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WEAK CONVERGENCE THEOREMS FOR ALMOST-ORBITS OF AN ASYMPTOTICALLY NONEXPANSIVE SEMIGROUP IN BANACH SPACES

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Abstract. In this paper, we deal with the asymptotic behavior for the almost-orbits $\{u(t)\}$ of an asymptotically nonexpansive semigroup $\mathcal{S} = \{S(t) : t \in G\}$ for a right reversible semitopological semigroup G, defined on a suitable subset C of Banach spaces with the Opial's condition, locally uniform Opial condition, or uniform Opial condition.

1. Introduction

In [21], Opial established the following weak convergence theorem in a Hilbert space: Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive asymptotically regular mapping for which the set $\mathcal{F}(T)$ of fixed points is nonempty. Then, for any x in C, the sequence of successive approximations $\{T^n x\}$ is weakly convergent to an element of $\mathcal{F}(T)$ (cf. [3], [22]).

Similar results were also obtained in [4], [5], [13], and [17] in uniformly convex Banach spaces. Corresponding theorems for asymptotically nonexpansive mappings and asymptotically nonexpansive semigroups were investigated by many mathematicians ([2], [7], [19], [24], [31]). Other related results may be found in [1], [6], [25], [26], [32], and [33].

And also, Lau-Takahashi ([14]) proved the following theorem: Let C be a closed convex subset of a uniformly convex Banach space X with Fréchet differentiable norm, G a right reversible semitopological semigroup, and $\mathcal{S} = \{ S(t) : t \in G \}$ a nonexpansive semigroup on C. If

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$\mathcal{F}(\mathcal{S}) \neq \emptyset$ and $\mathcal{W}(x) \subseteq \mathcal{F}(\mathcal{S})$ for $x \in C$, then the net $\{S(t)x\}$ converges weakly to some $p \in \mathcal{F}(\mathcal{S})$ (see Theorem 2 and 3 in [14]).

Bruck ([4]) introduced the concept of an almost-orbit of a nonexpansive mapping. Miyadera-Kobayashi ([18]) extended the notion to the case of a nonexpansive semigroup. We can find the case for the general commutative semigroup ([20], [28]). Takahashi-Zhang ([29], [30]) established the weak convergence of an almost-orbit of a noncommutative semigroup.

Recently, Lin-Tan-Xu ([15]) proved the convergence of iterates $\{T^n x\}$ of an asymptotically nonexpansive mapping T in Banach spaces without the uniform convexity. And also, Kim ([9]) proved the corresponding theorems for the net $\{S(t)x\}$ of an asymptotically nonexpansive semigroup for a right reversible semitopological semigroup G.

In this paper, we prove the result of Takahashi-Zhang ([30]) in Banach spaces without the uniform convexity. The results of this paper are also extensions of Lin-Tan-Xu ([15]) and Kim ([9]).

2. Preliminaries and notations

Let C be a nonempty closed convex subset of a real Banach space X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* ([6]) if there exists a sequence $\{\alpha_n\}$ of nonnegative real numbers with $\lim_{n\to\infty} \alpha_n = 0$ such that

$$\| T^n x - T^n y \| \leq (1 + \alpha_n) \| x - y \|$$

for all $x, y \in C$. In particular if $\alpha_n = 0$ for all $n \geq 1$, then T is said to be *nonexpansive.* Let $\mathcal{S} = \{ S(t) : t \geq 0 \}$ be a family of mappings from C into itself. $\mathcal{S}$ is called an *asymptotically nonexpansive semigroup* on C if $S(t+s) = S(t)S(s)$ for every $t, s \geq 0$, and there exists a function $\alpha(\cdot) : R^+ \to R^+$ with $\lim_{t\to\infty} \alpha(t) = 0$ such that

$$\| S(t)x - S(t)y \| \leq (1 + \alpha(t)) \| x - y \|$$

for all $x, y \in C$ and $t \geq 0$, where R^+ is the set of all nonnegative real numbers. In particular, if $\alpha(t) = 0$ for all $t \geq 0$, then $\mathcal{S}$ is called a *nonexpansive semigroup* on C.

Let G be a semitopological semigroup, i.e., G is a semigroup with a Hausdorff topology such that for each $s \in G$ the mappings $s \to as$ and $s \to sa$ from G to G are continuous. G is called *right reversible*, if any two closed left ideals of G have nonvoid intersection. In this case, $(G, \succcurlyeq)$ is a directed system when the binary relation " $\succcurlyeq$ " on G is defined by $t \succcurlyeq s$ if and only if

$$\{t\} \cup \overline{Gt} \subseteq \{s\} \cup \overline{Gs}, \quad t, s \in G.$$

Right reversible semitopological semigroups include all commutative semigroups which are right amenable as discrete semigroups ([8]). Left reversibility of G is defined similarly. G is called *reversible* if it is both left and right reversible.

