Comm. Korean Math. Soc. **13** (1998), No. 3, pp. 545–560

THE JACOBI OPERATOR OF REAL HYPERSURFACES IN A COMPLEX SPACE FORM

U-HANG KI, HE-JIN KIM AND AN-AYE LEE*

ABSTRACT. Let ϕ and A be denoted by the structure tensor field of type (1,1) and by the shape operator of a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$ respectively. The main purpose of this paper is to prove that if a real hypersurface in $M_n(c)$ satisfies $R_\xi \phi A = A\phi R_\xi$, then the structure vector field ξ is principal, where R_ξ is the Jacobi operator with respect to ξ.

0. Introduction

A Kaehlerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. The complete and simply connected complex space form is a complex projective space P_nC, a complex Euclidean space C_n, or a complex hyperbolic space H_nC according as $c > 0, c = 0$ or $c < 0$.

The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ will be denoted by (ϕ, g, ξ, η).

Typical examples of real hypersurfaces in P_nC are homogeneous ones. Takagi ([22]) classified homogeneous real hypersurfaces of a complex projective space P_nC as the following six types.

THEOREM A. *Let M be a homogeneous real hypersurface of P_nC. Then M is locally congruent to one of the followings:*

(A_1) *a geodesic hypersphere (that is, a tube over a hyperplane $P_{n-1}C$),*
(A_2) *a tube over a totally geodesic $P_kC (1 \leq k \leq n-2)$,*

Received January 14, 1998. Revised May 19, 1998.
1991 Mathematics Subject Classification: 53C15, 53C45.
Key words and phrases: homogeneous real hypersurface, principal curvature vector field, Jacobi operator.
* The authors wish to acknowledge the financial support of the Korea Research Foundation made in the program year of 1997.

(B) *a tube over a complex quadric* Q_{n-1},
(C) *a tube over* $P_1C \times P_{(n-1)/2}C$ *and* $n(\geq 5)$ *is odd,*
(D) *a tube over a complex Grassman* $G_{2,5}C$ *and* $n = 9$,
(E) *a tube over a Hermitian symmetric space* $SO(10)/U(5)$ *and* $n = 15$.

This result was generalized by many authors ([4], [7], [10], [11], [13] and [16] etc.). One of them, Kimura ([10]) asserts that M has constant principal curvatures and the structure vector field ξ is principal if and only if M is locally congruent to a homogeneous real hypersurface.

On the other hand, real hypersurfaces of H_nC have been also investigated by many geometers([2], [9], [17] and [18] etc.) from different points of view. In particular, Berndt ([3]) proved the following:

THEOREM B. *Let* M *be a real hypersurface of* H_nC. *Then* M *has constant principal curvatures and* ξ *is principal if and only if* M *is locally congruent to one of the followings:*

(A_0) *a self-tube, that is, a horosphere,*
(A_1) *a geodesic hypersphere or a tube over a hyperplane* $H_{n-1}C$,
(A_2) *a tube over a totally geodesic* $H_kC(1 \leq k \leq n-2)$,
(B) *a tube over a totally real hyperbolic space* H_nR.

Let M be a real hypersurface of type A_1 or type A_2 in a complex projective space P_nC or that of type A_0, A_1 or A_2 in a complex hyperbolic space H_nC. Then M is said to be of *type* A for simplicity. By a theorem due to Okumura ([19]) and to Montiel and Romero ([18]) we have

THEOREM C. *If the shape operator* A *and the structure tensor* ϕ *commute to each other, then a real hypersurface of a complex space form* $M_n(c)$, $c \neq 0$ *is locally congruent to be of type* A.

We denote by ∇ the Levi-Civita connection with respect to g. The curvature tensor field R on M is defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, where X and Y are vector fields on M. We define the Jacobi operator field $R_X = R(\cdot, X)X$ with respect to a unit vector field X. Then we see that R_X is a self-adjoint endomorphism of the tangent space. It is related with the Jacobi vector fields, which are solutions of the second order differential equation (the Jacobi equation) $\nabla_{\dot\gamma}(\nabla_{\dot\gamma}Y) + R(Y, \dot\gamma)\dot\gamma = 0$ along a geodesic γ. It is well-known that the notion of Jacobi vector fields involve many important geometric properties. In the preceding work [5], we investigated the Jacobi operators on real hypersurfaces in a complex projective space. Particularly, Cho and one of the present authors proved the following ([6])

THEOREM D. *Let M be a connected real hypersurface of P_nC. If M satisfies $R_\xi\phi A = A\phi R_\xi$, then M is locally congruent to one of the following spaces:*

(A_1) *a geodesic hypersphere (that is, a tube of radius r over a hyperplane $P_{n-1}C$, where $0 < r < \frac{\pi}{2}$);*

(A_2) *a tube of radius r over a totally geodesic $P_kC(1 \le k \le n-2)$, where $0 < r < \frac{\pi}{2}$.*

The purpose of the present paper is to improve above Theorem D. Namely we will prove the following:

THEOREM. *Let M be a real hypersurface satisfying $R_\xi\phi A = A\phi R_\xi$ in a complex space form $M_n(c), c \neq 0$. Then M is locally congruent to be of type* A.

