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HIGHER EIGENVALUE ESTIMATE ON MANIFOLD

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ABSTRACT. In this paper we will estimate the lower bound of k-th Dirichlet eigenvalue λ_k of Laplace equation on bounded domain in sphere.

1. Introduction

Let M be an n-dimensional compact Riemannian manifold. In terms of local coordinate $(x^1, x^2, \cdots, x^n)$, the metric can be expressed as $ds^2 = \sum_{i,j=1}^n g_{ij}dx^i dx^j$ and the Laplacian operator is defined by $\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^n \frac{\partial}{\partial x_j}(\sqrt{g} g^{ij} \frac{\partial}{\partial x^i})$. We consider the Laplace equation

$$(1) \qquad \begin{aligned} \Delta u &= -\lambda u \quad in \quad M \\ u &= 0 \qquad on \quad \partial M. \end{aligned}$$

It is well known that the set of eigenvalues consists of a sequence $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \uparrow +\infty$ and each associated eigenspace is finite dimensional. Eigenspace belonging to distinct eigenvalues are orthogonal in $L^2(M)$, and $L^2(M)$ is the direct sum of all the eigenspaces. Furthermore, each eigenfunction is C^∞ on M. One seeks information about the eigenvalues and the eigenfunctions of the Laplacian in terms of geometrical data. It turns out that lower bounds are more interesting from both the mathematical and physical points of view. We will consider the estimate of k-th Dirichlet eigenvalue of Laplace equation. In case of Dirichlet problem for bounded domain Ω in R^n, H.Weyl [4] proved in 1912 the asymptotic formula $\lambda_k \sim C_n(\frac{k}{V})^{\frac{2}{n}}$, $\quad k \to \infty$, where $V =$ Volume of(Ω), $\quad C_n =$

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$\frac{(2\pi)^2(n)^{\frac{2}{n}}}{(\omega_{n-1})^{\frac{2}{n}}}$, $\omega_{n-1} =$ Area(S^{n-1}). Based on this formula, in 1960, Polya [3] conjectured that $\lambda_k \leq C_n(\frac{k}{V})^{\frac{2}{n}}$ for all k. In the case $n = 2$, Polya proved that the conjecture holds for some special planar domains. In 1980, E. Lieb [1] proved that there exists a constant $\tilde{C_n} < C_n$ such that $\lambda_k \geq \tilde{C_n}(\frac{k}{V})^{\frac{2}{n}}$. In 1983, Li-Yau [2] proved that $\lambda_k \geq \frac{n}{n+2}C_n(\frac{k}{V})^{\frac{2}{n}}$. The purpose of this paper is to estimate the lower bound of k-th Dirichlet eigenvalue λ_k on bounded domain in $S^n(r)$.

2. Main theorem

In [2], Li-Yau used Fourier transform to estimate the lower bound of λ_k on bounded domain in R^n. Let M be an n-dimensional compact Riemannian manifold embedded to R^d. Taking the tubular neighborhood of M in R^d, we tried the method of Li-Yau [2].

LEMMA 1. [5] *Given $f : R^{n+1} \to R$ such that $0 \leqq f \leqq M_1$, if $\int_{R^{n+1}} f(z)|z|^2\,dz \leqq M_2$ where M_1, M_2 are constants, then*

$$\int_{R^{n+1}} f(z)\,dz \leqq (M_1\frac{\omega_n}{n+1})^{\frac{2}{n+3}}(M_2)^{\frac{n+1}{n+3}}(\frac{n+3}{n+1})^{\frac{n+1}{n+3}}.$$

LEMMA 2. *Let M be an n-dimensional compact Riemannian manifold and embedded to R^d. Let $\{\eta_j\}_{j=1,2,\cdots,d-n}$ be the orthonormal basis of $(TM)^{\perp}$ in R^d. Let $B_\varepsilon(M) = \{Z \in R^d | Z = X + Y, X \in M, Y = \sum_{j=1}^{d-n}\xi^j\eta_j\}$ be the tubular neighborhood of M in R^d. Then it holds that*

$$\begin{aligned}&\int_{B_\varepsilon(M)} dv\\ &= \int_{U\times B_\varepsilon(R^{d-n})}(1-\lambda_1)\cdots(1-\lambda_n)du^1du^2\cdots du^n d\xi^1 d\xi^2\cdots d\xi^{d-n}.\end{aligned}$$

