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STABILITY THEOREM FOR THE FEYNMAN INTEGRAL VIA ADDITIVE FUNCTIONALS

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ABSTRACT. Recently, a stability theorem for the Feynman integral as a bounded linear operator on $L_2(\mathbb{R}^d)$ with respect to measures whose positive and negative variations are in the generalized Kato class was proved. We study a stability theorem for the Feynman integral with respect to measures whose positive variations are in the class of σ-finite smooth measures and negative variations are in the generalized Kato class. This extends the recent result in the sense that the class of σ-finite smooth measures properly contains the generalized Kato class.

0. Introduction

Since Feynman path integral was introduced in 1948 in [7], its existence theory has been developed by many mathematicians. In recent years the scope of the existence theory for the analytic operator-valued Feynman integral was widely extended using the theory of additive functionals in the framework of Dirichlet forms [1]. Existence theorems of the analytic operator-valued Feynman integral of the functions determined by smooth measures were proved under some conditions in [1]. So it is natural to ask the corresponding operator-valued Feynman integrals are stable under perturbations of these smooth measures. In [5], a stability theorem for the Feynman integral with respect to measures whose positive and negative variations are in the generalized Kato class, denoted by GK_d, was proved. It is a partial extension of Lapidus' result in [15].

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In this paper, we extend the results in [5]. In fact, we prove a stability theorem for the Feynman integral with respect to signed measures $\mu, \mu_n, n = 1, 2, \cdots$ satisfying the following conditions: For each Borel set E in $\mathbb{R}^d$, $\{\mu_n(E)\}_{n=1}^{\infty}, \{\mu_n^-(E)\}_{n=1}^{\infty}$ are nonincreasing sequences and $\mu_n(E)$ converges to $\mu(E)$ and there exist $\nu \in S_\sigma$ and $\eta \in GK_d$ such that $\mu_n^+ \leq \nu$ and $\mu_n^- \leq \eta$ for all $n \in \mathbb{N}$. Here S_σ stands for the class of all σ-finite smooth measures. It is presumably well known fact that GK_d is properly contained in S_σ [1,3]. So Theorem 3.6 which is our main theorem in this paper is a natural extension of Theorem 4.2 in [5].

1. Preliminaries

Our main concern in this paper lies on the specific functionals determined by signed smooth measures (See section 2). Generalized Kato class measures were considered in connection with Schrödinger semigroups [19] and the concept of smooth measures was introduced by M. Fukushima in the description of the class of Revuz measures associated with positive continuous additive functionals in the Dirichlet space setting [8]. Moreover, the relation between generalized Kato class measures and smooth measures was examined in [2]. Now we need to recall definitions and results related to Brownian motion, positive continuous additive functionals, measures in the generalized Kato class, smooth measures, closed forms and their associated operators.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P_x)$ be the canonical Brownian motion on $\mathbb{R}^d$ [4]. Let t be a nonnegative real number. For each ω in $\Omega = C([0,\infty), \mathbb{R}^d)$, the collection of all continuous functions from $[0,\infty)$ to $\mathbb{R}^d$, we define a function $\theta_t\omega : [0,\infty) \to \mathbb{R}^d$ by $(\theta_t\omega)(s) = \omega(t+s)$ for all s in $[0,\infty)$.

DEFINITION 1.1. A function $A : [0,\infty) \times \Omega \to \mathbb{R}$ is called a positive continuous additive functional (abbreviated by PCAF) if $A(t,\cdot) = A_t$ is $\mathcal{F}_t$-measurable for each t and there exists $\Lambda \in \mathcal{F}$ (called a defining set of A) satisfying the following properties:

(i) $P_x(\Lambda) = 1$ for all x in $\mathbb{R}^d$.

(ii) $\theta_t\omega \in \Lambda$ for all ω in Λ.

(iii) For each ω in Λ, the function $A_\cdot(\omega) : [0,\infty) \to \mathbb{R}$ is continuous, increasing and vanishes at 0 and is additive in the sense that

$$A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$$

for all $t, s \geq 0$.

