Comm. Korean Math. Soc. **13** (1998), No. 3, pp. 561–577

ON CONTACT THREE CR SUBMANIFOLDS OF A $(4m+3)$-DIMENSIONAL UNIT SPHERE

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Abstract. We study $(n+3)$-dimensional contact three CR submanifolds of a Riemannian manifold with Sasakian three structure and investigate some characterizations of $S^{4r+3}(a) \times S^{4s+3}(b)$ $(a^2+b^2 = 1,\ 4(r+s) = n-3)$ as a contact three CR submanifold of a $(4m+3)$-dimensional unit sphere.

1. Introduction

Let S^{4m+3} be a $(4m+3)$-dimensional unit sphere, that is,

$$S^{4m+3} = \{q \in Q^{m+1} \ : \ \|q\| = 1\},$$

where Q^{m+1} is the real $4(m+1)$-dimensional quaternionic number space. For any point q in S^{4m+3}, we put

$$\xi = Iq, \quad \eta = Jq, \quad \zeta = Kq,$$

where $\{I, J, K\}$ denotes the canonical quaternionic Kähler structure of Q^{m+1}. Then $\{\xi, \eta, \zeta\}$ becomes a Sasakian three structure, that is, ξ, η and ζ are mutually orthogonal unit Killing vector fields which satisfy

$$\begin{cases} \overline{\nabla}_Y\overline{\nabla}_X\xi = g(X,\xi)Y - g(Y,X)\xi, \\ \overline{\nabla}_Y\overline{\nabla}_X\eta = g(X,\eta)Y - g(Y,X)\eta, \\ \overline{\nabla}_Y\overline{\nabla}_X\zeta = g(X,\zeta)Y - g(Y,X)\zeta \end{cases} \tag{1.1}$$

Received February 14, 1998. Revised May 19, 1998.

1991 Mathematics Subject Classification: 53C40, 53C15.

Key words and phrases: contact three CR submanifold, contact three CR dimension, Sasakian three structure, $(4m+3)$-dimensional unit sphere.

The first author was supported (in part) by the Taegu University Research Grant and the second by BSRI-98-1404 and TGRC-KOSEF.

for any vector fields X, Y tangent to S^{4m+3}, where g denotes the canonical metric on S^{4m+3} induced from that of Q^{m+1} and $\overline{\nabla}$ the Riemannian connection with respect to g. In this case, putting

$$\phi X = \overline{\nabla}_X \xi, \quad \psi X = \overline{\nabla}_X \eta, \quad \theta X = \overline{\nabla}_X \zeta, \tag{1.2}$$

it follows that

$$\begin{aligned} &\phi\xi = 0, \ \psi\eta = 0, \ \theta\zeta = 0, \\ &\theta\eta = -\psi\zeta = \xi, \ \phi\zeta = -\theta\xi = \eta, \ \psi\xi = -\phi\eta = \zeta, \\ &[\eta, \zeta] = 2\xi, \ [\zeta, \xi] = 2\eta, \ [\xi, \eta] = 2\zeta \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} &\phi^2 = -I + f_\xi \otimes \xi, \quad \psi^2 = -I + f_\eta \otimes \eta, \quad \theta^2 = -I + f_\zeta \otimes \zeta, \\ &\psi\theta = \phi + f_\eta \otimes \zeta, \quad \theta\phi = \psi + f_\zeta \otimes \xi, \quad \phi\psi = \theta + f_\xi \otimes \eta, \\ &\theta\psi = -\phi + f_\zeta \otimes \eta, \quad \phi\theta = -\psi + f_\xi \otimes \zeta, \quad \psi\phi = -\theta + f_\eta \otimes \xi, \end{aligned} \tag{1.4}$$

where I denotes the identity transformation and

$$f_\xi(X) = g(X, \xi), \quad f_\eta(X) = g(X, \eta), \quad f_\zeta(X) = g(X, \zeta) \tag{1.5}$$

(cf. [4,5,7,8,10]). Moreover, from (1.3) and (1.4), we have

$$\begin{aligned} &\mathcal{L}_\xi\phi = 0, \quad \mathcal{L}_\eta\phi = -2\theta, \quad \mathcal{L}_\zeta\phi = 2\psi, \\ &\mathcal{L}_\xi\psi = 2\theta, \quad \mathcal{L}_\eta\psi = 0, \quad \mathcal{L}_\zeta\psi = -2\phi, \\ &\mathcal{L}_\xi\theta = -2\psi, \quad \mathcal{L}_\eta\theta = 2\phi, \quad \mathcal{L}_\zeta\theta = 0, \end{aligned} \tag{1.6}$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to X.

Let M be an $(n+3)$-dimensional submanifold tangent to the structure vectors ξ, η and ζ of S^{4m+3}. If there exists a subbundle ν of the normal bundle $TM^\perp$ such that

$$\phi\nu_x \subset \nu_x, \quad \psi\nu_x \subset \nu_x, \quad \theta\nu_x \subset \nu_x, \tag{1.7}$$

$$\phi\nu_x^\perp \subset T_xM, \quad \psi\nu_x^\perp \subset T_xM, \quad \theta\nu_x^\perp \subset T_xM \tag{1.8}$$

for each x in M, where $\nu^\perp$ is the complementary orthogonal subbundle to ν in $TM^\perp$ and TM the tangent bundle of M, then the submanifold is called a *contact three CR submanifold* of S^{4m+3} and the dimension of ν *contact three CR dimension*. A typical example of contact three CR submanifold with zero contact three CR dimension is a real hypersurface.

