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ZERO SCALAR CURVATURE ON OPEN MANIFOLDS

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ABSTRACT. Let (M, g) be a noncompact complete Riemannian manifold of dimension $n \geq 3$ with scalar curvature S, which is close to 0. With conditions on a conformal invariant and scalar curvature of (M, g), we show that there exists a conformal metric $\overline{g}$, near g, whose scalar curvature $\overline{S} = 0$ by gluing solutions of the corresponding partial differential equation on each bounded subsets K_i with $\cup K_i = M$.

1. Introduction

Let (M, g) be a noncompact complete Riemannian manifold of dimension $n \geq 3$ with scalar curvature S. In this paper, we look for a conformal metric $\overline{g} = u^{4/(n-2)}g$, near g, whose scalar curvature $\overline{S} = 0$. This problem is equivalent to finding a smooth positive solution u, which is close to 1, of the following partial differential equation:

$$-c_n \Delta u + Su = 0, \tag{A}$$

where $c_n = 4(n-1)/(n-2)$.

For compact Riemannian manifolds, conformal changes to a constant scalar curvature have been studied separately, based on the sign of $Q(M, g)$, where

$$Q(M,g) \equiv \inf_{u \in C_0^\infty(M)} \frac{\int_M |\nabla u|^2 + \frac{n-2}{4(n-1)} S\, u^2 \; dV_g}{\left(\int_M u^{2n/(n-2)} \; dV_g\right)^{(n-2)/n}}.$$

$Q(M, g)$ is a conformal invariant which can be used for a study of open manifolds. Note that $Q(M, g) \leq Q(S^n, g_0)$ where g_0 is the standard metric on S^n. Now we state our Theorem.

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THEOREM 1. *Let (M,g) be a noncompact complete Riemannian manifold of dimension $n \geq 3$ with scalar curvature S and infinite volume. Assume that $Q(M,g) > 0$ and $\int_M |S|^{2n/(n+2)} + |S|^{n/2}\, dV_g < \infty$. Then, there exists a conformal metric $\overline{g} = u^{4/(n-2)}g$ whose scalar curvature is 0. Moreover, u satisfies the followings:*

$$\text{(B)} \qquad \int_M |\nabla(u-1)|^2 + |u-1|^{2n/(n-2)}\, dV_g < \infty$$

and

$$\int_M |\nabla(u-1)|^2 + |u-1|^{2n/(n-2)}\, dV_g \to 0 \text{ as}$$
$$\int_M |S|^{2n/(n+2)} + |S|^{n/2}\, dV_g \to 0.$$

Note that $Q(M,g) \geq 0$ is a necessary condition for (M,g) to have a conformal metric with zero scalar curvature (see [3]). When (M,g) is conformal to a subdomain $(K - \Gamma, h)$ of a compact Riemannian manifold (K,h) and Γ is a smooth submanifold of (K,h), $Q(M,g) > 0$ is a necessary condition for (M,g) to have a complete conformal metric with zero scalar curvature (see [2]). Conformal changes of a metric to zero scalar curvature on a subdomain of a compact Riemannian manifold have been studied by Ma and McOwen [6]. Recently, Li [5] studied conformal changes of a metric to a constant negative scalar curvature on noncompact complete Riemannian manifolds with a lower bound of the first eigenvalue of the conformal Laplacian and a lower bound of scalar curvature. In this paper, we study conformal metrics with zero scalar curvature using integrals of scalar curvature.

2. Proof of main results

First we show the existence of a solution of (A). Using the Fredholm alternative, the following existence of a conformal metric on a smooth bounded domain with nonzero boundary data can be shown (see [3]).

LEMMA 1. *Let K_i be a smooth bounded domain with boundary ∂K_i. If $Q(M,g) > 0$, then there exists a unique positive solution u_i of (A) with $u_i = 1$ on ∂K_i.*

Assume that there exists a sequence $\{K_i\}$ of smooth bounded domains with $K_i \subset K_{i+1}$ and $\cup K_i = M$. By Lemma 1, there exists a smooth positive solution u_i of (A) on each K_i and $u_i = 1$ on ∂K_i. We extend the domain of u_i by defining $u_i = 1$ on the outside of K_i and use the same notation u_i for this extension.

We construct a positive solution of (A) on M by gluing solutions u_i of equation (A) on each K_i. For this, the behavior of u_i should be studied. Note that $h_i > -1$ where $h_i = u_i - 1$.

