Comm. Korean Math. Soc. **13** (1998), No. 3, pp. 515–523

UNIQUENESS FOR THE NONHARMONIC FOURIER SERIES OF DISTRIBUTIONS

MUN-JA CHO AND SOON-YEONG CHUNG

ABSTRACT. We generalize the uniqueness theorem of K.Yoneda[9] for nonharmonic series under a much weaker condition as follows:

Let $\{\lambda_k\}_{k=0}^{\infty}$ be a discrete sequence in $\mathbb{R}^n$. If $\sum_{k=0}^{\infty} a_k e^{i\langle\lambda_k, x\rangle} = 0$ for all $x \in \mathbb{R}^n$ and there exists a number $N > 0$ such that

$$\sum_{k=0}^{\infty} \frac{|a_k|}{1 + |\lambda_k|^N} < \infty$$

then $a_k = 0$ for all $k \in \mathbb{N}_0$.

1. Introduction

In this paper we shall discuss the uniqueness of the representation by the nonharmonic series

$$\sum_{k=0}^{\infty} a_k e^{i\lambda_k x}$$

for a system of complex exponentials $\{e^{i\lambda_k x}\}_{k=0}^{\infty}$. This is closely related to the completeness of the system $\{e^{i\lambda_k x}\}$ in the corresponding space. That is to say, every theorem on the uniqueness generates a theorem on the completeness, and vice versa.

In $L^2[-\pi, \pi]$, the situation is particularly simple. It is well known that if $|\lambda_k - k| \le 1/4$ then every function f in $L^2[-\pi, \pi]$ has a *unique*

Received January 24, 1998. Revised May 5, 1998.
1991 Mathematics Subject Classification: Primary 46F23; Secondary 46F15.
Key words and phrases: nonharmonic series, tempered distribution.
Partially supported by GARC-KOSEF and Ministry of Education.

nonharmonic series expansion

$$f(x) = \sum_{k=0}^{\infty} a_k e^{i\lambda_k x}$$

with $\sum |a_k|^2 < \infty$, since the trigonometric system $\{e^{ikx}\}$ is *stable* in $L^2[-\pi, \pi]$ under sufficiently small perturbations of the integers (see [10]). But, in general, the system $\{e^{i\lambda_k x}\}$ is not complete even in $L^2[-\pi, \pi]$, even if the sequence $\{\lambda_k\}$ is discretely distributed on $\mathbb{R}$.

Recently in the paper [9] of K. Yoneda, he proved a uniqueness of the representation by nonharmonic series

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos \lambda_k x + b_k \sin \lambda_k x)$$

where $\lambda_k > 0$, $\lambda_k \to \infty$ as $k \to \infty$ and

$$\frac{1}{2}|a_0| + \sum_{k=1}^{\infty} \frac{|a_k| + |b_k|}{\lambda_k^2} < \infty.$$

The purpose of this paper is to generalize the uniqueness theorem for the nonharmonic series under much weaker condition than Yoneda's. Moreover, we shall give it in the n-dimensional version. Actually, one of the main theorems of this paper is as follows:

THEOREM 3.4. *Let $\{\lambda_k\}_{k=0}^{\infty}$ be a discrete sequence in $\mathbb{R}^n$. If $\sum_{k=0}^{\infty} a_k e^{i\langle \lambda_k, x\rangle} = 0$ for all $x \in \mathbb{R}^n$ and there exists a number $N > 0$ such that*

$$\sum_{k=0}^{\infty} \frac{|a_k|}{1 + |\lambda_k|^N} < \infty$$

then $a_k = 0$ for all $k \in \mathbb{N}_0$.

We shall prove this theorem by using the Schwartz distribution theory, which is quite a different method from that of Yoneda's.