PROOF. Let $x : U \to M$ be a coordinate chart such that $x(u) = x(u^1, \cdots, u^n) = (x^1, x^2, \cdots, x^d)$. We define $C^\infty - map$ $\Phi : U \times$

$B_\varepsilon(R^{d-n}) \to B_\varepsilon(M)$ by

$$\Phi(u,\xi) = \Phi(u^1, u^2, \cdots, u^n, \xi^1, \xi^2, \cdots, \xi^{d-n}) = x(u) + \sum_{j=1}^{d-n} \xi^j \eta_j(x(u))$$
$$= \sum_{i=1}^{d} (x^i(u) + \sum_{j=1}^{d-n} \xi^j \eta_j^i(x(u))) e_i.$$

On the other hand

$$\int_{B_\varepsilon(M)} dv = \int_{U \times B_\varepsilon(R^{d-n})} \sqrt{\mathrm{Det}({}^t D\Phi D\Phi)} du^1 \cdots du^n d\xi^1 \cdots d\xi^{d-n}.$$

Let ${}^t D\Phi D\Phi = (a_{ij})$. Then we have the following:

$$a_{ij} = \begin{cases} \delta_i^j & \text{for} \quad n+1 \leqq i, \ j \leqq n+1 \\ 0 & \text{for} \quad n+1 \leqq i, \ 1 \leqq j \leqq n \\ 0 & \text{for} \quad 1 \leqq i \leqq n, \ n+1 \leqq j. \end{cases}$$

In particular, if $1 \leqq p, q \leqq n$, we have

$$a_{pq} = \sum_{i=1}^{d} x^i_{,p} x^i_{,q} + \xi^j \eta^i_{j,q} x^i_{,p} + \xi^j \eta^i_{j,p} x^i_{,q} + \sum_{j.s=1}^{k} \xi^j \xi^s \eta^i_{j,p} \eta^i_{s,q}.$$

Using the facts $\langle \eta_j, \ x_{,pq} \rangle = h_{jpq}$, $-h_{jpq} = \langle \eta_{j,p}, \ x_q \rangle = \sum_{l=1}^n h^l_{jp} g_{lq}$, we have $a_{pq} = g_{pq} - 2\sum_{j=1}^k \xi^j h_{jpq} - \sum_{j,s=1}^k \xi^j \xi^s \sum_{l=1}^n h^l_{sq} h_{jpl}$.

Since $\eta = \sum_{j=1}^k \xi^j \eta_j$ is a normal vector, we can choose tangent vectors $\{x_p\}$ pointwisely such that $\langle \eta, \ x_{,pq} \rangle = \sum_{j=1}^k \xi^j h_{jpq} = \lambda_p \delta_p^q$ and $g_{pq} = \delta_p^q$.

Then $a_{pp} = 1 - 2\lambda_p(\xi \, , x) + \lambda_p^2(\xi \, , x) = (1-\lambda_p)^2$ and $\mathrm{Det}({}^t D\Phi D\Phi) = (1-\lambda_1)^2 (1-\lambda_2)^2 \cdots (1-\lambda_n)^2$ where λ_i depends ξ and x.