The first author wishes to express his gratitude to TGRC-KOSEF who gave him the opportunity to study at Chiba University.

1. Preliminaries

Let $M_n(c)$ be a real 2n-dimensional complex space form equipped with parallel almost complex structure J and a Riemannian metric tensor G which is J-Hermitian, and covered by a system of coordinate neighborhoods $\{\overline{V}; x^A\}$.

Let M be a real (2n-1)-dimensional hypersurface of $M_n(c)$ covered by a system of coordinate neighborhoods $\{V; y^h\}$ and immersed isometrically in $M_n(c)$ by the immersion $i : M \to M_n(c)$. Throughout the present paper the following convention on the range of indices are used, unless otherwise stated:

$$A, B, \cdots = 1, 2, \ldots, 2n; \;\; i, j, \cdots = 1, 2, \ldots, 2n-1.$$

The summation convention will be used with respect to those system of indices. When the argument is local, M need not be distinguished from

$i(M)$. Thus, for simplicity, a point p in M may be identified with the point $i(p)$ and a tangent vector X at p may also be identified with the tangent vector $i_*(X)$ at $i(p)$ via the differential i_* of i. We represent the immersion i locally by $x^A = x^A(y^h)$ and $B_j = (B_j{}^A)$ are also (2n-1)-linearly independent local tangent vectors of M, where $B_j{}^A = \partial_j x^A$ and $\partial_j = \partial/\partial y^j$. A unit normal C to M may then be chosen. The induced Riemannian metric g with components g_{ji} on M is given by $g_{ji} = G_{BA}B_j{}^B B_i{}^A$ because the immersion is isometric.

For the unit normal C to M, the following representations are obtained in each coordinate neighborhoods:

$$JB_i = \phi_i{}^h B_h + \xi_i C, \quad JC = -\xi^i B_i,$$

where we have put $\phi_{ji} = G(JB_j, B_i)$ and $\xi_i = G(JB_i, C), \xi^h$ being components of a vector field ξ associated with ξ_i and $\phi_{ji} = \phi_j{}^r g_{ri}$. By the properties of the almost Hermitian structure J, it is clear that ϕ_{ji} is skew-symmetric. A tensor field of type (1,1) with components $\phi_i{}^h$ will be denoted by ϕ. By the properties of the almost complex structure J, the following relations are then given:

$$\phi_i{}^r \phi_r{}^h = -\delta_i{}^h + \xi_i \xi^h, \quad \xi^r \phi_r{}^h = 0, \quad \xi_r \phi_i{}^r = 0, \quad \xi_i \xi^i = 1,$$

that is, the aggregate (ϕ, g, ξ) defines an almost contact metric structure.

Denoting by ∇_j the operator of van der Waerdern-Bortolotti covariant differentiation with respect to the induced Riemannian metric, equations of the Gauss and Weingarten for M are respectively obtained:

$$\nabla_j B_i = A_{ji} C, \quad \nabla_j C = -A_j{}^r B_r,$$

where $H = (A_{ji})$ is a second fundamental form and $A = (A_j{}^h)$, which is related by $A_{ji} = A_j{}^r g_{ri}$ is the shape operator derived from C. By means of above equations the covariant derivatives of the structure tensors are yielded:

$$\nabla_j \phi_i{}^h = -A_{ji}\xi^h + A_j{}^h \xi_i, \quad \nabla_j \xi_i = -A_{jr}\phi_i{}^r. \tag{1.1}$$

Since the ambient space is a complex space form, equations of the Gauss and Codazzi for M are respectively given by

$$\begin{aligned} R_{kjih} =& \frac{c}{4}(g_{kh}g_{ji} - g_{jh}g_{ki} + \phi_{kh}\phi_{ji} - \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih}) \\ &+ A_{kh}A_{ji} - A_{jh}A_{ki}, \end{aligned} \tag{1.2}$$

$$(1.3)\qquad \nabla_k A_{ji} - \nabla_j A_{ki} = \frac{c}{4}(\xi_k \phi_{ji} - \xi_j \phi_{ki} - 2\xi_i \phi_{kj}),$$

where R_{kjih} are components of the Riemannian curvature tensor R of M.

