned} y' &= (1 - \nu - \theta)y - y^2 - zy + y(\theta y - (\nu + \zeta)q) \\ q' &= (1 + q)(\theta y - (\nu + \zeta)q) \\ z' &= -\nu z + \zeta q + z(\theta y - (\nu + \zeta)q). \end{aligned} \tag{2.5}$$

We can easily show that if we choose ν, $\theta > 0$ with $\nu + \theta < 1$, the system (2.5) has the unique nonnegative fixed point $(y^*, q^*, z^*) = (\nu(\nu + \zeta)\kappa, \nu\theta\kappa, \theta\zeta\kappa)$ ([5]) in addition to the origin, where

$$\kappa = \frac{1 - \nu - \theta}{\theta\zeta + \nu(\nu + \zeta)}.$$

3. Center manifold reduction and normal form calculation

Consider the case that $\nu = 0$ and $\theta = 1$. Then the system (2.5) becomes

$$\begin{aligned} y' &= -yz - \zeta yq \\ q' &= (1 + q)(y - \zeta q) \\ z' &= \zeta q + z(y - \zeta q). \end{aligned} \tag{3.1}$$

Transform

$$\begin{pmatrix} y \\ q \\ z \end{pmatrix} = \begin{pmatrix} \zeta & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{3.2}$$

Then the system (3.1) becomes

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 0 \\ \zeta & 0 & 0 \\ 0 & 0 & -\zeta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} -\zeta u^2 - uv + (1-\zeta)uw \\ \zeta u^2 + uv - uw - \zeta vw \\ \zeta u^2 - \zeta w^2 + uv - uw \end{pmatrix}. \tag{3.3}$$

Linearizing about the origin, we see that uv plane is associated with a pair of zero eigenvalues, while the w axis corresponds to an eigenvalues of $-\zeta$. Thus the center manifold is a 2-dimensional surface that is tangent to the uv plane at the origin. By using the center manifold reduction,

$$w = w(u,v) = (1 - \frac{1}{\zeta})u^2 + \frac{uv}{\zeta} + O(3) \tag{3.4}$$

is a center manifold for (3.1) and

$$\begin{aligned} u' &= -\zeta u^2 - uv + (1-\zeta)(\frac{u^2 v}{\zeta} + \frac{(\zeta - 1)u^3}{\zeta}) + O(4) \\ v' &= \zeta u + uv + \zeta u^2 - uv^2 - (\zeta - 1 + \frac{1}{\zeta})u^2 v - \frac{\zeta - 1}{\zeta}u^3 + O(4) \end{aligned} \tag{3.5}$$

is the approximate system for the flow on the center manifold given by (3.4). Now let us determine the stability of the fixed point at the origin. Note that the linearization about the origin is no help since (3.5) has a double zero eigenvalue at the origin. Take the near-identity transformation ([7]):

$$\begin{aligned} u &= m - lm + \frac{2\zeta - \zeta^2 - 1}{\zeta^2} lm^2 + \frac{2\zeta^2 - 2\zeta + 1}{2\zeta^2} l^2 m \\ v &= l + lm + \frac{1-\zeta}{2\zeta} l^2 + \frac{1-\zeta}{\zeta^2} lm^2 + \frac{2\zeta - 2\zeta^2 - 1}{\zeta^2} l^2 m + \frac{2\zeta^2 - 5\zeta + 2}{6\zeta^2} l^3. \end{aligned}$$

Then the normal form is

$$l' = \zeta m + O(4) \tag{3.6}$$
$$m' = -lm + \frac{\zeta - 1}{2\zeta} l^2 m + O(4).$$

Bogdanov [1] has shown that the system which has a linear part

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

has a normal form

$$x' = y$$
$$y' = \sum_{n=2} a_n y^n + y \sum_{n=2} n b_n x^{n-1}.$$

Thus the normal form (3.6) can be written by

$$l' = \zeta m \tag{3.7}$$
$$m' = -lm + \frac{\zeta - 1}{2\zeta} l^2 m + O(4).$$

Note that the system (3.1) has a line of fixed points $y = q = 0$ for all z. Hence the $O(4)$ terms in (3.7) is of the form of $mg(l)$, where $g(l)$ is third or higher order polynomial in l. Therefore the system (3.7) does not have an isolated fixed point; instead, it has a line of fixed points $m = 0$ for all l.