Hence it holds that

$$\int_{B_\varepsilon(M)} dv$$
$$= \int_{U \times B_\varepsilon(R^{d-n})} (1-\lambda_1)(1-\lambda_2) \cdots (1-\lambda_n) du^1 du^2 \cdots du^n d\xi^1 \cdots d\xi^{d-n}.$$

$\square$

COROLLARY 1. *Let M be a bounded domain in n-dimensional r-sphere $S^n(r)$.*

Let $B_\varepsilon(M) = \{z \in R^{n+1} \mid z = x + \frac{\xi}{r}x, -\varepsilon \leqq \xi \leqq \varepsilon\}$. Then

$$\int_{B_\varepsilon(M)} dv = Vol(M) \int_{-\varepsilon}^{\varepsilon} (1 + \frac{\xi}{r})^n \, d\xi.$$

PROOF. For a coordinate map $\alpha : U \subset R^n \to M$, we define $\Phi : U \times (-\varepsilon, \varepsilon) \to B_\varepsilon(M)$ by $\Phi(u,\xi) = (1+\frac{\xi}{r})\alpha(u)$. Let $ds^2 = \sum_{i=1}^n g_{ij} dx^i dx^j$ be the metric on M and g be the determinant of (g_{ij}). Using $\mathrm{Det}({}^tD\Phi D\Phi) = (1 + \frac{\xi}{r})^{2n} g$, we have

$$\begin{aligned}\int_{B_\varepsilon(M)} dv &= \int_{U\times(-\varepsilon,\varepsilon)} (1 + \frac{\xi}{r})^n \sqrt{g} du d\xi \\ &= \int_U \sqrt{g} du \int_{-\varepsilon}^{\varepsilon} (1 + \frac{\xi}{r})^n d\xi \\ &= \mathrm{Vol}(M) \int_{-\varepsilon}^{\varepsilon} (1 + \frac{\xi}{r})^n d\xi.\end{aligned}$$

□

THEOREM. *Let M be a bounded domain with boundary in n-dimensional sphere of radius r.*

Let λ_k be the k-th Dirichlet eigenvalue of Laplace equation (1). *Then*

$$\begin{aligned}&\sum_{i=1}^k \lambda_i \\ &\geqq k(2\pi)^2 Vol(M)^{-\frac{2}{n+1}} \left(\frac{n+1}{n+3}\right) \left(\frac{n+1}{\omega_n}\right)^{\frac{2}{n+1}} \left(\int_{-\varepsilon}^{\varepsilon} (1 + \frac{\xi}{r})^n \, d\xi\right)^{\frac{-2}{n+1}} \\ &\times \left(\int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1 + \frac{\xi}{r})^n \, d\xi\right) \left(\int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1 + \frac{\xi}{r})^{n-2} \, d\xi\right)^{-1} \\ &-k \left(\int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1 + \frac{\xi}{r})^{n-2} \, d\xi\right)^{-1} \left(\int_{-\varepsilon}^{\varepsilon} \eta'(\xi)^2(1 + \frac{\xi}{r})^n \, d\xi\right)\end{aligned}$$

for some C^∞-function $\eta(\xi) : [-\varepsilon, \varepsilon] \to R$ such that $\eta(0) = 1, \eta(-\varepsilon) = \eta(\varepsilon) = 0$.

PROOF. Let U be an open set in R^n and $\alpha : U \to M$ be a coordinate patch of M. We consider a tubular neighborhood of M, $B_\varepsilon(M)$, which is expressed by $B_\varepsilon(M) = \{x + y \in R^{n+1} | x \in M, y \in (TM)^\perp, \|y\| \leqq \varepsilon\}$. Then there is a submersion $\pi : B_\varepsilon(M) \to M$ such that $\pi(z)$ is a unique closest point from z in M . We define C^∞- map $\psi : U \times (-\varepsilon, \varepsilon) \to B_\varepsilon(M)$ by $\psi(u, \xi) = (1 + \frac{\xi}{r})\alpha(u)$.