For a nonnegative bounded Borel measurable function V on $\mathbb{R}^d$, we consider a function A^V defined on $[0,\infty) \times \Omega$ by

$$A^V(t,\omega) = A_t^V(\omega) = \int_0^t V(\omega(s))\,ds \tag{1.1}$$

for all (t,ω) in $[0,\infty) \times \Omega$. This is a typical example of a positive continuous additive functional.

DEFINITION 1.2. A positive Borel measure μ on $\mathbb{R}^d$ is said to be in the generalized Kato class if

$$\lim_{\alpha\to 0^+} \sup_{x\in\mathbb{R}^d} \int_{|x-y|\leqslant\alpha} \frac{\mu(dy)}{|x-y|^{d-2}} = 0\,, \quad d \geqslant 3,$$

$$\lim_{\alpha\to 0^+} \sup_{x\in\mathbb{R}^d} \int_{|x-y|\leqslant\alpha} (\log|x-y|^{-1})\mu(dy) = 0\,, \quad d = 2,$$

$$\sup_{x\in\mathbb{R}^d} \int_{|x-y|\leqslant 1} \mu(dy) < \infty\,, \quad d = 1.$$

We denote by GK_d the generalized Kato class.

Let $H^1(\mathbb{R}^d)$ be the standard Sobolev space, i.e.,

$$H^1(\mathbb{R}^d) \equiv \{u \in L_2(\mathbb{R}^d,m) |\ \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^d,m), 1 \leqslant i \leqslant d\} \tag{1.2}$$

where $L_2(\mathbb{R}^d,m)$ denotes the space of $\mathbb{R}$-valued functions on $\mathbb{R}^d$ which are square integrable with respect to the Lebesgue measure m and the derivatives are taken in the distributional sense. In this paper, we adopt $L_2(\mathbb{R}^d)$ instead of $L_2(\mathbb{R}^d,m)$. For a form q and an operator H, $D(q)$ and $D(H)$ stand for the domains of q and H, respectively. We let $\mathcal{E}$ denote the classical Dirichlet form, that is, the bilinear form acting on $D(\mathcal{E}) \equiv H^1(\mathbb{R}^d)$:

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{\mathbb{R}^d} \nabla u \cdot \nabla v\, dm \tag{1.3}$$

and for $u, v \in D(\mathcal{E})$, we define

$$\mathcal{E}_1(u,v) = \frac{1}{2}\int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, dm + \int_{\mathbb{R}^d} uv \, dm . \tag{1.4}$$

We now give the definition of capacity and the definition of smooth measure.

DEFINITION 1.3. Given an open set G in $\mathbb{R}^d$, let

$$Cap(G) = inf\{\mathcal{E}_1(u,u) \mid u \in H^1(\mathbb{R}^d) \text{ and } u \geq 1 \text{ a.e. on } G\} . \tag{1.5}$$

For an arbitrary set A in $\mathbb{R}^d$, let

$$Cap(A) = inf\{Cap(G) \mid A \subset G \subset \mathbb{R}^d, \ G \text{ is open}\} . \tag{1.6}$$

DEFINITION 1.4. A (positive) Borel measure μ on $\mathbb{R}^d$ is called smooth if μ charges no set of zero capacity and if there exists an increasing sequence $\{F_n\}$ of compact sets such that

$$\mu(F_n) < \infty \text{ for } n \geq 1, \text{and} \tag{1.7}$$

$$\lim_n Cap(K - F_n) = 0 \text{ for any compact set } K \subset \mathbb{R}^d . \tag{1.8}$$

We shall denote by S the family of all smooth measures and by S_σ the family of all σ-finite smooth measures.

Noting that every generalized Kato class measure is a Radon measure, the following proposition was established in [3, Theorem 2.1].

PROPOSITION 1.5. $GK_d \subset S_\sigma \subset S$.