In this paper we shall study $(n+3)$-dimensional contact three CR submanifolds with $(p-1)$ contact three CR dimension of S^{4m+3}, where p is $4m-n$ the codimension. In this case the maximal $\{\phi, \psi, \theta\}$-invariant subspace

$$\mathcal{D}_x = T_xM \cap \phi T_xM \cap \psi T_xM \cap \theta T_xM$$

of T_xM has constant dimension $n-3$ because the orthogonal complement $\mathcal{D}_x^\perp$ to $\mathcal{D}_x$ in T_xM has constant dimension 6 at any point x in M (for details, see section 2).

We shall investigate some geometric characterizations of

$$S^{4r+3}(a) \times S^{4s+3}(b) \ (a^2 + b^2 = 1, \ r + s = (n-3)/4)$$

as a contact three CR submanifold of a $(4m+3)$-dimensional unit sphere.

2. Preliminaries

Let M be an $(n+3)$-dimensional contact three CR submanifold in a $(4m+3)$-dimensional Riemannian manifold $\overline{M}$ with Sasakian three structure $\{\xi, \eta, \zeta\}$ which satisfies (1.1). Then, by definition, we may set $\nu^\perp = Span\ \{N_1\}$ for a unit normal vector field N_1 to M. Here and in the sequel we use the same notations as shown in section 1. Put

$$U = -\phi N_1, \quad V = -\psi N_1, \quad W = -\theta N_1. \tag{2.1}$$

Then from (1.3), (1.4) and (1.8) we can see that U, V, W are mutually orthogonal unit tangent vector fields to M and satisfy

$$\begin{aligned} &g(\xi, U) = 0, \quad g(\xi, V) = 0, \quad g(\xi, W) = 0, \\ &g(\eta, U) = 0, \quad g(\eta, V) = 0, \quad g(\eta, W) = 0, \\ &g(\zeta, U) = 0, \quad g(\zeta, V) = 0, \quad g(\zeta, W) = 0. \end{aligned} \tag{2.2}$$

Moreover ξ, η, ζ, U, V and W are all contained in $\mathcal{D}_x^\perp$ and consequently $dim\ \mathcal{D}_x^\perp \geq 6$ at any point x in M. But we can prove that $dim\ \mathcal{D}_x^\perp = 6$ at any point x in M. In fact, if there is a non-zero vector $S \in \mathcal{D}_x^\perp$ which is orthogonal to all of ξ, η, ζ, U, V and W, then it is clear that $g(U, S) = 0$ and thus $g(N_1, \phi S) = 0$ because of (2.1). Hence, if $\phi S \in T_xM^\perp$, then $\phi S \in \nu_x$ and consequently it follows from (1.7) that $S \in \nu_x$ which is a contradiction. So $\phi S \in T_xM$. Similarly we can prove that ϕS, ψS, $\theta S \in T_xM$, let say

$$Z_1 := \phi S, \quad Z_2 := \psi S, \quad Z_3 := \theta S.$$

Then $\phi Z_1 = \psi Z_2 = \theta Z_3 = -S$ and consequently $S \in \mathcal{D}_x$, which is also a contradiction. Hence we have $dim\ \mathcal{D}_x^\perp = 6$. Therefore, for any tangent vector field X and for a local orthonormal basis $\{N_\alpha\}_{\alpha=1,\ldots,p}$ $(p = 4m - n)$ of normal vectors to M, we have the following decomposition in tangential and normal components:

$$\phi X = FX + u^1(X)N_1, \quad \psi X = GX + v^1(X)N_1, \quad \theta X = HX + w^1(X)N_1, \tag{2.3}$$

$$\phi N_\alpha = -U_\alpha + P_\phi N_\alpha, \quad \psi N_\alpha = -V_\alpha + P_\psi N_\alpha, \quad \theta N_\alpha = -W_\alpha + P_\theta N_\alpha, \quad \alpha = 1, \ldots, p. \tag{2.4}$$

It follows easily from (1.4) that $\{F, G, H\}$ and $\{P_\phi, P_\psi, P_\theta\}$ are respectively skew-symmetric linear endomorphisms acting on T_xM and $T_xM^\perp$. Since the Sasakian three structure $\{\xi, \eta, \zeta\}$ is tangent to M, the equations (1.4), (2.3) and (2.4) imply

$$\begin{cases} F^2X = -X + f_\xi(X)\xi + u^1(X)U_1, & u^1(FX) = 0, \\ G^2X = -X + f_\eta(X)\eta + v^1(X)V_1, & v^1(GX) = 0, \\ H^2X = -X + f_\zeta(X)\zeta + w^1(X)W_1, & w^1(HX) = 0, \end{cases} \tag{2.5}$$