In what follows, to write our formulas in convention forms, we denote by $A_{ji}{}^2 = A_{jr}A_i{}^r, h = g_{ji}A^{ji}, \alpha = A_{ji}\xi^j\xi^i$ and $\beta = A_{ji}{}^2\xi^j\xi^i$. If we put $U_j = \xi^r\nabla_r\xi_j$, then U is orthogonal to the structure vector field ξ. Because of the properties of the almost contact metric structure and the second equation of (1.1), we can get

$$(1.4)\qquad \phi_{jr}U^r = A_{jr}\xi^r - \alpha\xi_j,$$

which shows that $g(U,U) = \beta - \alpha^2$. By the definition of U and the second equation of (1.1), we easily see that

$$(1.5)\qquad U^r\nabla_j\xi_r = A_{jr}{}^2\xi^r - \alpha A_{jr}\xi^r.$$

On the other hand, differentiating (1.4) covariantly along M and making use of (1.1), we find

$$(1.6)\quad \xi_j A_{kr}U^r + \phi_{jr}\nabla_k U^r = \xi^r\nabla_k A_{jr} - A_{jr}A_{ks}\phi^{rs} - \alpha_k\xi_j + \alpha A_{kr}\phi_j{}^r,$$

which shows that

$$(1.7)\qquad (\nabla_k A_{ji})\xi^j\xi^i = 2A_{kr}U^r + \alpha_k,$$

where $\alpha_k = \partial_k\alpha$.

Transforming (1.6) by $\phi_i{}^j$ and taking account of (1.1) and (1.5), we find

$$(1.8)\qquad \nabla_k U_i + \xi_i A_{kr}{}^2\xi^r + \xi^r(\nabla_k A_{sr})\phi_i{}^s = (\nabla_k\xi^r)(\nabla_r\xi_i) + \alpha A_{ki}.$$

By the definition of U, (1.1), (1.7) and (1.8) it is verified that

$$(1.9)\qquad \xi^r\nabla_r U_j = -3U^s A_{rs}\phi_j{}^r + \alpha A_{jr}\xi^r - \beta\xi_j - \phi_{jr}\alpha^r.$$

We put

$$(1.10)\qquad A_{jr}\xi^r = \alpha\xi_j + \mu W_j,$$

where W is a unit vector field orthogonal to ξ. Then from (1.4) we see that $U = -\mu\phi W$, and W is also orthogonal to U. We assume that $\mu \neq 0$ on M, that is, ξ is not a principal curvature vector field and we put $\Omega = \{p \in M | \mu(p) \neq 0\}$. Then Ω is an open subset of M and hereafter we discuss our argument on Ω otherwise stated.

2. Real hypersurfaces satisfying $R_\xi \phi A = A\phi R_\xi$

Let M be a real hypersurface of $M_n(c), c \neq 0$. Suppose that $R_\xi \phi A = A\phi R_\xi$. Then from (1.2) we have

$$\frac{c}{4}(A_{jr}\phi_i{}^r + A_{ir}\phi_j{}^r) = (A_{jr}\xi^r)(A_{is}U^s) + (A_{ir}\xi^r)(A_{js}U^s). \tag{2.1}$$

Transvecting (2.1) with ξ^i, we find

$$\alpha A_{jr}U^r = -\frac{c}{4}U_j. \tag{2.2}$$

Thus (2.1) turns out to be

$$\alpha(A_{jr}\phi_i{}^r + A_{ir}\phi_j{}^r) + U_i A_{jr}\xi^r + U_j A_{ir}\xi^r = 0. \tag{2.3}$$

This means that $R_\xi \phi = \phi R_\xi$. By transvecting U^i, we have

$$\alpha A_{jr}{}^2\xi^r = (\beta - \frac{c}{4})A_{jr}\xi^r + \frac{c}{4}\alpha\xi_j. \tag{2.4}$$

We put $\beta = \alpha\lambda$ and $\theta\alpha = \alpha\lambda - \frac{c}{4}(\alpha \neq 0)$. Then (2.2) becomes

$$A_{jr}U^r = (\theta - \lambda)U_j \tag{2.5}$$

because $\alpha \neq 0$ on Ω, and hence (2.4) reduces to

$$A_{jr}{}^2\xi^r = \theta A_{jr}\xi^r + \frac{c}{4}\xi_j. \tag{2.6}$$

Therefore (1.8) becomes

$$\nabla_k U_j + \xi^r(\nabla_k A_{rs})\phi_j{}^s = (\nabla_k\xi^r)(\nabla_r\xi_j) + \alpha A_{kj} - \xi_j(\theta A_{kr}\xi^r + \frac{c}{4}\xi_k). \tag{2.7}$$