For a signed Borel measure $\mu = \mu^+ - \mu^-$ on $\mathbb{R}^d$ where μ^+ and μ^- are the usual positive and negative variations of μ, respectively, we say that μ is in $S_\sigma - GK_d$ if μ^+ is in S_σ and μ^- is in GK_d. For μ in $S_\sigma - GK_d$, we define $\mathcal{Q}_\mu$ and $\mathcal{E}_\mu$ as follows:

$$\mathcal{Q}_\mu(u,v) \equiv \int_{\mathbb{R}^d} uv \, d\mu = \int_{\mathbb{R}^d} uv \, d\mu^+ - \int_{\mathbb{R}^d} uv \, d\mu^- \tag{1.9}$$

for all u, v in $D(\mathcal{Q}_\mu) \equiv L_2(\mathbb{R}^d, |\mu|) \cap L_2(\mathbb{R}^d)$ and

$$\mathcal{E}_\mu(u,v) \equiv \mathcal{E}(u,v) + \mathcal{Q}_\mu(u,v) \tag{1.10}$$

for all u, v in $D(\mathcal{E}_\mu) \equiv D(\mathcal{E}) \cap D(\mathcal{Q}_\mu)$.

For μ in $S_\sigma - GK_d$, let A^{μ^+} and A^{μ^-} be PCAF's corresponding to μ^+ and μ^-, respectively. (The existence of A^{μ^+} and A^{μ^-} are guaranteed by [1, Theorem 3.3.10] and [1, Theorem 3.2.3], respectively). We let $A_t^\mu = A_t^{\mu^+} - A_t^{\mu^-}$. Then $(A_t^\mu)_{t\geqslant 0}$ is a continuous additive functional which has finite variation on every bounded interval [8]. Let us introduce the notation

$$p_t^\mu f(x) = E_x[e^{-A_t^\mu} f(\omega(t))] \tag{1.11}$$

provided that the right-hand side in (1.11) makes sense for $f \in L_2(\mathbb{R}^d)$ where E_x stands for the expectation with respect to P_x and P_x is the probability measure associated with the Brownian paths in $\mathbb{R}^d$ which start at x at time 0.

Let $\mathcal{H}$ be a real or complex Hilbert space equipped with the inner product $\langle\,,\rangle$ and the norm $\|\cdot\|$. From [13], we have the following theorem.

THEOREM 1.6. *Let q be a densely defined, symmetric closed form in $\mathcal{H}$ which is bounded below by γ. Then there exists a unique bounded below self-adjoint operator H satisfying that for any $\xi \leqslant \gamma$, $D(q) = D((H-\xi)^{\frac{1}{2}})$ and $q(u,v) = \langle (H-\xi)^{\frac{1}{2}}u, (H-\xi)^{\frac{1}{2}}v\rangle + \xi\langle u,v\rangle$, for all u, v in $D(q)$. Furthermore, $q(u,v) = \langle Hu, v\rangle$ for all u in $D(H)$, v in $D(q)$.*

From [1, Proposition 3.4.3 and Proposition 3.4.4], we have the following proposition.

PROPOSITION 1.7. *Let $\mu = \mu^+ - \mu^-$ be in $S_\sigma - GK_d$. Then*

(i) *$\mathcal{E}_\mu$ is a densely defined symmetric bilinear form with domain $D(\mathcal{E}_\mu) = D(\mathcal{E}) \cap D(Q_\mu)$.*
(ii) *$\mathcal{E}_\mu$ is closed and bounded below.*
(iii) *$(p_t^\mu)_{t\geqslant 0}$ is a strongly continuous symmetric semigroup on $L_2(\mathbb{R}^d)$.*

Moreover, let H^μ be the bounded below self-adjoint operator corresponding to $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ whose existence is guaranteed by Theorem 1.6 *and let $\tilde{H}^\mu$ be the infinitesimal generator of $(p_t^\mu)_{t\geqslant 0}$. Then*

(iv) $H^\mu = -\tilde{H}^\mu$

and hence we have

$$p_t^{\mu} f(x) = e^{-tH^{\mu}} f(x) \tag{1.12}$$

for all f in $L_2(\mathbb{R}^d)$.

REMARK. By (1.11) and (1.12), we obtain the Feynman-Kac formula

$$e^{-tH^{\mu}} f(x) = E_x[e^{-A_t^{\mu}(\omega)} f(\omega(t))] \tag{1.13}$$

for every f in $L_2(\mathbb{R}^d)$, *m-a.e.* x in $\mathbb{R}^d$ and for all $t \geq 0$.