Let $\{\phi_i\}_{i=1}^k$ be an orthonormal family of eigenfunctions corresponding to eigenvalues $\{\lambda_i\}_{i=1}^k$. We define C^∞-map $\tilde{\phi_i} : R^{n+1} \to R$ by

$$\tilde{\phi_i}(z) = \begin{cases} \phi_i(\pi(z))\eta(\rho(z)), & \text{for } z \in B_\varepsilon(M), \\ 0, & \text{for } z \in R^{n+1} - B_\varepsilon(M). \end{cases}$$

where $\rho : B_\varepsilon(M) \to R$ is the distance function from z to $\pi(z)$ and $\eta(\xi) : [-\varepsilon, \varepsilon] \to R$ is a C^∞-function such that $\eta(-\varepsilon) = \eta(\varepsilon) = 0$ and $\eta(0) = 1$. Then $\{\tilde{\phi_i}\}$ is an orthogonal family.

Let $\Phi(x, y) = \sum_{i=1}^k \tilde{\phi_i}(x)\tilde{\phi_i}(y)$. Then the Fourier transform of $\Phi(x, y)$ is

$$\begin{aligned} \tilde{\Phi}(z, y) &= (2\pi)^{-\frac{n+1}{2}} \int_{R^{n+1}} \Phi(x, y) e^{i<x,z>}\, dx \\ &= (2\pi)^{-\frac{n+1}{2}} \int_{B_\varepsilon(M)} \Phi(x, y) e^{i<x,z>}\, dx. \end{aligned}$$

By the Planchel formula,

$$\int_{R^{n+1}} |\Phi(x, y)|^2\, dx = \int_{R^{n+1}} |\tilde{\Phi}(z, y)|^2\, dz.$$

We define the function $F : R^{n+1} \to R$ by

$$\begin{aligned} F(z) &= \int_{R^{n+1}} |\tilde{\Phi}(z, y)|^2\, dy \\ &= \int_{R^{n+1}} \left|(2\pi)^{-\frac{n+1}{2}} \int_{R^{n+1}} \Phi(x, y) e^{i\langle x,z\rangle}\, dx\right|^2 dy. \end{aligned}$$

We want to estimate the upper bound of $F(z)$. By Hölder's inequality

$$
\begin{aligned}
&\left|\int_{B_\varepsilon(M)} \Phi(x,y)e^{i\langle x,z\rangle}\,dx\right|^2 \\
&\leqq \int_{B_\varepsilon(M)} |\Phi(x,y)|^2\,dx \int_{B_\varepsilon(M)} |e^{i\langle x,z\rangle}|^2\,dx \\
&\leqq \int_{B_\varepsilon(M)} \sum_{i,j=1}^{k} \tilde\phi_i(x)\tilde\phi_i(y)\tilde\phi_j(x)\tilde\phi_j(y)\,dx\cdot \mathrm{Vol}(M)\cdot\int_{-\varepsilon}^{\varepsilon}(1+\frac{\xi}{r})^n\,d\xi \\
&=\sum_{i=1}^{k}\tilde\phi_i^2(y)\int_{-\varepsilon}^{\varepsilon}\eta^2(\xi)(1+\frac{\xi}{r})^n\,d\xi\cdot\mathrm{Vol}(M)\cdot\int_{-\varepsilon}^{\varepsilon}(1+\frac{\xi}{r})^n\,d\xi.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
F(z) &= (2\pi)^{-(n+1)}\int_{B_\varepsilon(M)}\left|\int_{B_\varepsilon(M)}\Phi(x,y)e^{i<x,z>}\,dx\right|^2 dy \\
&\leqq (2\pi)^{-(n+1)}k\left(\int_{-\varepsilon}^{\varepsilon}\eta^2(\xi)(1+\frac{\xi}{r})^n\,d\xi\right)^2\cdot\mathrm{Vol}(M)\cdot\int_{-\varepsilon}^{\varepsilon}(1+\frac{\xi}{r})^n\,d\xi.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\int_{R^{n+1}} F(z)\,dz &= \int_{R^{n+1}}\int_{R^{n+1}}|\hat\Phi(z,y)|^2\,dydz \\
&= \int_{R^{n+1}}\int_{R^{n+1}}|\Phi(x,y)|^2 dxdy \\
&= \int_{B_\varepsilon(M)}\int_{B_\varepsilon(M)}\sum_{i,j=1}^{k}\tilde\phi_i(x)\tilde\phi_i(y)\tilde\phi_j(x)\tilde\phi_j(y)\,dxdy \\
&= k\left(\int_{-\varepsilon}^{\varepsilon}\eta^2(\xi)(1+\frac{\xi}{r})^n\,d\xi\right)^2.
\end{aligned}
$$