On the other hand, if we transvect $\nabla_k\xi^j$ to (2.3) and use (1.1), (1.5) and (2.6), then we obtain

$$(\nabla_k\xi^r)(\nabla_r\xi_i) + A_{ki}{}^2 = \frac{1}{\alpha}\{-U_i A_{kr}U^r + \theta(A_{kr}\xi^r)(A_{is}\xi^s) + \frac{c}{4}\xi_k A_{ir}\xi^r\}.$$

Consequently (2.7) becomes

$$\begin{aligned}(2.8)\qquad &\nabla_k U_j + \xi^r(\nabla_k A_{sr})\phi_j{}^s + \xi_j(\theta A_{kr}\xi^r + \frac{c}{4}\xi_k)\\ &= \alpha A_{jk} - A_{jk}{}^2 - \frac{1}{\alpha}\{U_j A_{kr}U^r + \theta(A_{kr}\xi^r)(A_{js}\xi^s) + \frac{c}{4}\xi_k A_{jr}\xi^r\},\end{aligned}$$

which together with (2.5) implies

$$(2.9)\qquad \alpha^2\{U^k\nabla_k U_j + \phi_j{}^i(\nabla_s A_{ir})U^s\xi^r\} = \frac{c}{4}\{\alpha(\lambda-\alpha) - \frac{c}{4} - 1\}U_j.$$

Differentiating (2.2) covariantly along Ω, we find

$$(2.10)\qquad -\frac{c}{4}\alpha_k U_j + \alpha^2(\nabla_k A_{jr})U^r + \alpha^2 A_{jr}\nabla_k U^r = -\frac{c}{4}\alpha\nabla_k U_j.$$

If we transvect ξ^k to this and make use of (1.3), (1.9) and (2.6), then we obtain

$$\begin{aligned}\alpha^2(\nabla_r A_{js})U^r\xi^s =& \frac{c}{4}\{d\alpha(\xi)U_j + \alpha\phi_{jr}\alpha^r\} - \alpha^2 A_j{}^k\phi_{rk}\alpha^r\\ &+ \{\frac{c}{4}\alpha^2 - \frac{3}{4}c(\alpha\theta - \alpha^2 + \frac{c}{4})\}A_{jr}\xi^r + \frac{c}{4}\alpha^2(\lambda - 2\alpha)\xi_j,\end{aligned}$$

where we have defined $d\alpha(\xi) = \alpha_t\xi^t$ and used $\alpha(\theta-\lambda) = -\frac{c}{4}$. Substituting this into (2.9) and using (2.3), we find

$$\begin{aligned}(2.11)\qquad &\alpha^2 U^k\nabla_k U_j\\ &= -\{\alpha g(A\xi,\nabla\alpha) + \frac{c}{4}d\alpha(\xi)\}A_{jr}\xi^r + \alpha^2 A_{jr}\alpha^r + \frac{c}{4}\alpha\alpha_j\\ &\quad + \{\alpha d\alpha(U) + \frac{c}{4}(3\alpha^2 - 2\alpha\theta - \frac{3}{4}c - 1)\}U_j,\end{aligned}$$

where $g(A\xi,\nabla\alpha) = A_{ji}\xi^j\alpha^i$. From (2.10) we have

$$\begin{aligned}&\frac{c}{4}(\alpha_j U_k - \alpha_k U_j) + \frac{c}{4}\alpha^2(\xi_k\phi_{jr}U^r - \xi_j\phi_{kr}U^r) + \alpha^2(A_{jr}\nabla_k U^r - A_{kr}\nabla_j U^r)\\ &= -\frac{c}{4}\alpha(\nabla_k U_j - \nabla_j U_k).\end{aligned}$$

Taking the inner product the last equation with U^k and using (2.2) and (2.11), we get

$$\begin{aligned}(2.12)\qquad &\alpha^2 A_{jr}{}^2\alpha^r + \frac{c}{2}\alpha A_{jr}\alpha^r + \frac{c}{4}\{\frac{c}{4} + \alpha(\lambda - \alpha)\}\alpha_j \\ &= \{\alpha g(A\xi, \nabla\alpha) + \frac{c}{4}d\alpha(\xi)\}(\lambda A_{jr}\xi^r + \frac{c}{4}\xi_j) + \frac{c}{4}d\alpha(U)U_j.\end{aligned}$$