Now we extend $\mathcal{E}_\mu$ to the subspace $D(\mathcal{E}_\mu^{\mathbb{C}}) \equiv D(\mathcal{E}_\mu) + iD(\mathcal{E}_\mu)$ of $L_2(\mathbb{R}^d, \mathbb{C}) \equiv L_2(\mathbb{R}^d) + iL_2(\mathbb{R}^d)$ where $i = \sqrt{-1}$.
Define $\mathcal{E}_\mu^{\mathbb{C}} : D(\mathcal{E}_\mu^{\mathbb{C}}) \longrightarrow \mathbb{C}$ by

$$\mathcal{E}_\mu^{\mathbb{C}}(u, v) \equiv \int_{\mathbb{R}^d} \nabla u \cdot \overline{\nabla v}\, dm + \int_{\mathbb{R}^d} u\bar{v}\, d\mu \tag{1.14}$$

for all u, v in $D(\mathcal{E}_\mu^{\mathbb{C}})$. From [1], we have the following propositions.

PROPOSITION 1.8. *Let μ be in $S_\sigma - GK_d$. Then for $u = u_1 + iu_2, v = v_1 + iv_2$ in $D(\mathcal{E}_\mu^{\mathbb{C}})$, $\mathcal{E}_\mu^{\mathbb{C}}$ is represented as follows:*

$$\mathcal{E}_\mu^{\mathbb{C}}(u, v) = \mathcal{E}_\mu(u_1, v_1) + \mathcal{E}_\mu(u_2, v_2) + i[\mathcal{E}_\mu(u_2, v_1) - \mathcal{E}_\mu(u_1, v_2)]\,. \tag{1.15}$$

PROPOSITION 1.9. *Let $\mu = \mu^+ - \mu^-$ be in $S_\sigma - GK_d$. Then*

(i) *$\mathcal{E}_\mu^{\mathbb{C}}$ is a densely defined symmetric sesquilinear form.*
(ii) *$\mathcal{E}_\mu^{\mathbb{C}}$ is bounded below and closed.*

Moreover, let $H_{\mathbb{C}}^{\mu}$ be the bounded below self-adjoint operator corresponding to $(\mathcal{E}_\mu^{\mathbb{C}}, D(\mathcal{E}_\mu^{\mathbb{C}}))$ whose existence is guaranteed by Theorem 1.6. *Then we obtain*

$$(e^{-tH_{\mathbb{C}}^{\mu}} u)(x) = E_x[e^{-A_t^{\mu}(\omega)} u(\omega(t))] \tag{1.16}$$

for every u in $L_2(\mathbb{R}^d, \mathbb{C})$, m-a.e. x in $\mathbb{R}^d$ and for all $t \geqslant 0$.

2. The existence of the analytic (in time) operator-valued Feynman integral

Now we introduce the definition and the existence theorem of the analytic (in time) operator-valued Feynman integral of functions that we are especially interested in. Given ω in $\Omega = C([0,\infty),\mathbb{R}^d)$, let

$$F_t^\mu(\omega) = F^\mu(\omega) = e^{-A_t^\mu(\omega)} \tag{2.1}$$

where μ is in $S_\sigma - GK_d$ and A_t^μ is given in Section 1. Let $\mathbb{C}, \mathbb{C}_+$ and $\overline{\mathbb{C}}_+$ be the set of all complex numbers, all complex numbers with positive real part and all nonzero complex numbers with nonnegative real part, respectively.