Using the fact $(\widehat{i\frac{\partial}{\partial x_j}\Phi})(z,y) = z_j\hat\Phi(z,y)$ and the Planchel formula, we calculate that

$$\int_{R^{n+1}} |z|^2 F(z)\, dz = \int_{R^{n+1}} \int_{R^{n+1}} |z|^2 |\hat{\Phi}(z,y)|^2\, dydz$$

$$= \int_{R^{n+1}} \int_{R^{n+1}} \sum_{l=1}^{n+1} z_l \hat{\Phi} z_l \hat{\Phi}\, dzdy$$

$$= \int_{R^{n+1}} \int_{R^{n+1}} |\nabla_x \Phi(x,y)|^2\, dydx$$

$$= \int_{B_\varepsilon(M)} \int_{B_\varepsilon(M)} |\nabla_x \Phi|^2\, dxdy$$

$$= \int_{B_\varepsilon(M)} \sum_{i,j=1}^{k} \tilde{\phi}_i(y)\tilde{\phi}_j(y)\, dy \cdot \int_{B_\varepsilon(M)} \sum_{l,q=1}^{n+1} a^{lq} \partial_l \tilde{\phi}_i(x) \partial_q \tilde{\phi}_j(x)\, dx.$$

$$= \sum_{i,j=1}^{k} \int_M \phi_i(\pi(y))\phi_j(\pi(y))\, dV_M \int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1+\frac{\xi}{r})^n\, d\xi$$

$$\cdot \int_{B_\varepsilon(M)} \left(\sum_{l,q=1}^{n} g^{lq}(1+\frac{\xi}{r})^{-2} \partial_l\phi_i \partial_q \phi_j \eta^2(\xi) + \phi_i\phi_j \eta'(\xi)^2 \right) dV$$

$$= \sum_{i=1}^{k} \int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1+\frac{\xi}{r})^n\, d\xi \int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1+\frac{\xi}{r})^{n-2}\, d\xi \int_M |\nabla \phi_i|^2\, dV_M$$

$$+ \sum_{i=1}^{k} \int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1+\frac{\xi}{r})^n\, d\xi \int_{-\varepsilon}^{\varepsilon} \eta'(\xi)^2(1+\frac{\xi}{r})^n\, d\xi$$

$$= -\sum_{i=1}^{k} \int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1+\frac{\xi}{r})^n\, d\xi \int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1+\frac{\xi}{r})^{n-2}\, d\xi \int_M \phi_i \Delta \phi_i\, dV_M$$

$$+ k \int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1+\frac{\xi}{r})^n\, d\xi \int_{-\varepsilon}^{\varepsilon} \eta'(\xi)^2(1+\frac{\xi}{r})^n\, d\xi$$

$$= \sum_{i=1}^{k} \lambda_i \int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1+\frac{\xi}{r})^n\, d\xi \int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1+\frac{\xi}{r})^{n-2}\, d\xi$$

$$+ k \int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1+\frac{\xi}{r})^n\, d\xi \int_{-\varepsilon}^{\varepsilon} \eta'(\xi)^2(1+\frac{\xi}{r})^n\, d\xi.$$