On the other hand, differentiating (2.6) covariantly along Ω and using (1.1), we find

$$\begin{aligned}(2.13)\qquad &(\nabla_k A_{js})A_r{}^s\xi^r + A_j{}^r(\nabla_k A_{rs})\xi^s - A_{jr}{}^2 A_{ks}\phi^{rs} + \theta A_{jr}A_{ks}\phi^{rs} \\ &= \theta_k A_{jr}\xi^r + \theta(\nabla_k A_{jr})\xi^r - \frac{c}{4}A_{kr}\phi_j{}^r,\end{aligned}$$

which together with (1.7) and (2.5) yields

$$(2.14)\qquad (\nabla_k A_{ts})\xi^t A_r{}^s\xi^r = \frac{1}{2}(\theta\alpha)_k + \theta A_{kr}U^r.$$

Transvecting (2.13) with ξ^k and making use of (1.7), we get

$$\begin{aligned}&\xi^k(\nabla_k A_{js})A_r{}^s\xi^r + 3(A_{jr}{}^2U^r - \theta A_{jr}U^r) + A_{jr}\alpha^r - \theta\alpha_j \\ &= d\theta(\xi)A_{jr}\xi^r + \frac{c}{4}U_j,\end{aligned}$$

or, using (1.3), (2.5) and (2.14)

$$(2.15)\qquad \frac{1}{2}(\theta\alpha)_j - \theta\alpha_j + A_{jr}\alpha^r = d\theta(\xi)A_{jr}\xi^r + (3\lambda - \theta)A_{jr}U^r + \frac{c}{2}U_j.$$

From (1.10) and (2.6) we have

$$\mu A_{jr}W^r = (\theta - \alpha)(\alpha\xi_j + \mu W_j) + \frac{c}{4}\xi_j$$

and hence

$$(2.16)\qquad \mu^2 = \alpha(\theta - \alpha) + \frac{c}{4}.$$

Thus it follows that

$$(2.17)\qquad A_{jr}W^r = \mu\xi_j + (\theta - \alpha)W_j$$

because $\mu \neq 0$ on Ω, which together with (2.6) implies that

$$(2.18)\qquad \mu A_{jr}{}^2W^r = (\theta^2 - \alpha\theta + \frac{c}{4})A_{jr}\xi^r + \frac{c}{4}(\theta - \alpha)\xi_j.$$

LEMMA 1. *$g(A\xi, \nabla\alpha) = \lambda d\alpha(\xi)$ on Ω.*

PROOF. By transvecting (2.12) with μW^j and using (1.10) and (2.18), we can verify the required equation. □

Differentiating (2.17) covariantly along Ω and using (1.1), we find

$$(2.19)\quad \begin{aligned} &(\nabla_k A_{jr})W^r + A_{jr}\nabla_k W^r \\ &= \mu_k \xi_j - \mu A_{kr}\phi_j{}^r + (\theta_k - \alpha_k)W_j + (\theta - \alpha)\nabla_k W_j. \end{aligned}$$

Since ξ and W are mutually unit orthogonal vector fields, it is seen that $\mu\xi^r\nabla_j W_r = A_{jr}U^r$. Thus, if we transvect (2.19) with W^j and use (2.17), then we obtain

$$(2.20)\quad (\nabla_j A_{rs})W^r W^s = -2A_{jr}U^r + \theta_j - \alpha_j.$$

Applying (2.19) by ξ^j and making use of (1.10) and (2.16), we get

$$(2.21)\quad \mu(\nabla_k A_{rs})W^r\xi^s = (\theta - 2\alpha)A_{kr}U^r + \frac{1}{2}(\theta\alpha)_k - \alpha\alpha_k.$$

Because of (1.3), (1.7), (2.2), (2.20) and (2.21), we have

$$(2.22)\quad \begin{aligned} &(\nabla_k A_{jr})(\alpha\xi^k + \mu W^k)(\alpha\xi^r + \mu W^r) \\ &= -\frac{c}{4}(3\alpha + 2\theta - 2\lambda)U_j - \frac{c}{4}\alpha_j + \alpha\lambda\theta_j. \end{aligned}$$