CLAIM 1. *$\int_M |h_i|^{2n/(n-2)}\, dV_g$ is bounded.*

PROOF. Note that h_i satisfies the following equation

$$-c_n \Delta h_i + S h_i = -S \text{ on } K_i. \tag{1}$$

From the given condition of Theorem 1,

(2)

$$\begin{aligned}
&Q(M,g)\Big(\int_{K_i} |h_i|^{2n/(n-2)}\, dV_g\Big)^{(n-2)/n} \\
&\le \int_{K_i} (-c_n \Delta h_i + S h_i) h_i\, dV_g \\
&= \int_{K_i} -S h_i\, dV_g \\
&\le \Big(\int_{K_i} |S|^{2n/(n+2)}\, dV_g\Big)^{(n+2)/2n} \Big(\int_{K_i} |h_i|^{2n/(n-2)}\, dV_g\Big)^{(n-2)/2n}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
Q(M,g)&\Big(\int_{K_i} |h_i|^{2n/(n-2)}\, dV_g\Big)^{(n-2)/2n} \\
&\le \Big(\int_{K_i} |S|^{2n/(n+2)}\, dV_g\Big)^{(n+2)/2n} \\
&\le \Big(\int_{M} |S|^{2n/(n+2)}\, dV_g\Big)^{(n+2)/2n}.
\end{aligned} \tag{3}$$

Claim 1 is proved. □

Following Aviles and McOwen [1], we show that there is a uniform bound for u_i on each compact subset of M.

CLAIM 2. *For each given compact set $X \subset M$, there exists a constant C_0 such that*

$$\max_{x \in X} u_i(x) \leq C_0,$$

where C_0 is a constant independent of i.

PROOF. Since X is compact,there exist $R > 0$ and a finite number of balls $B_R(y_1) \cdots B_R(y_N)$ which cover X with $y_k \in X$ for $k = 1 \cdots N$. Let $W = \cup_{k=1}^{k=N} B_{2R}(y_k)$ and Y be smooth bounded domains with $W \subset\subset Y$. Since u_i satisfies $-c_n \Delta u_i + S u_i = 0$ on Y for large i,

$$\sup_{x \in X} u_i(x) \leq \sup_{x \in B_R(y_k)} u_i(x) \leq C R^{-n/p} ||u_i||_{L^p(B_{2R}(y_k))}$$

for some $k \in \{1, \cdots, N\}$, where $p > 1$ and C depends only on n, p and Y (see [4]). By Claim 1 and Minkowski's inequality,

$$\begin{aligned} ||u_i||_{L^p(B_{2R}(y_k))} &= ||h_i + 1||_{L^p(B_{2R}(y_k))} \\ &\leq ||h_i||_{L^p(B_{2R}(y_k))} + |W|^{(n-2)/2n} \\ &\leq ||h_i||_{L^p(M)} + |W|^{(n-2)/2n} \end{aligned}$$

for $p = 2n/(n-2)$, where $|W|$ is a Riemannian volume of W. Therefore, there exists a uniform bound on $\sup_{x \in X} u_i(x)$ for large i. □

Using the standard elliptic estimates (see [1]) and Claim 2, we have a convergent subsequence $\{u_{i_k}\}$ which converges to u in $C^{2,\alpha}$ on each compact subset. By the maximum principle, there exists a nonnegative solution u for (A).

Next we give a uniform bound on $\int_M |\nabla h_i|^2 \, dV_g$.

CLAIM 3. *$\int_M |\nabla h_i|^2 \, dV_g$ is bounded.*

PROOF. Using (2), (3) and Hölder inequality,

$$\begin{aligned}
&\int_{K_i} c_n|\nabla h_i|^2\, dV_g \\
&= \int_{K_i} -c_n h_i \Delta h_i\, dV_g \\
&= \int_{K_i} -Sh_i - Sh_i^2\, dV_g \\
&\leq \int_{K_i} |Sh_i| + |Sh_i^2|\, dV_g \\
&\leq \int_{K_i} |Sh_i|\, dV_g + \Big(\int_{K_i} |S|^{n/2} dV_g\Big)^{2/n} \Big(\int_{K_i} |h_i|^{2n/(n-2)}\, dV_g\Big)^{(n-2)/n} \\
&\leq \Big(\int_{K_i} |S|^{2n/(n+2)} dV_g\Big)^{(n+2)/n} / Q(M,g) \\
&\quad + \Big(\int_{K_i} |S|^{n/2} dV_g\Big)^{2/n} \Big(\int_{K_i} |S|^{2n/(n+2)} dV_g\Big)^{(n+2)/n} / Q(M,g)^2 \\
&\leq \Big(\int_{M} |S|^{2n/(n+2)} dV_g\Big)^{(n+2)/n} / Q(M,g) \\
&\quad + \Big(\int_{M} |S|^{n/2} dV_g\Big)^{2/n} \Big(\int_{M} |S|^{2n/(n+2)} dV_g\Big)^{(n+2)/n} / Q(M,g)^2.
\end{aligned}$$

We conclude that $\int_M |\nabla h_i|^2\, dV_g < \infty$. □

From Claim 1, Claim 3 and $u_i = 1 + h_i \to u$, we have (B) in Theorem 1. Since volume of (M, g) is infinite and (B) holds, u can not be identically zero. By the maximum principle, we have a positive solution of (A). The last part of Theorem 1 comes from the estimates of Claim 1 and Claim 3.

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