4. The unfolding analysis

Recall that at $\nu = 0$ and $\theta = 1$, the system (2.5) has a line of fixed points $y = q = 0$ for all z. Now, we want to analyze the system (2.5) near the singular point $\nu = 0$ and $\theta = 1$ by using a perturbation method ([7]). For this, let $\nu = \alpha$ and $\theta = \beta + 1$ in the system (2.5). Then (2.5) becomes

$$y' = -(\alpha + \beta)y - y^2 - zy + y((\beta + 1)y - (\alpha + \zeta)q)$$
$$q' = (\beta + 1)y - (\alpha + \zeta)q + (\beta + 1)yq - (\alpha + \zeta)q^2 \tag{4.1}$$
$$z' = -\alpha z + \zeta q + z((\beta + 1)y - (\alpha + \zeta)q).$$

With a transformation (3.2) and by using the center manifold reduction,

$$w = w(u, v, \alpha, \beta) = \frac{1+\zeta}{\zeta}\beta u + \frac{1}{\zeta}uv + \frac{\zeta - 1}{\zeta}u^2 + O(3) \tag{4.2}$$

is a center manifold and hence for small values α and β, the flow near the origin on the center manifold (4.2) is given by

$$\begin{aligned} u' &= -(\alpha+\beta)u + (\zeta\beta - \alpha - \zeta)u^2 + \frac{(1-\zeta)(1+\zeta)}{\zeta}\beta u^2 \\ &\quad - uv - \frac{(\zeta-1)^2}{\zeta}u^3 + \frac{1-\zeta}{\zeta}u^2 v + O(4) \\ v' &= (\beta + \zeta\beta + \zeta)u - \alpha v + \zeta u^2 + (1 + \beta\zeta - \alpha)uv - \frac{1+\zeta}{\zeta}\beta u^2 \\ &\quad - (1+\zeta)\beta uv - \frac{\zeta-1}{\zeta}u^3 - (\zeta - 1 + \frac{1}{\zeta})u^2 v - uv^2 + O(4). \end{aligned} \tag{4.3}$$

Note that u' in (4.3) is equal to zero when $u = 0$. Therefore the system (4.3) has an invariant straight line $u = 0$. Take the near-identity transformation which preserves the invariant line $u = 0$ (now $q = 0$):

$$\begin{aligned} u &= q - pq \\ v &= p + pq + \frac{1-\zeta}{2\zeta}p^2. \end{aligned} \tag{4.4}$$

Then the system (4.3) becomes

$$\begin{aligned} p' &= (\zeta + \beta\zeta + \beta)q - \alpha p - \frac{\alpha(\zeta-1)}{2\zeta}p^2 - \frac{\beta(1+\zeta+2\zeta^2)}{\zeta}pq \\ &\quad - \frac{\beta(\zeta+1)^2}{\zeta}q^2 + O(4) \\ q' &= -(\alpha+\beta)q - (1+\alpha)pq + \frac{\zeta^2\beta - \alpha\zeta + \beta + \zeta\beta}{\zeta}q^2 \\ &\quad + \frac{\zeta-1}{2\zeta}p^2 q + O(4). \end{aligned} \tag{4.5}$$

Recall that if $\alpha = \beta = 0$, then the system (4.5) becomes

$$p' = \zeta q + O(4)$$
$$q' = -pq + \frac{\zeta - 1}{2\zeta} p^2 q + O(4)$$

that actually is of the form of

$$\begin{aligned} p' &= \zeta q \\ q' &= -pq + \frac{\zeta - 1}{2\zeta} p^2 q + qg(p) \end{aligned} \tag{4.6}$$

where $g(p)$ is third or higher order polynomial in p (See section 3). Hence the term O(4) in the second equation of (4.5) is of the form of $qg(\alpha, \beta, p)$. Therefore the system (4.5) has an invariant straight line $q = 0$. Now we give the following theorem:

THEOREM 4.1. *If the unfolding system for the system*

$$\begin{aligned} x' &= y + O(2) \\ y' &= -xy + O(3) \end{aligned} \tag{4.7}$$

has an invariant straight line, then there cannot be a Hopf bifurcation locally.