Thus, applying (2.13) by $A_t{}^k\xi^t$, we find

$$\begin{aligned} &-\frac{c}{4}(3\alpha + 2\theta - 2\lambda)U_j - \frac{c}{4}\alpha_j + \alpha\lambda\theta_j + A_j{}^r\{\theta A_{rs}U^s - \frac{c}{2}U_r + \frac{1}{2}(\theta\alpha)_r\} \\ &+ \theta A_{jr}{}^2U^r - \theta^2 A_{jr}U^r - \frac{c}{4}\theta U_j \\ &= g(A\xi, \nabla\theta)A_{jr}\xi^r + \theta\{\theta A_{jr}U^r - \frac{c}{2}U_j + \frac{1}{2}(\theta\alpha)_j\}, \end{aligned}$$

where we have used (1.3), (1.10), (2.6), (2.14), (2.17), (2.21) and (2.22), or using (2.5)

$$(2.23)\quad \begin{aligned} \frac{1}{2}A_{jr}(\theta\alpha)^r - \frac{1}{2}\theta(\theta\alpha)_j =& g(A\xi, \nabla\theta)A_{jr}\xi^r + \frac{c}{4}\alpha_j - \alpha\lambda\theta_j \\ &+ \{\frac{c}{4}(3\alpha + 3\theta - 4\lambda) + 2\lambda\theta(\theta - \lambda)\}U_j. \end{aligned}$$

LEMMA 2. *$d\alpha(U) = 3(\lambda - \alpha)(\frac{c}{4} - \alpha^2)$ on Ω.*

PROOF. From (2.15), we have

$$(2\lambda - \theta)d\alpha(U) - \alpha d\theta(U) = \{2(\theta - \lambda)(\theta + \alpha - 3\lambda) - \frac{c}{2}\}\alpha(\lambda - \alpha)$$

because of (2.5) and $g(A\xi, U) = 0$.

By (2.23) we also have

$$\frac{1}{2}\alpha\lambda d\theta(U) + (\alpha\theta - \alpha\lambda - \frac{1}{2}\lambda\theta)d\alpha(U) + \{(\theta - \lambda)(3\alpha\theta - 2\alpha\lambda - 2\theta\lambda) + \frac{c}{2}\lambda - \frac{3}{4}c\alpha\}\alpha(\lambda - \alpha) = 0.$$

Combining with the last two equations, it follows that

$$(\theta - \lambda)d\alpha(U) = \frac{3}{4}c\alpha(\lambda - \alpha) - 3\alpha(\lambda - \alpha)(\theta - \lambda)^2,$$

which proves Lemma 2 because we have $\alpha(\lambda - \theta) = \frac{c}{4}$. □

Transvecting (2.12) with ξ^j, we have

$$2g(A\xi, \nabla\alpha) = \alpha d\theta(\xi) + \theta d\alpha(\xi),$$

which together with Lemma 1 gives

$$\alpha d\theta(\xi) = (2\lambda - \theta)d\alpha(\xi). \tag{2.24}$$

From Lemma 1 and (1.10) we also have

$$\mu d\alpha(W) = (\lambda - \alpha)d\alpha(\xi). \tag{2.25}$$

If we apply (2.15) by μW^j and take account of (2.24), then we find

$$\mu\alpha d\theta(W) - \mu\theta d\alpha(W) + 2\mu g(AW, \nabla\alpha) = 2(\lambda - \alpha)(2\lambda - \theta)d\alpha(\xi),$$

or make use of (2.17) and(2.25),

$$\mu\alpha d\theta(W) = (4\lambda^2 - 2\alpha\lambda - 3\theta\lambda + \theta\alpha - \frac{c}{2})d\alpha(\xi).$$

Thus we see that

$$\mu\alpha d\theta(W) = (\lambda - \alpha)(4\lambda - 3\theta)d\alpha(\xi) \tag{2.26}$$

because of $\alpha(\lambda - \theta) = \frac{c}{4}$.

3. Proof of theorem

In this section we shall prove that ξ is a principal curvature vector field on M. By Lemma 1 and Lemma 2, (2.12) turns out to be

$$\begin{aligned}&\alpha^2 A_{jr}{}^2\alpha^r + \frac{c}{2}\alpha A_{jr}\alpha^r + \frac{c}{4}\{\frac{c}{4} + \alpha(\lambda - \alpha)\}\alpha_j \\ &= \tau(\alpha\lambda + \frac{c}{4})(\lambda A_{jr}\xi^r + \frac{c}{4}\xi_j) + \frac{3}{4}c(\lambda - \alpha)(\frac{c}{4} - \alpha^2)U_j,\end{aligned}$$

where we have put $\tau = d\alpha(\xi)$. Because of (2.2) and (2.24), the equation (2.15) is reduced to

$$\alpha A_{jr}\alpha^r + \frac{1}{2}\alpha(\theta\alpha)_j - \theta\alpha\alpha_j = \tau(2\lambda - \theta)A_{jr}\xi^r + \frac{c}{4}(2\alpha - 3\lambda + \theta)U_j, \tag{3.1}$$

which together with (2.6) implies

$$\begin{aligned}&\alpha A_{jr}{}^2\alpha^r + \frac{1}{2}\alpha A_{jr}(\theta\alpha)^r - \theta\alpha A_{jr}\alpha^r \\ &= \tau(2\lambda - \theta)(\theta A_{jr}\xi^r + \frac{c}{4}\xi_j) + \frac{c}{4}(2\alpha - 3\lambda + \theta)A_{jr}U^r.\end{aligned}$$