$$\begin{aligned} GFX &= -HX + f_\eta(X)\xi + u^1(X)V_1, \quad v^1(FX) = -w^1(X), \\ HFX &= GX + f_\zeta(X)\xi + u^1(X)W_1, \quad w^1(FX) = v^1(X), \\ FGX &= HX + f_\xi(X)\eta + v^1(X)U_1, \quad u^1(GX) = w^1(X), \\ HGX &= -FX + f_\zeta(X)\eta + v^1(X)W_1, \quad w^1(GX) = -u^1(X), \\ FHX &= -GX + f_\xi(X)\zeta + w^1(X)U_1, \quad u^1(HX) = -v^1(X), \\ GHX &= FX + f_\eta(X)\zeta + w^1(X)V_1, \quad v^1(HX) = u^1(X), \end{aligned} \tag{2.6}$$

$$
\begin{aligned}
(2.7)\qquad g(U_\alpha, X) &= u^1(X)\delta_{1\alpha}, \quad g(V_\alpha, X) = v^1(X)\delta_{1\alpha},\\
&g(W_\alpha, X) = w^1(X)\delta_{1\alpha}, \quad \alpha = 1, \dots, p,
\end{aligned}
$$

which yields

$$
(2.8)\qquad \begin{aligned}
&g(U_1, X) = u^1(X),\ g(V_1, X) = v^1(X),\ g(W_1, X) = w^1(X),\\
&U_\alpha = 0,\ V_\alpha = 0,\ W_\alpha = 0,\ \alpha = 2, \dots, p,
\end{aligned}
$$

$$
(2.9)\qquad \begin{cases}
g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - g(P_\phi N_\alpha, P_\phi N_\beta),\\
g(V_\alpha, V_\beta) = \delta_{\alpha\beta} - g(P_\psi N_\alpha, P_\psi N_\beta),\\
g(W_\alpha, W_\beta) = \delta_{\alpha\beta} - g(P_\theta N_\alpha, P_\theta N_\beta).
\end{cases}
$$

From (1.3) and (2.3), it follows that

$$
(2.10)\qquad \begin{aligned}
&F\xi = 0,\ G\eta = 0,\ H\zeta = 0,\ F\eta = -\zeta,\ F\zeta = \eta,\\
&G\xi = \zeta,\ G\zeta = -\xi,\ H\xi = -\eta,\ H\eta = \xi,\\
&u^1(\xi) = 0,\ u^1(\eta) = 0,\ u^1(\zeta) = 0,\ v^1(\xi) = 0,\ v^1(\eta) = 0,\\
&v^1(\zeta) = 0,\ w^1(\xi) = 0,\ w^1(\eta) = 0,\ w^1(\zeta) = 0.
\end{aligned}
$$

Using (1.4) and (2.1)-(2.4), we have

$$
(2.11)\qquad \begin{aligned}
&FU_1 = 0,\ GV_1 = 0,\ HW_1 = 0,\ FV_1 = W_1,\\
&FW_1 = -V_1,\ GU_1 = -W_1,\ GW_1 = U_1,\ HU_1 = V_1,\\
&HV_1 = -U_1,\ P_\phi N_1 = 0,\ P_\psi N_1 = 0,\ P_\theta N_1 = 0,
\end{aligned}
$$

which together with (2.1) and (2.4) implies

$$
U = U_1, \quad V = V_1, \quad W = W_1.
$$

Therefore we may put

$$
\begin{aligned}
(2.12)\qquad P_\phi N_\alpha &= \sum_{\beta=2}^{p} P^\phi_{\alpha\beta} N_\beta, \quad P_\psi N_\alpha = \sum_{\beta=2}^{p} P^\psi_{\alpha\beta} N_\beta,\\
&P_\theta N_\alpha = \sum_{\beta=2}^{p} P^\theta_{\alpha\beta} N_\beta, \quad \alpha = 2, \dots, p,
\end{aligned}
$$

where $(P^{\phi}_{\alpha\beta})$, $(P^{\psi}_{\alpha\beta})$ and $(P^{\theta}_{\alpha\beta})$ are skew-symmetric matrices which satisfy

$$(2.13)\qquad \sum_{\gamma=2}^{p} P^{\phi}_{\alpha\gamma}P^{\phi}_{\gamma\beta} = -\delta_{\alpha\beta}, \quad \sum_{\gamma=2}^{p} P^{\psi}_{\alpha\gamma}P^{\psi}_{\gamma\beta} = -\delta_{\alpha\beta},$$
$$\sum_{\gamma=2}^{p} P^{\theta}_{\alpha\gamma}P^{\theta}_{\gamma\beta} = -\delta_{\alpha\beta}.$$