DEFINITION 2.1. Given $t > 0$, $u \in L_2(\mathbb{R}^d, \mathbb{C})$ and $x \in \mathbb{R}^d$, consider the expression

$$\begin{aligned}(J^t(F^\mu)u)(x) &= E_x\{e^{-A_t^\mu(\omega)}u(\omega(t))\} \\ &= \int_{\Omega_x} e^{-A_t^\mu(\omega)}u(\omega(t))\, dP_x(\omega),\end{aligned} \tag{2.2}$$

where Ω_x is the set of ω in $C([0,\infty),\mathbb{R}^d)$ such that $\omega(0) = x$ and P_x is the probability measure associated with the Brownian paths in $\mathbb{R}^d$ which start at x at time 0. We say that the operator-valued function space integral $J^t(F^\mu)$ exists for $t > 0$ if (2.2) defines $J^t(F^\mu)$ as an element of $\mathcal{L}(L_2(\mathbb{R}^d,\mathbb{C}))$, the space of bounded linear operators on $L_2(\mathbb{R}^d,\mathbb{C})$. If $J^t(F^\mu)$ exists for every $t > 0$ and, in addition, has an extension as a function of t to an analytic operator-valued function on $\mathbb{C}_+$, and a strongly continuous function on $\overline{\mathbb{C}}_+$, we say that $J^t(F^\mu)$ exists for all $t \in \overline{\mathbb{C}}_+$. When t is purely imaginary, $J^t(F^\mu)$ is called the analytic (in time) operator-valued Feynman integral of F^μ.

The following theorem comes from [1]. We state it and give a sketch of its proof for convenience.

THEOREM 2.2. *Let $\mu = \mu^+ - \mu^-$ be in $S_\sigma - GK_d$ and let $\mathcal{E}_\mu^{\mathbb{C}}$ be given by (1.14) and $H_{\mathbb{C}}^\mu$ be the self-adjoint operator corresponding* *to* $(\mathcal{E}_{\mu}^{\mathbb{C}}, D(\mathcal{E}_{\mu}^{\mathbb{C}}))$. *Then* $J^t(F^{\mu})$ *exists for all* $t \in \overline{\mathbb{C}}_+$ *and has the representation*

$$J^t(F^{\mu}) = e^{-tH_{\mathbb{C}}^{\mu}} \tag{2.3}$$

for all $t \in \overline{\mathbb{C}}_+$, *where* $e^{-tH_{\mathbb{C}}^{\mu}}$ *is given meaning via the Spectral Theorem applied to the self-adjoint operator* $H_{\mathbb{C}}^{\mu}$. *In particular, for* $t \in \mathbb{R}$, *the analytic (in time) operator-valued Feynman integral* $J^{it}(F^{\mu})$ *exists and we have*

$$J^{it}(F^{\mu}) = e^{-itH_{\mathbb{C}}^{\mu}} \tag{2.4}$$

where $\{e^{-itH_{\mathbb{C}}^{\mu}}, t \in \mathbb{R}\}$ *is the unitary group corresponding to the self-adjoint operator* $H_{\mathbb{C}}^{\mu}$.

PROOF. By Proposition 1.7, $\mathcal{E}_{\mu}$ given by (1.10) is a densely defined, symmetric closed bilinear form which is bounded below and the continuous additive functional A_t^{μ} is related to the operator H^{μ} by the Feynman-Kac formula (1.13). Hence in the light of Theorem 2.2.5 in [1], the proof is complete. □

3. Stability theorem

The present paper owes our preceding paper [5], especially section 3. In order to prove Theorem 3.6, the main result of this paper, some known results in operator theory and a perturbation theorem which was proved in [5] are necessary. In fact, the main theorem is closely related to perturbation theories for closed forms. So we collect some important results of [5] first, and then we state a theorem concerned with form cores for closed forms (See [2] for precise proofs.) which is also important to exhibit our main theorem.

Throughout this paper, let $\mathcal{H}$ denote a complex Hilbert space with the inner product $\langle\, , \rangle$ and the norm $\|\cdot\|$. Furthermore, for x_n, x in $\mathcal{H}$, let $x_n \to x$ denote that x_n is strongly convergent to x and $x_n \xrightarrow{w} x$ denote that x_n is weakly convergent to x. For operators A_n, A on $\mathcal{H}$, let $A_n \to A$ indicate that A_n converges to A in the strong operator topology.

DEFINITION 3.1. Let $A, A_m, m = 1, 2, \cdots$ be self-adjoint operators on $\mathcal{H}$. We say that $\{A_m\}_{m=1}^{\infty}$ converges to A in the strong resolvent sense if

$$[I + iA_m]^{-1} \to [I + iA]^{-1},$$

where I denotes the identity operator and $i = \sqrt{-1}$.