Using (1.10), (2.24), (2.26) and the fact that $\alpha(\lambda - \theta) = \frac{c}{4}$, we have

$$\alpha g(A\xi, \nabla\theta) = (4\lambda^2 - 3\theta\lambda - \frac{c}{2})\tau.$$

Therefore (2.23) implies

$$\begin{aligned}&\frac{1}{2}\alpha A_{jr}(\theta\alpha)^r - \frac{1}{2}\alpha\theta(\theta\alpha)_j - \frac{c}{4}\alpha\alpha_j + \lambda\alpha^2\theta_j \\ &= \tau(4\lambda^2 - 3\theta\lambda - \frac{c}{2})A_{jr}\xi^r + \frac{c}{4}(3\alpha^2 + 3\theta\alpha - 4\alpha\lambda - 2\lambda\theta)U_j.\end{aligned}$$

Combining with the last four equations, we can verify by directly computation that

$$\alpha\alpha_j = \tau A_{jr}\xi^r + 3(\frac{c}{4} - \alpha^2)U_j, \tag{3.2}$$

which implies

$$\alpha\phi_{jr}\alpha^r = -\tau U_j + 3(\frac{c}{4} - \alpha^2)(A_{jr}\xi^r - \alpha\xi_j). \tag{3.3}$$

On the other hand, from (2.8) we have

$$\begin{aligned} &\nabla_k U_j - \nabla_j U_k + (\lambda - 2\theta)(\xi_k A_{jr}\xi^r - \xi_j A_{kr}\xi^r) \\ &= \xi^r(\nabla_j A_{rs})\phi_k{}^s - \xi^r(\nabla_k A_{rs})\phi_j{}^s. \end{aligned} \tag{3.4}$$

Transvecting (3.4) with ξ^k and using (1.7), (2.2) and (3.3), we find

$$\alpha\xi^k(\nabla_k U_j - \nabla_j U_k) = (\alpha\lambda - \frac{3}{4}c + 3\alpha^2)(A_{jr}\xi^r - \alpha\xi_j) + \tau U_j. \tag{3.5}$$

Differentiating (3.2) covariantly along Ω and taking account of (1.1), we obtain

$$\begin{aligned} \alpha\nabla_k\alpha_j + \alpha_k\alpha_j =& \tau_k A_{jr}\xi^r + \tau(\nabla_k A_{jr})\xi^r - \tau A_{jr}A_{ks}\phi^{rs} \\ &- 6\alpha\alpha_k U_j + 3(\frac{c}{4} - \alpha^2)\nabla_k U_j, \end{aligned}$$

from which, taking the skew-symmetric part with respect to k and j,

$$\begin{aligned} &\tau_k A_{jr}\xi^r - \tau_j A_{kr}\xi^r - \frac{c}{2}\tau\phi_{kj} - 2\tau A_{jr}A_{ks}\phi^{rs} \\ &- 6\tau(U_j A_{kr}\xi^r - U_k A_{jr}\xi^r) + 3(\frac{c}{4} - \alpha^2)(\nabla_k U_j - \nabla_j U_k) = 0, \end{aligned} \tag{3.6}$$

where we have used (1.3) and (3.2).

Applying (3.6) by ξ^k and making use of (1.4), (2.5), (3.5) and $\alpha(\lambda - \theta) = \frac{c}{4}$, we find

$$\begin{aligned} \alpha\tau_j =& \{d\tau(\xi) + 3(\lambda - \alpha - \theta)(\alpha\lambda - \frac{3}{4}c + 3\alpha^2)\}A_{jr}\xi^r \\ &+ \tau(\lambda - \theta - 9\alpha)U_j - 3\alpha(\lambda - \alpha - \theta)(\alpha\lambda - \frac{3}{4}c + 3\alpha^2)\xi_j. \end{aligned}$$

Substituting this into (3.6), we can reduce

$$(\frac{c}{4} - \alpha^2)\alpha(\nabla_k U_j - \nabla_j U_k) - \frac{c}{2}\tau\phi_{kj} - 2\tau\alpha A_{jr}A_{ks}\phi^{rs}$$
$$= \tau(\lambda - \theta - 3\alpha)(U_j A_{kr}\xi^r - U_k A_{jr}\xi^r)$$
$$+ 3\alpha(\lambda - \theta - \alpha)(\alpha\lambda - \frac{3}{4}c + 3\alpha^2)(\xi_k A_{jr}\xi^r - \xi_j A_{kr}\xi^r).$$