By Lemma 1, we have

$$
\begin{aligned}
&\sum_{i=1}^{k} \lambda_i \\
&\geqq k(2\pi)^2 \mathrm{Vol}(M)^{-\frac{2}{n+1}} \left(\frac{n+1}{n+3}\right) \left(\frac{n+1}{\omega_n}\right)^{\frac{2}{n+1}} \left(\int_{-\varepsilon}^{\varepsilon} (1+\frac{\xi}{r})^n \, d\xi\right)^{\frac{-2}{n+1}} \\
&\times \left(\int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1+\frac{\xi}{r})^n \, d\xi\right) \left(\int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1+\frac{\xi}{r})^{n-2} \, d\xi\right)^{-1} \\
&-k \left(\int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1+\frac{\xi}{r})^{n-2} \, d\xi\right)^{-1} \left(\int_{-\varepsilon}^{\varepsilon} \eta\prime(\xi)^2(1+\frac{\xi}{r})^n \, d\xi\right)
\end{aligned}
$$

for some C^∞-function $\eta(\xi) : [-\varepsilon, \varepsilon] \to R$ such that $\eta(0) = 1, \eta(-\varepsilon) = \eta(\varepsilon) = 0$. $\square$

COROLLARY 2. *Let M and λ_k satisfy the hypothesis of Main Theorem. Then*

$$
\begin{aligned}
\lambda_k \geqq\ & (2\pi)^2 Vol(M)^{-\frac{2}{n+1}} \left(\frac{n+1}{n+3}\right) \left(\frac{n+1}{\omega_n}\right)^{\frac{2}{n+1}} \left(\int_{-\varepsilon}^{\varepsilon} (1+\frac{\xi}{r})^n \, d\xi\right)^{\frac{-2}{n+1}} \\
&\times \left(\int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1+\frac{\xi}{r})^n \, d\xi\right) \left(\int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1+\frac{\xi}{r})^{n-2} \, d\xi\right)^{-1} \\
&- \left(\int_{-\varepsilon}^{\varepsilon} \eta^2(\xi)(1+\frac{\xi}{r})^{n-2} \, d\xi\right)^{-1} \left(\int_{-\varepsilon}^{\varepsilon} \eta\prime(\xi)^2(1+\frac{\xi}{r})^n \, d\xi\right).
\end{aligned}
$$

COROLLARY 3. *Let r be bigger than $\frac{1}{3}\pi^4 - \frac{3}{5}(\frac{3}{4})^{\frac{4}{3}}\pi^{\frac{2}{3}}(\pi^2 - 1)$. Let M be a bounded domain in 2-dimensional r-sphere $S^2(r)$.*

Let λ_k be the k-th Dirichlet eigenvalue of Laplace equation (1). Then

$$
\lambda_k \geqq c\Big(\frac{1}{r\,Vol(M)}\Big)^{\frac{2}{3}} - \frac{4}{9}\frac{\pi^2}{r^2} + \frac{4}{3}\frac{1}{\pi^2 r}
$$

where $c = \frac{16}{5}(\frac{3}{8})^{\frac{2}{3}}(\pi^2 - 1)(\frac{3}{\omega_2})^{\frac{2}{3}}$.

PROOF. We define C^∞-function $\eta(x) : [-\varepsilon, \varepsilon] \to R$ by $\eta(x) = \frac{1}{2}cos\frac{\pi}{\varepsilon}x + \frac{1}{2}$. Then $\eta(x)$ satisfies that $\eta(0) = 1$ and $\eta(-\varepsilon) = \eta(\varepsilon) = 1$. Also, in r-sphere $S^n(r)$, we can take $\varepsilon = r$. By Corollary 2, it holds that $\lambda_k \geqq \frac{16}{5}(\frac{3}{8})^{\frac{2}{3}}(\pi^2 - 1)(\frac{3}{\omega_2})^{\frac{2}{3}}(\frac{1}{r\mathrm{Vol}(M)})^{\frac{2}{3}} - \frac{4}{9}\frac{\pi^2}{r^2} + \frac{4}{3}\frac{1}{\pi^2 r}$. Since $\omega_2 = 4\pi$ and $\mathrm{Vol}(M) \leqq 4\pi r^2$, we have the conclusion. □

REMARK. In r-sphere $S^n(r)$, it is possible to take $\varepsilon = r$. Therefore, in Corollary 2, the lower bound of λ_k depends on the curvature $\frac{1}{r}$ of M, the volume of M and k .

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