From [14] and [13], we have the following two theorems, respectively.

THEOREM 3.2. (Trotter, Kato, Rellich, Neveu) *Let H, H_m, $m = 1, 2, \cdots$ be self-adjoint operators on $\mathcal{H}$. Then the following statements are equivalent :*

(a) *$\{H_m\}_{m=1}^{\infty}$ converges to H in the strong resolvent sense.*
(b) *$e^{-itH_m} \to e^{-itH}$ for all t in $\mathbb{R}$.*
(c) *$[I + i\lambda H_m]^{-1} \to [I + i\lambda H]^{-1}$ for all λ in $\mathbb{R}$, $\lambda \neq 0$.*
(d) *$e^{-itH_m} \to e^{-itH}$, uniformly in t on any compact subset of $\mathbb{R}$.*

If, in addition, the operators H_m and H are uniformly bounded below, then (a) implies :

(e) *$e^{-tH_m} \to e^{-tH}$, uniformly in t on any compact subset of $[0, +\infty)$.*

THEOREM 3.3. *Let $\{t_n\}$ be a nonincreasing sequence of densely defined, closed symmetric forms in $\mathcal{H}$ which are uniformly bounded below by γ. If H_n is the self-adjoint operator associated with t_n, then H_n converges to a self-adjoint operator $H \geq \gamma$ strongly in the generalized sense. Furthermore, $(H_n - \xi)^{1/2}u \xrightarrow{w} (H - \xi)^{1/2}u$ for all u in $\bigcup_n D(t_n)$ and $\xi < \gamma$. If, in particular, the symmetric form t defined by $t(u, u) = \lim_{n\to\infty} t_n(u, u)$ with $D(t) = \bigcup_n D(t_n)$ is closable, then H is the self-adjoint operator associated with $\tilde{t}$, the closure of t, and $(H_n - \xi)^{1/2}u \to (H - \xi)^{1/2}u$ for all u in $D(t)$ and $\xi < \gamma$.*

In [5], the following theorem was proved.

THEOREM 3.4. *Let t, t_n, $n = 1, 2, \cdots$ be densely defined, symmetric closed forms in $\mathcal{H}$ satisfying the following properties where H and H_n are the self-adjoint operators associated with t and t_n, respectively:*

(i) *$t(u, u) \geq \gamma\langle u, u\rangle$, for all u in $D(t)$,*
$t_n(u, u) \geq \gamma\langle u, u\rangle$, for all u in $D(t_n)$, $n = 1, 2, \cdots$ with $\gamma < 0$.

(ii) *There is a core D' of t such that $D' \subset \liminf D(t_n)$ and for some $\alpha < \gamma$, $(H_n - \alpha)^{1/2}u \to (H - \alpha)^{1/2}u$ for all u in D'.*

Then $\{H_n\}_{n=1}^{\infty}$ converges to H in the strong resolvent sense.

In general, a σ-finite smooth measure may not be a Radon measure. Furthermore, if μ is a nowhere Radon smooth measure, then it may happen that $D(\mathcal{E}_\mu)$ contains no non-trivial continuous functions. Thus it is necessary to find out a relatively nice form core for a closed form. To this end we define a class of functions $C_q(\mathbb{R}^d)$ as follows:

(3.1)
$$C_q(\mathbb{R}^d) = \{f \,|\, f \text{ is bounded Borel measurable, quasi-continuous, and with compact support}\}.$$

PROPOSITION 3.5. *Let μ be a measure in $S_\sigma - GK_d$. Then $D(\mathcal{E}_\mu) \cap C_q(\mathbb{R}^d)$ is a core of $\mathcal{E}_\mu$ and hence $H^1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d, |\mu|) \cap C_q(\mathbb{R}^d)$ is a core of $\mathcal{E}_\mu$.*