2. Preliminaries

We first introduce the Schwartz distribution. By $\mathcal{S}(\mathbb{R}^n)$ we denote the set of infinitely differentiable functions φ in $\mathbb{R}^n$ satisfying the condition that for any α and $\beta \in \mathbb{N}_0^n$

$$\|\varphi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial^\beta \varphi(x)\right|$$

is finite. This condition is equivalent to

$$\sup_{|\beta|\le m} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^m |\partial^\beta \varphi(x)| < \infty$$

for all $m \in \mathbb{N}_0$. Here we use the conventional multi-index notations such as; $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$, $\partial_j = \dfrac{\partial}{\partial x_j}$, $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_n$ etc. for any multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}_0^n$ where $\mathbb{N}_0$ is the set of all nonnegative integers.

By $\mathcal{S}'(\mathbb{R}^n)$ we denote the set of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$ under the topology given by the semi-norms $\|\cdot\|_{\alpha,\beta}$ for each α and $\beta \in \mathbb{N}_0^n$. We usually call the element of $\mathcal{S}(\mathbb{R}^n)$ a rapidly decreasing function and the element of $\mathcal{S}'(\mathbb{R}^n)$ a tempered distribution (see [1] for more details).

For a function φ in $\mathcal{S}(\mathbb{R}^n)$, the Fourier transform is defined by

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} \varphi(x) e^{-i\langle x,\xi\rangle} dx, \qquad \xi \in \mathbb{R}^n.$$

For a tempered distribution u in $\mathcal{S}'(\mathbb{R}^n)$ its Fourier transform $\widehat{u}$ is defined by

$$\langle \widehat{u}, \varphi\rangle = \langle u, \widehat{\varphi}\rangle, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Then it is well known that the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ is an isomorphism onto itself, respectively.

3. Main results

Here we shall give a more generalized uniqueness theorem for nonharmonic series. Recently K. Yonena proved the following theorem:

THEOREM ([9]). *Let* $\lambda_k > 0$, $\lambda_k \to \infty$ *as* $k \to \infty$ *and*

$$\frac{1}{2}|a_0| + \sum_{k=1}^{\infty} \frac{|a_k| + |b_k|}{\lambda_k^2} < \infty. \tag{3.1}$$

If $\frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos \lambda_k x + b_k \sin \lambda_k x) = 0$ *for all* $x \in \mathbb{R}$, *then* $a_0 = a_1 = b_1 = \cdots = a_k = b_k = \cdots = 0$.

In this section we generalize this theorem by considering much weaker conditions than (3.1). We say that a sequence $\{\lambda_k\}_{k=0}^{\infty}$ in $\mathbb{R}^n$ is discrete if any subsequence of $\{\lambda_k\}$ does not converge.

THEOREM 3.1. *Let* $\{\lambda_k\}_{k=0}^{\infty}$ *be a discrete sequence of real numbers. If* $\sum_{k=0}^{\infty} a_k e^{i\lambda_k x} = 0$ *for all* $x \in \mathbb{R}$ *and there exists a number* $N > 0$ *such that*

$$\sum_{k=0}^{\infty} \frac{|a_k|}{1 + |\lambda_k|^N} < \infty \tag{3.2}$$

then $a_k = 0$ *for all* $k \in \mathbb{N}_0$.

PROOF. First, let $u_N(x) = \sum_{k=0}^{N} a_k e^{i\lambda_k x}$ and define $\langle u, \varphi \rangle = \lim_{N\to\infty} \langle u_N(x), \varphi \rangle$, for all $\varphi \in \mathcal{S}(\mathbb{R})$. Then this means that

$$\begin{aligned} \langle u, \varphi \rangle &= \lim_{N\to\infty} \left\langle \sum_{k=0}^{N} a_k e^{i\lambda_k x}, \varphi(x) \right\rangle \\ &= \lim_{N\to\infty} \sum_{k=0}^{N} a_k \left\langle e^{i\lambda_k x}, \varphi(x) \right\rangle \\ &= \sum_{k=0}^{\infty} a_k \left\langle e^{i\lambda_k x}, \varphi(x) \right\rangle, \qquad \varphi \in \mathcal{S}(\mathbb{R}). \end{aligned} \tag{3.3}$$