3. Fundamental equations for contact three CR submanifold

Let M be as in section 2. We denote by ∇ the Levi-Civita connection on M and denote by D the normal connection induced from $\overline{\nabla}$ in $TM^{\perp}$. Then the Gauss and Weingarten equations are of the form

$$(3.1)\qquad \overline{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$(3.2)\qquad \overline{\nabla}_X N_\alpha = -A_\alpha X + D_X N_\alpha, \quad \alpha = 1,\dots,p$$

for any tangent vector fields X, Y to M. Here h denotes the second fundamental form and A_α is the shape operator corresponding to N_α. They are related by

$$h(X,Y) = \sum_{\alpha=1}^{p} g(A_\alpha X, Y)N_\alpha.$$

Furthermore we may put

$$(3.3)\qquad D_X N_\alpha = \sum_{\beta=1}^{p} s_{\alpha\beta}(X)N_\beta,$$

where $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of D. Finally the equations of Gauss, Codazzi and Ricci (cf. [2,6]) are given by

$$(3.4)\qquad g(\overline{R}_{XY}Z,W) = g(R_{XY}Z,W) + \sum_{\alpha}\{g(A_\alpha X,Z)g(A_\alpha Y,W) - g(A_\alpha Y,Z)g(A_\alpha X,W)\},$$

$$(3.5) \qquad g(\overline{R}_{XY}Z, N_\alpha) = g((\nabla_X A_\alpha)Y - (\nabla_Y A_\alpha)X, Z) + \sum_\beta \{g(A_\beta Y, Z)s_{\beta\alpha}(X) - g(A_\beta X, Z)s_{\beta\alpha}(Y)\},$$

$$(3.6) \qquad g(\overline{R}_{XY}N_\alpha, N_\beta) = g(R^\perp_{XY}N_\alpha, N_\beta) + g([A_\beta, A_\alpha]X, Y)$$

for any tangent vector fields X, Y, Z to M, where $\overline{R}$ and R denote the Riemannian curvature tensor of $\overline{M}$ and M respectively and $R^\perp$ is the curvature tensor of the normal connection D.

Differentiating (2.3) covariantly and using (1.1), (1.2), (2.8), (2.11), (3.1) and (3.2), we have

$$(3.7) \qquad \begin{aligned} &(\nabla_Y F)X = g(X,\xi)Y - g(X,Y)\xi - g(A_1X,Y)U + u^1(X)A_1Y, \\ &(\nabla_Y u^1)X = -g(A_1FX, Y), \end{aligned}$$

$$(3.8) \qquad \begin{aligned} &(\nabla_Y G)X = g(X,\eta)Y - g(X,Y)\eta - g(A_1X,Y)V + v^1(X)A_1Y, \\ &(\nabla_Y v^1)X = -g(A_1GX, Y), \end{aligned}$$

$$(3.9) \qquad \begin{aligned} &(\nabla_Y H)X = g(X,\zeta)Y - g(X,Y)\zeta - g(A_1X,Y)W + w^1(X)A_1Y, \\ &(\nabla_Y w^1)X = -g(A_1HX, Y). \end{aligned}$$

Differentiating (2.1) covariantly and using (1.1), (1.2), (2.8) and (3.1)-(3.3), we have

$$(3.10) \qquad \begin{cases} \nabla_X U = FA_1X, \\ g(A_\alpha U, X) = -\sum_{\beta=2}^p s_{1\beta}(X)P^\phi_{\beta\alpha}, \quad \alpha = 2, \ldots, p, \end{cases}$$

$$(3.11) \qquad \begin{cases} \nabla_X V = GA_1X, \\ g(A_\alpha V, X) = -\sum_{\beta=2}^p s_{1\beta}(X)P^\psi_{\beta\alpha}, \quad \alpha = 2, \ldots, p, \end{cases}$$

$$(3.12) \qquad \begin{cases} \nabla_X W = HA_1X, \\ g(A_\alpha W, X) = -\sum_{\beta=2}^p s_{1\beta}(X)P^\theta_{\beta\alpha}, \quad \alpha = 2, \ldots, p. \end{cases}$$

On the other hand, since ξ, η and ζ are tangent to M, it follows from (1.2) that

$$\left\{\begin{array}{l} \nabla_X\xi = FX, \\ g(A_1\xi, X) = u^1(X), \quad \text{that is,} \quad A_1\xi = U, \\ A_\alpha\xi = 0, \quad \alpha = 2, \ldots, p, \end{array}\right. \tag{3.13}$$

$$\left\{\begin{array}{l} \nabla_X\eta = GX, \\ g(A_1\eta, X) = v^1(X), \quad \text{that is,} \quad A_1\eta = V, \\ A_\alpha\eta = 0, \quad \alpha = 2, \ldots, p, \end{array}\right. \tag{3.14}$$

$$\left\{\begin{array}{l} \nabla_X\zeta = HX, \\ g(A_1\zeta, X) = w^1(X), \quad \text{that is,} \quad A_1\zeta = W, \\ A_\alpha\zeta = 0, \quad \alpha = 2, \ldots, p. \end{array}\right. \tag{3.15}$$

In the rest of this paper *we suppose that $\overline{M}$ is of constant curvature* 1 *and that N_1 is parallel with respect to the normal connection D.* Hence it follows from (3.3) that

$$s_{1\beta} = 0, \quad \beta = 2, \ldots, p, \tag{3.16}$$

which together with (3.10)-(3.12) implies

$$A_\alpha U = 0, \quad A_\alpha V = 0, \quad A_\alpha W = 0, \quad \alpha = 2, \ldots, p. \tag{3.17}$$