PROOF. Since μ is a measure in $S_\sigma - GK_d$, there exist real constant $\lambda > 1$ and real constants c and β such that $||p_t^{\mu^+ - \lambda\mu^-} f||_2 \le ce^{\beta t}||f||_2$ for all $t \ge 0$ and $f \in L_2(\mathbb{R}^d)$. (See [1, Proposition 3.4.7 and Theorem 3.4.8]). Noting that $\mathcal{E}_\mu$ is a closed form, we conclude that $D(\mathcal{E}_\mu) \cap C_q(\mathbb{R}^d)$ is a core of $\mathcal{E}_\mu$. (See [2, Theorem 5.7]). □

THEOREM 3.6. *Let μ, $\mu_n, n = 1, 2, \cdots$ be signed measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ satisfying the following properties:*

(i) *For each Borel set E in $\mathbb{R}^d$, $\{\mu_n(E)\}_{n=1}^{\infty}$, $\{\mu_n^-(E)\}_{n=1}^{\infty}$ are non-increasing sequences and $\mu_n(E)$ converges to $\mu(E)$.*

(ii) *There exist Radon measures ν and η such that*

$$\mu_n^+ \le \nu \in S_\sigma \quad , \quad \mu_n^- \le \eta \in GK_d$$

for all $n \in \mathbb{N}$.

For simplicity, let $t_n = \mathcal{E}_{\mu_n}^{\mathbb{C}}$ and $t = \mathcal{E}_{\mu}^{\mathbb{C}}$ where $\mathcal{E}_{\mu_n}^{\mathbb{C}}$ and $\mathcal{E}_{\mu}^{\mathbb{C}}$ are given in Section 1. Assume that t_n is uniformly bounded below by $\alpha < 0$. Then $\{H_n\}_{n=1}^{\infty}$ converges to H in the strong resolvent sense where H_n and H are self-adjoint operators associated with t_n and t, respectively.

REMARK 3.7. Using the hypothesis (i) in the above theorem, we get $\{\mu_n^+(E)\}_{n=1}^{\infty}$ is a nonincreasing sequence for each Borel set E in $\mathbb{R}^d$. Then the definitions of GK_d and S_σ and the hypothesis (ii) implies that for all $n \in \mathbb{N}$, μ_n is in $S_\sigma - GK_d$.

REMARK 3.8. A simple proof shows that the limiting measure μ is in $S_\sigma - GK_d$. To prove this, let $E \in \mathcal{B}(\mathbb{R}^d)$. Then, $\mu_n^+(E) \to inf\{\mu_n^+(E)\}$ and $\mu_n^-(E) \to inf\{\mu_n^-(E)\}$ as $n \to \infty$ by the monotone convergence theorem for sequences. Since $\mu_n(E) \to \mu(E) = \mu^+(E) - \mu^-(E)$ we get $\mu = \mu^+ - \mu^- = inf\mu_n^+ - inf\mu_n^-$ and this implies that $inf\mu_n^+ \geq \mu^+$ and $inf\mu_n^- \geq \mu^-$. Hence we conclude that $\mu^+ \in S_\sigma$ and $\mu^- \in GK_d$.

PROOF OF THEOREM 3.6. For each $n \in \mathbb{N}$, t_n is a densely defined, closed symmetric form which is bounded below by Remark 3.7 and Proposition 1.7. Using (1.15) and hypotheses on measures μ_n, a direct calculation shows that t_n is a nonincreasing sequence of forms. Since t_n is uniformly bounded below by α, we can define

$$q(f,f) = \lim_{n\to\infty} t_n(f,f) \tag{3.2}$$

for all f in $D(q) = \bigcup_n D(t_n)$. Let $f = g + ih$ be in $\bigcup_n D(t_n)$. By (1.15) and (3.2), we have

$$q(f,f) = \mathcal{E}(g,g) + \mathcal{E}(h,h) + \lim_{n\to\infty} \Big[\int_{\mathbb{R}^d} |g|^2 \, d\mu_n + \int_{\mathbb{R}^d} |h|^2 \, d\mu_n \Big] . \tag{3.3}$$