Throughout the rest of this paper, G is a right reversible semitopological semigroup.

A family $\mathcal{S} = \{ S(t) : t \in G \}$ of mappings from C into itself is said to be a *continuous representation of* G on C if it satisfies the followings:

(1) $S(ts) = S(t)S(s)$ for all $t, s \in G$,
(2) For every $x \in C$, the mapping $(s, x) \to S(s)x$ from $G \times C$ into C is continuous when $G \times C$ has the product topology.

A continuous representation $\mathcal{S}$ of G on C is said to be an *asymptotically nonexpansive semigroup* on C if each $t \in G$, there exists $k_t > 0$ such that

$$||S(t)x - S(t)y|| \leq (1 + k_t) \parallel x - y \parallel$$

for all $x, y \in C$, where $\lim_{t \in G} k_t = 0$.

Let $\mathcal{F}(S)$ denote the set of all common fixed points of mappings $S(t)$ for $t \in G$ in C, that is,

$$\mathcal{F}(\mathcal{S}) = \bigcap_{t \in G} \mathcal{F}(S(t)).$$

A continuous function $u : G \to C$ is said to be an *almost-orbit* of $\mathcal{S} = \{S(t) : t \in G\}$ if

$$\lim_{t \in G} (\sup_{s \in G} ||u(st) - S(s)u(t)||) = 0.$$

Clearly, for each $x \in C$, the orbit $\{S(t)x : t \in G\}$ of $\mathcal{S}$ at x is an almost-orbit of $\mathcal{S}$.

Some rudiments in the geometry of Banach spaces are necessary for the proofs of the main theorems in this paper. In the sequel, we give the notations: $\overline{\lim} = \limsup, \underline{\lim} = \liminf$, " $\rightharpoonup$ " for weak convergence, and " $\to$ " for strong convergence. Also, a space X is always understood to be an infinite dimensional Banach space without Schur's property, i.e., the weak and strong convergence doesn't coincide for nets.

A Banach space X is said to satisfy *Opial's condition* with respect to G if for each net $\{x_\alpha\}_{\alpha \in G}$ in X, the condition $x_\alpha \rightharpoonup x$ implies that

$$\overline{\lim_{\alpha \in G}} \parallel x_\alpha - x \parallel < \overline{\lim_{\alpha \in G}} \parallel x_\alpha - y \parallel$$

for all $y \neq x$ (see [21] for the same notion for a sequence $\{x_n\}$). Spaces possessing that property include the Hilbert spaces and the l^p spaces for $1 \leq p < \infty$. However, $L^p (p \neq 2)$ do not satisfy that property ([16]).

Recently, Prus ([23]) introduced the notion of the uniform Opial condition for any sequence $\{x_n\}$ in X. A Banach space X is said to satisfy *the uniform Opial condition* if for each G and for each $c > 0$, there exists an $r_G > 0$ such that

$$1 + r_G \leq \underline{\lim_{\alpha \in G}} ||x_\alpha + x||$$

for each $x \in X$ with $||x|| \geq c$ and each net $\{x_\alpha\}_{\alpha \in G}$ in X such that $x_\alpha \rightharpoonup 0$ and $\underline{\lim}_{\alpha \in G} ||x_\alpha|| \geq 1$. We now define *Opial's modulus of* X, denoted by $r_X(\cdot)$ as follows

$$r_X(c) = \inf\{\underline{\lim_{\alpha \in G}} \parallel x + x_\alpha \parallel -1\},$$

where $c \geq 0$ and the infimum is taken over all $x \in X$ with $\parallel x \parallel \geq c$ and nets $\{x_\alpha\}_{\alpha \in G}$ in X such that $x_\alpha \rightharpoonup 0$ and $\underline{\lim}_{\alpha \in G} \parallel x_\alpha \parallel \geq 1$. It is easy to see that $r_X(0) = 0$, and that $r_X(\cdot)$ is nondecreasing and continuous. And also, we know that X satisfies the uniform Opial condition if and only if $r_X(c) > 0$ for all $c > 0$.

We now introduce the notion of the locally uniform Opial condition. A Banach space X is said to satisfy the *locally uniform Opial condition* if for any weak null net $\{x_\alpha\}_{\alpha\in G}$ in X with $\underline{\lim}_{\alpha\in G} \| x_\alpha \| \geq 1$ and any $c > 0$, there is an $r_G > 0$ such that

$$1 + r_G \leq \underline{\lim}_{\alpha\in G} \| x_\alpha + x \|$$

for all $x \in X$ with $\| x \| \geq c$ (see [15] for the same notion for a sequence $\{x_n\}$). We can easily see that each "$\underline{\lim}$" can be replaced by "$\overline{\lim}$" in the definition of the (locally) uniform Opial condition. Clearly, uniform Opial condition implies locally uniform Opial condition, which in turn implies Opial's condition ([15]).