Transvecting $U^k W^j$ to this and using (1.4), (2.5) and (2.17), we have

$$(3.7)\quad (\frac{c}{4} - \alpha^2)\alpha(\nabla_k U_j - \nabla_j U_k)U^k W^j = \mu\tau\{(\lambda - \theta - 3\alpha)\alpha(\lambda - \alpha) + \frac{c}{2}\theta\}.$$

By the way, (2.11) turns out to be

$$\alpha^2 U^k \nabla_k U_j = \tau(\alpha\lambda + \frac{c}{4})A_{jr}\xi^r + \alpha^2 A_{jr}\alpha^r + \frac{c}{4}\alpha_j$$
$$+ \{3\alpha(\lambda - \alpha)(\frac{c}{4} - \alpha^2) + \frac{c}{4}(3\alpha^2 - 2\alpha\theta - \frac{3}{4}c - 1)\}U_j$$

because of Lemma 1 and Lemma 2, which implies

$$\alpha W^j U^k \nabla_k U_j = \mu\tau(\theta - \lambda),$$

where we have used (2.17) and (2.25).

Since U and W are mutually orthogonal, by transvecting $W^k U^j$ to (2.8) and making use of (1.4), (2.20) and (2.24), we have

$$\alpha W^k U^j \nabla_k U_j = \mu\tau(2\lambda - \theta - \alpha).$$

Because of the last two equations, it is clear that

$$\alpha W^j U^k(\nabla_k U_j - \nabla_j U_k) = \mu\tau(2\theta - 3\lambda + \alpha).$$

From this and (3.7), we see that

$$\mu\tau\{(\lambda - \theta - 3\alpha)\alpha(\lambda - \alpha) + \frac{c}{2}\theta + 3\alpha(\lambda - \theta - \alpha)(2\theta - 3\lambda + \alpha)\} = 0,$$

or, using the fact that $\alpha(\lambda - \theta) = \frac{c}{4}$, it follows that $\tau(\alpha^2 - \frac{c}{4}) = 0$.

Let Ω_1 be the set of points such that $\tau \neq 0$ in Ω and suppose that Ω_1 be non-empty. Then we have $\alpha^2 = \frac{c}{4}$ on Ω_1 and therefore $A\xi = 0$ because of (3.2), which is a contradiction. Hence Ω_1 is empty and thus $\tau = 0$ on Ω. Accordingly (3.2) and (3.6) are respectively reduced to

$$\alpha\alpha_j = 3(\frac{c}{4} - \alpha^2)U_j, \tag{3.8}$$

$$(\frac{c}{4} - \alpha^2)(\nabla_k U_j - \nabla_j U_k) = 0.$$

Now, let $\Omega_2 = \{p \in \Omega | \alpha^2(p) \neq \frac{c}{4}\}$ and suppose that Ω_2 be non void. We then have $\nabla_k U_j - \nabla_j U_k = 0$. Therefore (3.5) means

$$\alpha\lambda - \frac{3}{4}c + 3\alpha^2 = 0 \tag{3.9}$$

on Ω_2. Since we have $\alpha(\lambda - \theta) = \frac{c}{4}$, it follows that $\theta\alpha + 3\alpha^2 - \frac{c}{2} = 0$ on Ω_2. Differentiation gives

$$\alpha^2\theta_j + 3(\frac{c}{4} - \alpha^2)(\theta + 6\alpha)U_j = 0 \tag{3.10}$$

by virtue of (3.2).

On the other hand, (3.1) implies $\alpha^2 = c$, where we have used (2.5), (3.9) and (3.10). Thus α is constant on Ω_2 and hence $U = 0$ because of (3.2) and consequently the set Ω_2 is void. Therefore we have $\alpha^2 = \frac{c}{4}$ on Ω and thus α is constant. Accordingly (2.15) turns out to be $\theta_j = -2(2\theta + \alpha)U_j$, which unable us to be $(2\theta + \alpha)(\nabla_j U_i - \nabla_i U_j) = 0$. From this and (3.5) we have $2\theta + \alpha = 0$ and hence $2\alpha(\lambda - \alpha) + \alpha^2 = 0$. Therefore Ω is empty and thus ξ is a principal curvature vector field. Consequently (2.1) means that $A\phi = \phi A$. This completes the proof because of Theorem C.

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U-Hang Ki and He-Jin Kim
Topology and Geometry Research Center
Kyungpook National University
Taegu 702-701, Korea

An-Aye Lee
NAJU College
Naju 523-930, Korea