Since the curvature tensor $\overline{R}$ of $\overline{M}$ is of the form

$$\overline{R}_{XY}Z = g(Y, Z)X - g(X, Z)Y,$$

the equations (3.5) and (3,16) give

$$(\nabla_X A_1)Y - (\nabla_Y A_1)X = 0. \tag{3.18}$$

4. Some properties of the shape operator A_1

In this section we assume that A_1 and $\{F, G, H\}$ are commute on M, that is,

$$A_1F = FA_1, \quad A_1G = GA_1, \quad A_1H = HA_1. \tag{4.1}$$

Then (2.11) and (4.1) yield

$$FA_1U = 0, \quad GA_1V = 0, \quad HA_1W = 0,$$

from which together with (2.5) and (3.13)-(3.15), it follows that

$$A_1U = \xi + \lambda_1 U, \quad A_1V = \eta + \lambda_2 V,$$
$$A_1W = \zeta + \lambda_3 W,$$

where $\lambda_1 = u^1(A_1U) = g(A_1U, U)$, $\lambda_2 = v^1(A_1V) = g(A_1V, V)$ and $\lambda_3 = w^1(A_1W) = g(A_1W, W)$. By the way, from (2.6) and (4.1) we have

$$\lambda_1 = u^1(A_1U) = v^1(HA_1U) = v^1(A_1HU),$$

which and (2.11) imply $\lambda_1 = v^1(A_1V) = \lambda_2$. Similarly we have $\lambda_1 = \lambda_2 = \lambda_3$ and consequently

$$A_1U = \xi + \lambda U, \quad A_1V = \eta + \lambda V, \quad A_1W = \zeta + \lambda W, \tag{4.2}$$

where here and in the sequel we put $\lambda = \lambda_1$.

Differentiating (4.2) covariantly and using (3.10) and (3.13), we obtain

$$(\nabla_X A_1)U + A_1FA_1X = FX + (X\lambda)U + \lambda FA_1X$$

and therefore

$$\begin{aligned} g((\nabla_X A_1)Y, U) =& g(A_1^2X, FY) + g(FX, Y) \\ &+ (X\lambda)g(U, Y) + \lambda g(FA_1X, Y) \end{aligned} \tag{4.3}$$

with the aid of (4.1). Moreover, from (3.18), (4.1) and (4.3), it follows that

$$\begin{aligned} &2g(A_1^2X, FY) + 2g(FX, Y) \\ &+ (X\lambda)g(U, Y) - (Y\lambda)g(U, X) + 2\lambda g(FA_1X, Y) = 0. \end{aligned} \tag{4.4}$$

Putting $X = U$ or $Y = U$ in (4.4), we may have

$$X\lambda = (U\lambda)u^1(X), \quad Y\lambda = (U\lambda)u^1(Y) \tag{4.5}$$

because of (2.11) and (4.1), and hence (4.4) reduces to

$$FA_1^2X = \lambda FA_1X + FX. \tag{4.6}$$

On the other hand, from the first equation of (4.5), it is clear that

$$\nabla_X(grad\ \lambda) = (X\mu)U + \mu FA_1X,$$

where $\mu = U\lambda$. Since $g(\nabla_X(grad\ \lambda), Y) = g(\nabla_Y(grad\ \lambda), X)$, we have

$$(X\mu)u^1(Y) - (Y\mu)u^1(X) + 2\mu g(FA_1X, Y) = 0, \tag{4.7}$$

from which, putting $X = U$ or $Y = U$, we find

$$X\mu = (U\mu)u^1(X), \quad Y\mu = (U\mu)u^1(Y).$$

Hence (4.7) gives $\mu FA_1X = 0$, from which together with (4.1) and (4.6) we can easily see that μ must be zero and λ is constant. Now we prove

LEMMA 4.1. *Let M be an $(n+3)$-dimensional contact three CR submanifold with $(p-1)$ contact three CR dimension in an $(n+p+3)$-dimensional Riemannian manifold with Sasakian three structure and of constant curvature 1. If*

$$A_1F = FA_1, \quad A_1G = GA_1, \quad A_1H = HA_1$$

and N_1 is parallel with respect to the normal connection, then

$$A_1^2 = \lambda A_1 + I, \tag{4.8}$$

$$\nabla A_1 = 0, \tag{4.9}$$

where $\lambda = u^1(A_1U)$ is constant.

PROOF. Applying F to (4.6) and using (2.5), (3.13) and (4.2), we have

$$A_1^2X = \lambda A_1X + X, \tag{4.10}$$

which implies (4.8). Next, differentiating (4.10) covariantly and using the fact that λ is constant, we have

$$(\nabla_Y A_1)A_1X + A_1(\nabla_Y A_1)X = \lambda(\nabla_Y A_1)X, \tag{4.11}$$

from which, taking account of (3.18), we obtain

$$(\nabla_Y A_1)A_1X = (\nabla_X A_1)A_1Y,$$

and consequently

$$g((\nabla_Y A_1)A_1X, Z) = g((\nabla_X A_1)A_1Y, Z) = g(A_1(\nabla_X A_1)Z, Y).$$