Note that $\xi^m\widehat{\varphi}(\xi) = \left[(-i\frac{d}{dx})^m\varphi\right]\widehat{\ }(\xi)$ for any $\varphi \in \mathcal{S}(\mathbb{R})$ and $m \in \mathbb{N}_0$. Then it follows that by integration by parts

$$
\begin{aligned}
(3.4) \qquad |\xi|^m\,|\widehat{\varphi}(\xi)| &= \left|\left[\left(-i\frac{d}{dx}\right)^m \varphi\right]\widehat{\ }(\xi)\right| \\
&= \left|\int e^{-ix\xi}\left(\frac{d}{dx}\right)^m \varphi(x)\,dx\right| \\
&\le C\sup_{x\in\mathbb{R}}(1+x^2)\left|\left(\frac{d}{dx}\right)^m \varphi(x)\right|
\end{aligned}
$$

where $C = \displaystyle\int_{\mathbb{R}} \frac{dx}{1+x^2} < \infty$.

First we shall show that u defines a tempered distribution.

If we put $M = \displaystyle\sum_{k=0}^{\infty} \frac{|a_k|}{1+|\lambda_k|^N}$, then we obtain from (3.3) and (3.4) that for any $\varphi \in \mathcal{S}(\mathbb{R})$

$$
\begin{aligned}
|\langle u,\varphi\rangle| &\le \sum_{k=0}^{\infty} |a_k|\,\left|\langle e^{i\lambda_k x}, \varphi(x)\rangle\right| \\
&\le \sum_{k=0}^{\infty} |a_k|\,|\widehat{\varphi}(-\lambda_k)| \\
&= \sum_{k=0}^{\infty} \frac{|a_k|}{1+|\lambda_k|^N}(1+|\lambda_k|^N)\,|\widehat{\varphi}(-\lambda_k)| \\
&\le MC\left[\sup_{x\in\mathbb{R}}(1+x^2)\,|\varphi(x)| + \sup_{x\in\mathbb{R}}(1+x^2)\left|\left(\frac{d}{dx}\right)^N \varphi(x)\right|\right] < \infty.
\end{aligned}
$$

Thus, u defines a distribution in $\mathcal{S}'(\mathbb{R})$.

Suppose that $V_k \subset \mathbb{R}$ is a neighborhood of λ_k which does not contain λ_j for $j \neq k$. Let $\varphi_k \in C_0^\infty(\mathbb{R})$ such that $\mathrm{supp}\varphi_k \subset V_k$ and $\varphi_k \equiv 1$ near λ_k.

Thus the hypothesis $\displaystyle\sum_{k=0}^{\infty} a_k e^{i\lambda_k x} = 0$ for all $x \in \mathbb{R}$ implies that $u = 0$

as a tempered distribution. Thus using $\langle e^{i\lambda_k x}, \widehat{\varphi}_j(x)\rangle = 2\pi\varphi_j(\lambda_k)$ by the inversion formula we have

$$\begin{aligned} 0 = \langle u, \widehat{\varphi}_j\rangle &= \sum_{k=0}^{\infty} a_k \langle e^{i\lambda_k x}, \widehat{\varphi}_j(x)\rangle \\ &= 2\pi \sum_{k=0}^{\infty} a_k \varphi_j(\lambda_k) \\ &= 2\pi a_j \end{aligned}$$

for all $j \in \mathbb{N}_0$. So $a_j = 0$ for all $j \in \mathbb{N}_0$. Thus, this theorem is proved. □

REMARK. In the above theorem the condition (3.2) is much weaker than (3.1) in Yoneda's theorem. In fact, (3.1) is a particular case (3.2) with $N = 2$. By this consideration we can say that Yoneda proved the uniqueness of the nonharmonic series of distributions of order at most 2.