Next recall a generalization of uniform convex Banach spaces which is due to Sullivan ([27]). Let $k \geq 1$ be an integer. Then a Banach space X is said to be *k-uniformly rotund* (briefly $k - UR$) if for given any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that if $\{x_1, x_2, \cdots, x_{k+1}\} \subset B_X(1)$, the closed unit ball of X, satisfies $V(x_1, x_2, \cdots, x_{k+1}) \geq \varepsilon$, then

$$\frac{1}{k+1}\left\|\sum_{i=1}^{k+1} x_i\right\| \leq 1 - \delta(\varepsilon).$$

Here, $V(x_1, x_2, \cdots, x_{k+1})$ is the volume enclosed by the set $\{x_1, x_2, \cdots, x_{k+1}\}$, i.e.,

$$V(x_1, \cdots, x_{k+1}) = \sup\left\{\left|\begin{matrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+1}) \\ \vdots & & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+1}) \end{matrix}\right|\right\},$$

where the supremum is taken over all $f_1, f_2, \cdots, f_k \in B_{X^*}(1)$. The *modulus of k - uniform rotundity* of X is the function $\delta_X^{(k)}(\cdot)$ defined by

$$\delta_X^{(k)}(\varepsilon) = \inf\left\{1 - \frac{1}{k+1} \| \sum_{i=1}^{k+1} x_i \| : x_i \in B_X(1),\ V(x_1, x_2, \cdots, x_{k+1}) \geq \varepsilon\right\}.$$

Then it is seen that X is $k - UR$ if and only if $\delta_X^{(k)}(\varepsilon) > 0$ for $\varepsilon > 0$. It is obvious that if the modulus of $k-$ uniform rotundity $\delta_X^{(k)}(\varepsilon)$ is a nondecreasing function of ε (In fact, it is almost surely the case that $\delta_X^{(k)}$ is also continuous, but this require a detailed argument). It is known that the following implications hold ([11], [12], [27]).

(1) uniform convexity $\Longleftrightarrow 1-UR$.
(2) p-uniformly rotund is q-uniformly rotund if $p \leq q$.

Let $\mathcal{W}(u)$ denotes the set of all weak limits of subnets $\{u(t_\alpha)\}$ of the net $\{u(t)\}$ for a right reversible semitopological semigroup G.

3. Weak convergence theorems

In this section, we study the asymptotic behavior for the almost-orbits $\{u(t)\}$ of an asymptotically nonexpansive semigroup $\mathcal{S} = \{S(t) : t \in G\}$ in a Banach space X which satisfies the locally uniform Opial condition, uniform Opial condition, or Opial's condition.

We have the following equivalent statement for locally uniform Opial condition.

PROPOSITION 3.1. ([10]) *Let X be a Banach space and let G be a right reversible semitopological semigroup. Then the following two statements are equivalent.*

(1) *X satisfies the locally uniform Opial condition.*
(2) *If for any net $\{x_\alpha\}_{\alpha \in G}$ in X which converges weakly to $x \in X$ and for any net $\{y_\beta\}_{\beta \in G}$ in X,*

$$\overline{\lim_{\beta \in G}}\big(\overline{\lim_{\alpha \in G}} \| x_\alpha - y_\beta \|\big) \leq \overline{\lim_{\alpha \in G}} \| x_\alpha - x \|,$$

then $\{y_\beta\}$ converges strongly to $x \in X$ as $\beta \in G$.

We begin with the following lemma which plays a crucial role in the proof of our main theorems in this section.

LEMMA 3.2. *Let C be a nonempty weakly compact convex subset of a Banach space X satisfying the Opial's condition, G a right reversible semitopological semigroup, and $\mathcal{S} = \{S(t) : t \in G\}$ an asymptotically nonexpansive semigroup on C. If $\{u(t)\}$ is an almost-orbit of $\mathcal{S}$, then we have the following results.*

(1) $\mathcal{F}(\mathcal{S}) \subseteq E(u)$, *where* $E(u) = \{y \in C : \lim_{t \in G} \|u(t) - y\|$ *exist*$\}$.