Since $g((\nabla_Y A_1)A_1X,\ Z) = g((\nabla_Z A_1)A_1X,\ Y)$, the above equation yields $g((\nabla_Y A_1)A_1X, Z) = g(A_1(\nabla_X A_1)Y, Z)$, which implies

$$(\nabla_Y A_1)A_1X = A_1(\nabla_Y A_1)X.$$

Thus (4.11) reduces to $2A_1(\nabla_Y A_1)X = \lambda(\nabla_Y A_1)X$, from which, applying A_1 and using (4.8), we have $\frac{\lambda^2+4}{2}(\nabla_Y A_1)X = 0$ and consequently (4.9) follows. □

Let ρ be an eigenvalue of A_1. Then from (4.8) it is clear that ρ satisfies $\rho^2 - \lambda\rho - 1 = 0$ and consequently A_1 has exactly two constant eigenvalues

$$\rho_1 = (\lambda + \sqrt{\lambda^2+4})/2, \quad \rho_2 = (\lambda - \sqrt{\lambda^2+4})/2.$$

In fact, since ${\rho_k}^2 - \lambda\rho_k - 1 = 0$ $(k = 1,2)$, (3.13)-(3.15) and (4.2) imply

$$\begin{aligned} A_1(\rho_1U + \xi) &= \rho_1(\rho_1U + \xi), & A_1(\rho_2U + \xi) &= \rho_2(\rho_2U + \xi), \\ A_1(\rho_1V + \eta) &= \rho_1(\rho_1V + \eta), & A_1(\rho_2V + \eta) &= \rho_2(\rho_2V + \eta), \\ A_1(\rho_1W + \zeta) &= \rho_1(\rho_1W + \zeta), & A_1(\rho_2W + \zeta) &= \rho_2(\rho_2W + \zeta). \end{aligned}$$

Since the eigenvalues are constant, the eigenspaces define distributions on M. We denote them by T_k for $k = 1,2$, that is,

$$T_k = \{X \in TM \ :\ A_1X = \rho_kX\}.$$

By means of (4.9), we can easily see that the distribution T_1, T_2 are both involutive and that the integral submanifolds M_k of T_k, $k = 1,2$, are totally geodesic and parallel along T_j, $j \neq k$.

5. Main results

In this section let $\overline{M}$ be the $(4m+3)$-dimensional unit sphere S^{4m+3}, as already shown in section 1, in $\mathbb{R}^{4(m+1)}$ with center at the origin O of $\mathbb{R}^{4(m+1)}$. Let $\widetilde{N}$ be the inward unit normal to S^{4m+3}. Then the equations of Gauss and Weingarten for S^{4m+3} are given by

$$\widetilde{\nabla}_X Y = \overline{\nabla}_X Y + g(X,Y)\widetilde{N}, \tag{5.1}$$

$$\widetilde{\nabla}_X \widetilde{N} = -X \tag{5.2}$$

for any vector fields X, Y tangent to S^{4m+3}, where $\widetilde{\nabla}$ denotes the Euclidean connection of $\mathbb{R}^{4(m+1)}$. In particular, for $Y = N_1$ it follows from (3.2) and (3.3) with $\alpha = 1$, (3.16) and (5.1) that

$$\begin{aligned} \widetilde{\nabla}_X N_1 &= \overline{\nabla}_X N_1 + g(X, N_1)\widetilde{N} \\ &= -A_1 X + \sum_{\alpha=1}^{p} s_{1\alpha}(X) N_\alpha + g(X, N_1)\widetilde{N} \\ &= -A_1 X \end{aligned} \tag{5.3}$$

for any tangent vector field X to M.

From now on, we consider the integral submanifolds M_k, $k = 1, 2$. Let P_k be the position vector of M_k in $\mathbb{R}^{4(m+1)}$ and put

$$Q_k = P_k + (1 + {\rho_k}^2)^{-1}(\rho_k N_1 + \widetilde{N}), \quad k = 1, 2.$$

Then for $X \in T_k$ we have $\widetilde{\nabla}_X Q_k = 0$ because of $A_1 X = \rho_k X$, (5.2) and (5.3), and so Q_k is a fixed point for M_k. Moreover, it is clear that

$$\|Q_k - P_k\|^2 = (1 + {\rho_k}^2)^{-1}$$

which means that P_k belongs to a sphere S_k with radius $(1 + {\rho_k}^2)^{-1/2}$ and center Q_k.

Next, we consider M_k, $k = 1, 2$, as submanifolds of S^{4m+3}. Since M_k is totally geodesic in M, it is clear that $A_Y{}^{(k)} = 0$ where $A_Y{}^{(k)}$ is the shape operator of M_k in S^{4m+3} with respect to the tangent vector Y to M_j, $j \neq k$. This shows that the first normal space ([3]) of M_k is contained in $Span\{N_1, \ldots, N_p\}$. We now prove

LEMMA 5.1. *$Span\{N_1, \ldots, N_p\}$ $(p = 4m - n)$ is invariant under parallel translation with respect to the normal connection $D^{(k)}$ of M_k in S^{4m+3}.*