It is true that if a function f is almost everywhere zero, the tempered distribution T_f defined by f is zero in a distribution sense. So we get the following one:

COROLLARY 3.2. *Let $\{\lambda_k\}_{k=0}^{\infty}$ be a discrete sequence of real numbers. If $\sum_{k=0}^{\infty} a_k e^{i\lambda_k x} = 0$ a.e. $x \in \mathbb{R}$ and there exists a number $N > 0$ satisfying (3.2) then $a_k = 0$ for all $k \in \mathbb{N}_0$.*

In the proof of Theorem 3.1, if we consider $u = 0$ as a tempered distribution, then this result can be of the following form:

COROLLARY 3.3. *Let $\{\lambda_k\}_{k=0}^{\infty}$ be a discrete sequence of real numbers. If $\sum_{k=0}^{\infty} a_k e^{i\lambda_k x} = 0$ in $\mathcal{S}'(\mathbb{R})$ and there exists a number $N > 0$ satisfying (3.2) then $a_k = 0$ for all $k \in \mathbb{N}_0$.*

We have proved a uniqueness of nonharmonic Fourier series defined on $\mathbb{R}$ as a distribution of finite order. Now we shall give a uniqueness theorem for the nonharmonic Fourier series defined on $\mathbb{R}^n$.

THEOREM 3.4. *Let* $\{\lambda_k\}_{k=0}^{\infty}$ *be a discrete sequence in* $\mathbb{R}^n$. *If* $\sum_{k=0}^{\infty} a_k e^{i\langle\lambda_k,x\rangle} = 0$ *for all* $x \in \mathbb{R}^n$ *and there exists a number* $N > 0$ *such that*

$$\sum_{k=0}^{\infty} \frac{|a_k|}{1+|\lambda_k|^N} < \infty \tag{3.5}$$

then $a_k = 0$ *for all* $k \in \mathbb{N}_0$.

PROOF. Let $u_N(x) = \sum_{k=0}^{N} a_k e^{i\langle\lambda_k,x\rangle}$ and define $\langle u,\varphi\rangle = \lim_{N\to\infty} \langle u_N(x),\varphi\rangle$, for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

In the same way as the proof of theorem 3.1, we can express

$$\langle u,\varphi\rangle = \sum_{k=0}^{\infty} a_k \left\langle e^{i\langle\lambda_k,x\rangle}, \varphi(x)\right\rangle \tag{3.6}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Let $\xi = (\xi_1,\cdots,\xi_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1,\cdots,\alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| = N$. Since $\xi^\alpha\widehat{\varphi}(\xi) = ((-i)^{|\alpha|}\partial^\alpha\varphi)\widehat{\ }(\xi)$, we obtain that

$$\begin{aligned}
|\xi|^N|\widehat{\varphi}(\xi)| &\le |(|\xi_1|+\cdots+|\xi_n|)^N\widehat{\varphi}(\xi)| \\
&= \sum_{\alpha_1+\cdots+\alpha_n=N} \frac{N!}{\alpha_1!\cdots\alpha_n!}|\xi_1|^{\alpha_1}\cdots|\xi_n|^{\alpha_n}|\widehat{\varphi}(\xi)| \\
&= \sum_{|\alpha|=N} \frac{N!}{\alpha_1!\cdots\alpha_n!}|((-i)^{|\alpha|}\partial^\alpha\varphi)\widehat{\ }(\xi)| \\
&\le \sum_{|\alpha|=N} \frac{N!}{\alpha_1!\cdots\alpha_n!} C \sup_{x\in\mathbb{R}^n}(1+|x|^2)^{n+1}|\partial^\alpha\varphi(x)| \\
&\le C' \sup_{\substack{|\alpha|=N \\ x\in\mathbb{R}^n}} (1+|x|^2)^{n+1}|\partial^\alpha\varphi(x)|
\end{aligned} \tag{3.7}$$

where $C = \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{n+1}} < \infty$.

First we shall show that u defines a tempered distribution.

If we put $M = \sum_{k=0}^{\infty} \frac{|a_k|}{1+|\lambda_k|^N}$, then we obtain from (3.6) and (3.7) for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}
|\langle u, \varphi\rangle| &\le \sum_{k=0}^{\infty} |a_k| \left|\left\langle e^{i\langle \lambda_k, x\rangle}, \varphi(x)\right\rangle\right| \\
&\le \sum_{k=0}^{\infty} |a_k|\, |\widehat{\varphi}(-\lambda_k)| \\
&= \sum_{k=0}^{\infty} \frac{|a_k|}{1+|\lambda_k|^N} (1+|\lambda_k|^N)\, |\widehat{\varphi}(-\lambda_k)| \\
&\le MC \left[\sup_{x\in\mathbb{R}^n} (1+|x|^2)^{n+1} |\varphi(x)| + \sup_{\substack{|\alpha|=N \\ x\in\mathbb{R}^n}} (1+|x|^2)^{n+1} |\partial^\alpha \varphi(x)| \right] \\
&< \infty.
\end{aligned}$$

Thus, u defines a distribution in $\mathcal{S}'(\mathbb{R}^n)$.