PROOF. An unfolding of (4.7) can be written as

$$\begin{aligned} x' &= y + \alpha(x + f(x,y)) \\ y' &= -xy + \beta(y + yg(x,y)), \end{aligned} \tag{4.8}$$

where $f(x,y)$ and $g(x,y)$ are strictly nonlinear in x and y. The system (4.8) has an invariant straight line $y = 0$. Obviously the origin is a fixed point which we suppose always a saddle. Note that the system (4.8) has another fixed point (x^*, y^*) which goes to $(0,0)$ as α and β go to zero. The Jacobian matrix for the system (4.8) is given by

$$J = \begin{pmatrix} \alpha + \alpha f_x(x,y) & 1 + \alpha f_y(x,y) \\ -y + \beta y g_x(x,y) & -x + \beta + \beta g(x,y) + \beta y g_y(x,y) \end{pmatrix}.$$

Thus

$$\det J\mid_{(0,0)}=\alpha\beta \text{ and } \operatorname{tr} J\mid_{(0,0)}=\alpha+\beta.$$

Since the origin is assumed to be a saddle, $\alpha\beta<0$. From the system (4.8), we find that the nontrivial fixed point (x^*,y^*) satisfies

$$\begin{aligned}-x^*+\beta+\beta g(x^*,y^*)&=0\\ y^*+\alpha(x^*+f(x^*,y^*))&=0.\end{aligned}\tag{4.9}$$

Therefore we have

$$\begin{aligned}\det J\mid_{(x^*,y^*)}&=-\alpha(\beta+\beta g+f)(\alpha\beta f_x g_y+\alpha\beta g_y-\alpha\beta f_y g_x\\&\quad-\beta g_x+\alpha f_y+1)\\ \operatorname{tr} J\mid_{(x^*,y^*)}&=\alpha(1+f_x-\beta^2 g_y-\beta^2 g g_y-\beta f g_y),\end{aligned}\tag{4.10}$$

where f, g, f_x, g_x, f_y and g_y are evaluated at (x^*,y^*). Note that f and g are strictly nonlinear in x and y. Therefore the only local solution for $\operatorname{tr} J\mid_{(x^*,y^*)}=0$ is $\alpha=0$ from (4.10). However if $\alpha=0$, then $\det J\mid_{(x^*,y^*)}=0$ too. Therefore the nontrivial fixed point (x^*,y^*) of the system (4.8) cannot undergo a Hopf bifurcation locally. □

Now let's go back to the system (4.5). Recall that the system (4.5) has an invariant straight line $q=0$. Clearly the origin is a fixed point. Another fixed point can be found by using MAPLE to perform the multivariate Taylor series expansion with respect to α and β which is

$$(p^*,q^*)=(-\alpha-\beta+O(2),-\frac{\alpha(\alpha+\beta)}{\zeta}+O(3)).$$

The nontrivial fixed point (p^*,q^*) goes to the origin as α and β go to zero. For the Jacobian J of the system (4.5), we have

$$\det J\mid_{(0,0)}=\alpha(\alpha+\beta) \text{ and } \operatorname{tr} J\mid_{(0,0)}=-2\alpha-\beta.$$

By the multivariate Taylor series expansion with respect to α and β, we find that

$$\det J\mid_{(p^*,q^*)}=-\alpha(\alpha+\beta+O(2)).$$

On the other hand, the nontrivial fixed point (p^*, q^*) satisfies

$$-\alpha - \beta - (1+\alpha)p^* + \frac{\zeta^2\beta - \alpha\zeta + \beta + \beta\zeta}{\zeta}q^* + \frac{\zeta - 1}{2\zeta}p^{*^2} = 0.$$

Therefore with this information and by using the multivariate Taylor series expansion with respect to α and β, we find that

$$\text{(4.11)} \quad \begin{aligned} \mathrm{tr}J\mid_{(p^*,q^*)} = \alpha(-1 &+ \frac{(\zeta-1)(\alpha+\beta)}{\zeta} + \frac{(\zeta+1+2\zeta^2)\beta(\alpha+\beta)}{\zeta^2} \\ &- \frac{(\zeta^2\beta - \zeta\alpha + \beta + \zeta\beta)(\alpha+\beta)}{\zeta^2} + O(4)). \end{aligned}$$