PROOF. Since S^{4m+3} is of constant curvature 1, the equation (3.6) of Ricci implies

$$g([A_1, A_N]X, Y) = g(R^{\perp}_{XY}N_1, N) = 0$$

since $D_X N_1 = 0$. Hence $A_1 A_N = A_N A_1$ and so, for $X \in T_k$ we have $A_N X \in T_k$, that is,

$$A_N T_k \subset T_k, \quad k = 1, 2, \tag{5.4}$$

for any normal vector N to M. On the other hand, for any vector field X tangent to M_k, we have

$$\overline{\nabla}_X N_\alpha = -A_\alpha X + D_X N_\alpha.$$

But $D_X N_\alpha \in Span\{N_1, \ldots, N_p\}$ and $A_\alpha X \in T_k$ as a consequence of (5.4). Hence

$$D_X{}^{(k)} N_\alpha = D_X N_\alpha \in Span\{N_1, \ldots, N_p\},$$

which completes the proof. □

As a consequence of Lemma 5.1 we can apply Erbacher's reduction theorem ([3, p. 339]) and this yields that M_k belongs to a totally geodesic submanifold $S_k(1)$ of dimension $(dim M_k + 4m - n)$ in S^{4m+3}. Therefore M_k belongs to the intersection of this $S_k(1)$ and the sphere $S_k((1 + \rho_k{}^2)^{-1/2}, Q_k)$ obtained above. Note that Q_k belongs to the Euclidean space of dimension $(dim M_k + p + 1)$ passing through O and containing $S_k(1)$. Since $(dim M_k + 4m - n)$ is a multiple of 4, we may conclude

THEOREM 5.2. *Let M be an $(n + 3)$-dimensional contact three CR submanifold of $(p-1)$ contact three CR dimension in an $(n+p+3)(n+p = 4m)$-dimensional unit sphere S^{4m+3}. If*

$$A_1 F = F A_1, \quad A_1 G = G A_1, \quad A_1 H = H A_1$$

and N_1 is parallel with respect to the normal connection, then M is locally a product $M_1 \times M_2$, where $M_i (i = 1, 2)$ belongs to some $(4r_i + 3)$-dimensional sphere.

We next prove

LEMMA 5.3. *Let M be an $(n+3)$-dimensional compact, minimal, contact three CR submanifold of $(p-1)$ contact three CR dimension in S^{4m+3}. If N_1 is parallel with respect to the normal connection and the scalar curvature$\geq (n+1)(n+3)$ on M, then $A_1F = FA_1$, $A_1G = GA_1$, $A_1H = HA_1$ and $A_\alpha = 0$ $(\alpha = 2, \ldots, 4m-n)$.*

PROOF. The following integral formula is well known ([12]):

$$\begin{aligned}(5.5)\quad \int_M \{&Ric(U,U) + \frac{1}{2}\|\mathcal{L}_U g\|^2 - \|\nabla U\|^2 - (divU)^2 \\ &+ Ric(V,V) + \frac{1}{2}\|\mathcal{L}_V g\|^2 - \|\nabla V\|^2 - (divV)^2 \\ &+ Ric(W,W) + \frac{1}{2}\|\mathcal{L}_W g\|^2 - \|\nabla W\|^2 - (divW)^2\} * 1 = 0,\end{aligned}$$

where $*1$ denotes the volume element of M. Since S^{4m+3} is of constant curvature 1, the equation (3.4) of Gauss and (3.17) imply

$$\begin{aligned}&Ric(U,U) = n+2+(trA_1)g(A_1U,U) - g(A_1^2U,U),\\ (5.6)\qquad &Ric(V,V) = n+2+(trA_1)g(A_1V,V) - g(A_1^2V,V),\\ &Ric(W,W) = n+2+(trA_1)g(A_1W,W) - g(A_1^2W,W).\end{aligned}$$

On the other hand, it follows from (3.10)-(3.12) that

$$\begin{aligned}(5.7)\qquad divU = tr(FA_1) = 0, \quad divV = tr(GA_1) = 0,\\ divW = tr(HA_1) = 0.\end{aligned}$$

From (3.10)-(3.12), we also have

$$\begin{aligned}(\mathcal{L}_U g)(X,Y) &= g(\nabla_X U, Y) + g(\nabla_Y U, X) = g((FA_1 - A_1F)X,Y),\\ (\mathcal{L}_V g)(X,Y) &= g(\nabla_X V, Y) + g(\nabla_Y V, X) = g((GA_1 - A_1G)X,Y),\\ (\mathcal{L}_W g)(X,Y) &= g(\nabla_X W, Y) + g(\nabla_Y W, X) = g((HA_1 - A_1H)X,Y).\end{aligned}$$

Using (2.5) and (3.10)-(3.12), we also have

$$\begin{aligned}&\|\nabla U\|^2 = trA_1^2 - 1 - g(A_1^2U,U),\\ (5.8)\qquad &\|\nabla V\|^2 = trA_1^2 - 1 - g(A_1^2V,V),\\ &\|\nabla V\|^2 = trA_1^2 - 1 - g(A_1^2V,V).\end{aligned}$$