Suppose that $V_k \subset \mathbb{R}^n$ is a neighborhood of λ_k which does not contain λ_j for $j \neq k$. Let $\varphi_k \in C_0^\infty(\mathbb{R}^n)$ such that supp $\varphi_k \subset V_k$ and $\varphi_k \equiv 1$ near λ_k. Thus the hypothesis $\sum_{k=0}^{\infty} a_k e^{i\langle \lambda_k, x\rangle} = 0$ for all $x \in \mathbb{R}^n$ implies that $u = 0$ as a tempered distribution. Thus using $\left\langle e^{i\langle\lambda_k, x\rangle}, \widehat{\varphi}_j(x)\right\rangle = (2\pi)^n \varphi_j(\lambda_k)$ by the inversion formula we have

$$\begin{aligned}
0 = \langle u, \widehat{\varphi}_j\rangle &= \sum_{k=0}^{\infty} a_k \left\langle e^{i\langle \lambda_k, x\rangle}, \widehat{\varphi}_j(x)\right\rangle \\
&= (2\pi)^n \sum_{k=0}^{\infty} a_k \varphi_j(\lambda_k) \\
&= (2\pi)^n a_j
\end{aligned}$$

for all $j \in \mathbb{N}_0$. So $a_j = 0$ for all $j \in \mathbb{N}_0$. Thus we complete the proof. □

Let $\{\lambda_k\}_{k=0}^{\infty}$ be a sequence satisfying the condition of Theorem 3.4. It is for the same reason as corollary 3.2 and 3.3 that we take the followings: If $\sum_{k=0}^{\infty} a_k e^{i\langle\lambda_k, x\rangle} = 0$ a.e. $x \in \mathbb{R}^n$ and there exists a number $N > 0$ such that

$$\sum_{k=0}^{\infty} \frac{|a_k|}{1 + |\lambda_k|^N} < \infty$$

then $a_k = 0$ for all $k \in \mathrm{N}_0$. Furthermore, it is true when $\langle \sum a_k e^{i\langle\lambda_k, x\rangle}, \varphi \rangle = 0$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

References

[1] M. A. Al-Gwaiz, *The Theory of Distributions*, Marcel Dekker, INC., New York, 1992.

[2] T. Carleman, *Über die Approximation Analytischer Funktionen durch Lineare Aggregate von Vorgegebene Potenzen*, Ark. Math. **17** (1922), 1-30.

[3] S.-Y. Chung, D. Kim and E.-G. Lee, *Fourier series and periodic hyperfunctions*, Preprint.

[4] R. J. Duffin and J. J. Eachus, *Some notes on an expansion theorem of Paley and Wiener*, Bull. Amer. Math. Soc. **48** (1942), 850-855.

[5] B. N. José, *An introduction to the theory of distributions*, Marcel Dekker, INC., New York, 1973.

[6] N. Levinson, *Gap and Density Theorems*, Amer. Math. Soc. Colloq. Publ., New York, 1940.

[7] Paley, R. E. A. C., and N. Wiener, *Fourier Transforms in the Complex Domain*, Amer. Math. Soc. Colloq. Publ., New York, 1934.

[8] L. Schwartz, *Théorie des Distributions*, Hermann, Paris, 1966.

[9] K. Yoneda, *Uniqueness for non-harmonic trigonometric Series*, Proc. Amer. Math. Soc. **124** (1992), 1795-1800.

[10] R. M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, INC., 1980.

Department of Mathematics
Sogang University
Seoul 121-742, Korea
E-mail: sychung@ccs.sogang.ac.kr