Note that locally, $\det J\mid_{(p^*,q^*)} = -\alpha(\alpha+\beta)$. Therefore $\det J\mid_{(0,0)}$ and $\det J\mid_{(p^*,q^*)}$ have the same values but different sign locally and hence the one of theses fixed points is a saddle for small α and β. Without loss of generality, assume that (0,0) is a saddle. Then $\alpha(\alpha+\beta) < 0$ and so the possible cases can be reduced to two cases:

(1) $\alpha > 0$ and $\alpha + \beta < 0$.
(2) $\alpha < 0$ and $\alpha + \beta > 0$.

In both cases, $\det J\mid_{(p^*,q^*)} > 0$ and from (4.11), we find that $\alpha = 0$ is the only local solution for $\mathrm{tr}J\mid_{(p^*,q^*)} = 0$. However, if $\alpha = 0$, then $\det J\mid_{(p^*,q^*)} = 0$ from (4.10). Therefore the nontrivial fixed point (p^*, q^*) of the system (4.5) cannot undergo a Hopf bifurcation locally.

At this stage, we can compare between this normal form analysis and the analysis done by Feng [3]. Feng found that for the system (4.1), when $\alpha + \beta < 0$ and for small $\alpha > 0$, the nontrivial fixed point (endemic equilibrium) is stable for $0 < \zeta < \zeta_1$, unstable for $\zeta_1 < \zeta < \zeta_2$ and stable for $\zeta > \zeta_2$, where

$$\zeta_1 = -\frac{\alpha\beta(1+\sqrt{1+4(\beta+1)^2})}{2(\beta+1)^2} + O(\alpha^{\frac{3}{2}})$$
$$\zeta_2 = -\beta(\beta+1)^2 + O(\alpha^{\frac{1}{2}}).$$

Two Hopf bifurcations occur, both of them are supercritical to stable periodic orbits. However for the corresponding normal form (4.5), if both

α and β are small such that $\alpha > 0$ and $\alpha+\beta < 0$, then the nontrivial fixed point (p^*, q^*) stays stable for any finite $\zeta > 0$ and so there cannot be Hopf bifurcations on these values. Therefore, by using the center manifold reduction and normal form calculation for epidemiological system (4.1), we are lead to an apparent contradiction to Feng's result. For fixed β and small enough $\alpha > 0$ such that $O(\alpha^{\frac{1}{2}}) \longrightarrow 0$ and $O(\alpha^{\frac{3}{2}}) \longrightarrow 0$, the curve for ζ_1 and ζ_2 are shown in Figure 1. Furthermore, if β close to 0, the horizontal line for ζ_2 moves closer to the $\zeta = 0$ axis and the slope of ζ_1 closer to 0. However, since α, $\beta \in O(\epsilon)$ and $\zeta \in O(1)$ in the center manifold-normal form analysis, it has to be above the curve for ζ_2 (e.g. at the point indicated in Figure 2). Thus the nontrivial fixed point for the system (4.1) is always stable for α, $\beta \in O(\epsilon)$ and $\zeta \in O(1)$, where $\alpha > 0$ and $\alpha+\beta < 0$. This can be checked by calculation with DSTOOL ([6]). Therefore there is no contradiction between this analysis and Feng's analysis.

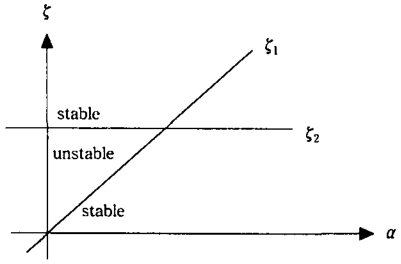

Figure 1.

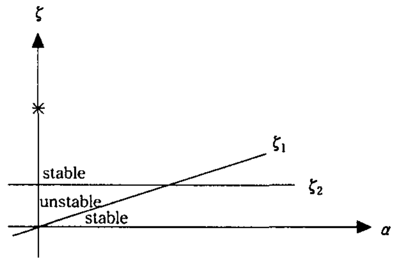

Figure 2.

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