We claim that $q \subset t$. In fact, $D(q) \subset D(t)$. (See Remark 3.8). And so for the proof of $q \subset t$, it remains to show that for all $f = g + ih$ in $D(q)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |g|^2 \, d\mu_n = \int_{\mathbb{R}^d} |g|^2 \, d\mu \tag{3.4}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |h|^2 \, d\mu_n = \int_{\mathbb{R}^d} |h|^2 \, d\mu . \tag{3.5}$$

If $g = \chi_E$, where χ_E denotes the characteristic function of a Borel set E, (3.4) is true by hypotheses on measures μ_n and μ. For a simple function g, (3.4) is easily proved by using the case of characteristic functions. Suppose that g is a nonnegative Borel measurable function. Then there exists a nonnegative and nondecresing sequence $\{g_m\}$ of simple functions converging to g. By the monotone convergence theorem, we have

$$\lim_{m\to\infty}\int_{\mathbb{R}^d}|g_m|^2\,d\mu_n=\int_{\mathbb{R}^d}|g|^2\,d\mu_n \tag{3.6}$$

for all sufficiently large n.
And so,

$$\lim_{n\to\infty}\Big[\lim_{m\to\infty}\int_{\mathbb{R}^d}|g_m|^2\,d\mu_n\Big]=\lim_{n\to\infty}\int_{\mathbb{R}^d}|g|^2\,d\mu_n. \tag{3.7}$$

Using the iterated limit theorem for a double sequence and the case of simple functions, we have

$$\begin{aligned}&\lim_{n\to\infty}\Big[\lim_{m\to\infty}\int_{\mathbb{R}^d}|g_m|^2\,d\mu_n\Big]\\&=\lim_{m\to\infty}\Big[\lim_{n\to\infty}\int_{\mathbb{R}^d}|g_m|^2\,d\mu_n\Big]\\&=\lim_{m\to\infty}\Big[\int_{\mathbb{R}^d}|g_m|^2\,d\mu\Big]\\&=\int_{\mathbb{R}^d}|g|^2\,d\mu.\end{aligned} \tag{3.8}$$

By (3.7) and (3.8), we conclude that

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}|g|^2\,d\mu_n=\int_{\mathbb{R}^d}|g|^2\,d\mu. \tag{3.9}$$

For a Borel measurable function $g=g^+-g^-$, we easily get (3.4) by using the case of nonnegative Borel measurable functions. By the essentially same method as in the proof of (3.4), we can prove (3.5).

Now note that t is a closed form. Hence q is closable. By proposition 3.5, $D=D(\mathcal{E}_\mu)\cap C_q(\mathbb{R}^d)$ is a core of $\mathcal{E}_\mu$ and hence $D'=D+iD$ is a core of $t = \mathcal{E}_{\mu}^{\mathbb{C}}$. Furthermore, $D' \subset D(q) = \underset{n}{\cup} D(t_n) \subset D(t)$. Consequently, it is easy to show that t is the closure of q and t is bounded below with lower bound α. Then in the light of Theorem 3.3, $(H_n - \xi)^{\frac{1}{2}} u \to (H - \xi)^{\frac{1}{2}} u$ for all u in $D(q)$ and $\xi < \alpha$. Hence we conclude that $\{H_n\}_{n=1}^{\infty}$ converges to H in the strong resolvent sense by Theorem 3.4. □

COROLLARY 3.9. *Under the same conditions as in* Theorem 3.6,

$$J^{it}(F^{\mu_n}) \to J^{it}(F^{\mu}) \tag{3.10}$$

for all $t \in \mathbb{R}$.

PROOF. By virtue of Theorem 2.2, we get

$$J^{it}(F^{\mu_n}) = e^{-itH_{\mathbb{C}}^{\mu_n}} \quad and \quad J^{it}(F^{\mu}) = e^{-itH_{\mathbb{C}}^{\mu}} \tag{3.11}$$

where $H_{\mathbb{C}}^{\mu_n}$ and $H_{\mathbb{C}}^{\mu}$ are self-adjoint operators associated with $\mathcal{E}_{\mu_n}^{\mathbb{C}}$ and $\mathcal{E}_{\mu}^{\mathbb{C}}$, respectively. By Theorem 3.6 and Theorem 3.2, we get (3.10). □

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