Since M is assumed to be minimal, $trA_\alpha = 0$ ($\alpha = 1, \ldots, 4m-n$). So the scalar curvature ρ is given by

$$\rho = (n+2)(n+3) - \sum_{\alpha=1}^{4m-n} trA_\alpha^2. \tag{5.9}$$

Therefore, substituting (5.6)-(5.9) into (5.5), we obtain

$$\int_M \{\frac{1}{2}(\|\mathcal{L}_U g\|^2 + \|\mathcal{L}_V g\|^2 + \|\mathcal{L}_W g\|^2) + 3\rho - 3(n+1)(n+3) + 3\sum_{\alpha=2}^{4m-n} trA_\alpha^2\} * 1 = 0,$$

which together with the assumption $\rho \geq (n+1)(n+3)$ yields

$$\mathcal{L}_U g = 0, \quad \mathcal{L}_V g = 0, \quad \mathcal{L}_W g = 0 \quad \text{and} \quad trA_\alpha^2 = 0,$$

where $\alpha = 2, \ldots, 4m-n$. Thus these give the required results. □

For the submanifold M given in Lemma 5.3, we can easily see that its first normal space is contained in $Span\{N_1\}$ which is invariant under parallel translation with respect to the normal connection from our assumption. Thus we may apply Erbacher's reduction theorem and this yields that there is an $(n+4)$-dimensional totally geodesic unit sphere S^{n+4} such that $M \subset S^{n+4}$. Here we note that $n+4$ is of type $4r+3$ for some positive integer r. Moreover, since the tangent space T_xS^{n+4} of the totally geodesic submanifold S^{n+4} at x in M is $T_xM \oplus Span\{N_1\}$, S^{n+4} is an invariant submanifold of S^{4m+3} with respect to the Sasakian three structure $\{\xi, \eta, \zeta\}$ (that is, ξ, η and ζ are all tangent to S^{n+4} and $\phi(T_xS^{n+4}) \subset T_xS^{n+4}$, $\psi(T_xS^{n+4}) \subset T_xS^{n+4}$ and $\theta(T_xS^{n+4}) \subset T_xS^{n+4}$ for any x in S^{n+4}), because of (2.1) and (2.3). Hence the submanifold M given in Lemma 5.3 can be regarded as a real hypersurface of S^{n+4} which is totally geodesic invariant submanifold of S^{4m+3}.

Tentatively we denote S^{n+4} by M' and by i_1 the immersion of M into M' and i_2 the totally geodesic immersion of M' onto S^{4m+3}. Then, from the Gauss equation (3.1), it follows that

$$\nabla'_{i_1X} i_1 Y = i_1 \nabla_X Y + h'(X,Y) = i_1 \nabla_X Y + g(A'X, Y)N', \tag{5.10}$$

where h' is the second fundamental form of M in M', A' is the corresponding shape operator and N' is a unit normal vector field to M in M'. Since $i = i_2 \circ i_1$, we have

$$\overline{\nabla}_{i_2 \circ i_1 X} i_2 \circ i_1 Y = i_2 \nabla'_{i_1 X} i_1 Y + \overline{h}(i_1 X, i_1 Y) \\ = i_2(i_1 \nabla_X Y + g(A'X, Y)N') \tag{5.11}$$

because M' is totally geodesic in S^{4m+3}. Comparing (5.11) with (3.1), we easily see that

$$N_1 = i_2 N', \quad A_1 = A'. \tag{5.12}$$

As M' is invariant submanifold of S^{4m+3}, for any $X' \in TM'$,

$$\phi i_2 X' = i_2 \phi' X', \quad \psi i_2 X' = i_2 \psi' X', \quad \theta i_2 X' = i_2 \theta' X' \tag{5.13}$$

is valid, where $\{\phi', \psi', \theta'\}$ is the induced Sasakian three structure of $M' = S^{n+4}$. Thus it follows from (2.3) that

$$\begin{aligned} \phi i X &= \phi i_2 \circ i_1 X = i_2 \phi' i_1 X = i_2(i_1 F'X + u'(X)N') \\ &= iF'X + u'(X) i_2 N' = iF'X + u'(X)N_1, \\ \psi i X &= \psi i_2 \circ i_1 X = i_2 \psi' i_1 X = i_2(i_1 G'X + v'(X)N') \\ &= iG'X + v'(X) i_2 N' = iG'X + v'(X)N_1, \\ \theta i X &= \theta i_2 \circ i_1 X = i_2 \theta' i_1 X = i_2(i_1 H'X + w'(X)N') \\ &= iH'X + w'(X) i_2 N' = iH'X + w'(X)N_1. \end{aligned}$$

Comparing this equation with (2.3), we have $F = F'$, $u' = u^1$; $G = G'$, $v' = v^1$ and $H = H'$, $w' = w^1$. By Lemma 5.3, we know that M is a real hypersurface of S^{n+4} which satisfies $F'A' = A'F'$, $G'A' = A'G'$ and $H'A' = A'H'$. Now applying a theorem due to Pak [10], we may conclude

THEOREM 5.4. *Let M be an $(n+3)$-dimensional compact, minimal, contact three CR submanifold of $(p-1)$ contact three CR dimension in S^{4m+3}. If N_1 is parallel with respect to the normal connection and the scalar curvature$\geq (n+1)(n+3)$ on M, then*

$$M = S^{4r+3}(a) \times S^{4s+3}(b), \quad r + s = \frac{n-3}{4}.$$

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