(2) *If $\mathcal{W}(u)(\neq \emptyset) \subseteq \mathcal{F}(\mathcal{S})$, then $\mathcal{W}(u)$ consists of one point and hence $\{u(t)\}$ converges weakly to a point of $\mathcal{F}(\mathcal{S})$.*

PROOF. (1). Let $\{v(t)\}$ be an another almost-orbit of $\mathcal{S}$. Let ϕ and ψ be as follows:

$$\phi(t) = \sup_{s\in G} ||u(st) - S(s)u(t)|| \quad \text{and} \quad \psi(t) = \sup_{s\in G} ||v(st) - S(s)v(t)||$$

for all $t \in G$. Then $\lim_{t\in G} \phi(t) = \lim_{t\in G} \psi(t) = 0$. Since, for all $s, t \in G$

$$||u(st) - v(st)|| \leq \phi(t) + \psi(t) + (1 + k_s)||u(t) - v(t)||,$$

$$\inf_{s\in G} \sup_{\tau \succcurlyeq s} ||u(\tau) - v(\tau)|| \leq \phi(t) + \psi(t) + ||u(t) - v(t)||$$

for all $t \in G$. It follows that

$$\inf_{s\in G} \sup_{\tau \succcurlyeq s} ||u(\tau) - v(\tau)|| \leq \sup_{s\in G} \inf_{t \succcurlyeq s} ||u(t) - v(t)||.$$

Therefore $\lim_{t\in G} ||u(t) - v(t)||$ exists. Let $y \in \mathcal{F}(\mathcal{S})$ and put $v(t) \equiv y$ for all $t \in G$. Then $v(t)$ is an almost-orbit of $\mathcal{S}$. Hence $\lim_{t\in G} ||u(t) - y||$ exists. This proves that $\mathcal{F}(\mathcal{S}) \subseteq E(u)$ as desired.

(2). Let y_1 and y_2 be two weak limits of subnets $\{u(t_\alpha)\}$ and $\{u(t_\beta)\}$ of the net $\{u(t)\}$, respectively. Since $\mathcal{W}(u) \subseteq \mathcal{F}(\mathcal{S})$, there are $d_1, d_2 \geq 0$ from (1) such that

$$d_1 = \lim_{t\in G} ||u(t) - y_1|| \quad \text{and} \quad d_2 = \lim_{t\in G} ||u(t) - y_2||.$$

If $y_1 \neq y_2$, then we have

$$\begin{aligned} d_1 &= \lim_{t\in G} ||u(t) - y_1|| = \overline{\lim_{\alpha\in G}} ||u(t_\alpha) - y_1|| \\ &< \overline{\lim_{\alpha\in G}} ||u(t_\alpha) - y_2|| = \overline{\lim_{\beta\in G}} ||u(t_\beta) - y_2|| \\ &< \overline{\lim_{\beta\in G}} ||u(t_\beta) - y_1|| = \lim_{t\in G} ||u(t) - y_1|| \\ &= d_1. \end{aligned}$$

This is a contradiction, which implies that $\mathcal{W}(u)$ is a singleton. This completes the proof. □

Now, we can prove the convergence theorem of the almost-orbits $\{u(t)\}$ of an asymptotically nonexpansive semigroup $\mathcal{S} = \{S(t) : t \in G\}$ for a right reversible semitopological semigroup G.

THEOREM 3.3. *Let C be a nonempty weakly compact convex subset of a Banach space X satisfying the locally uniform Opial condition, G a right reversible semitopological semigroup, $\mathcal{S} = \{S(t) : t \in G\}$ an asymptotically nonexpansive semigroup on C, and $\{u(t)\}$ an almost-orbit of $\mathcal{S}$. If $\mathcal{W}(u) \neq \emptyset$ and $u(t)$ is asymptotically regular, that is, $\lim_{t \in G}\{u(st) - u(t)\} = 0$ strongly for all $s \in G$, then $\{u(t)\}$ converges weakly to a point of $\mathcal{F}(\mathcal{S})$.*

PROOF. In view of Lemma 3.2, it suffices to show that $\mathcal{W}(u) \subseteq \mathcal{F}(\mathcal{S})$. Let $y \in \mathcal{W}(u)$. Then there exists a subnet $\{u(t_\alpha)\}$ of the net $\{u(t)\}$ such that $u(t_\alpha) \rightharpoonup y$. Since for all $s \in G$

$$||u(t_\alpha) - S(s)y|| \leq \phi(t_\alpha) + ||u(t_\alpha) - u(st_\alpha)|| + (1 + k_s)||u(t_\alpha) - y||,$$

this implies that

$$\overline{\lim_{s \in G}}\, \overline{\lim_{\alpha \in G}}\, ||u(t_\alpha) - S(s)y|| \leq \overline{\lim_{\alpha \in G}}\, ||u(t_\alpha) - y||$$

from the asymptotic regularity of $u(t)$. Hence, we have $\lim_{s \in G} S(s)y = y$ from Lemma 3.1. From the continuity of $S(t)$, we obtain

$$S(t)y = \lim_{s \in G} S(t)S(s)y = \lim_{s \in G} S(ts)y = \lim_{s \in G} S(s)y = y.$$

Therefore, $\mathcal{W}(u) \subseteq \mathcal{F}(\mathcal{S})$. This completes the proof. □

It is not clear whether the asymptotic regularity of $u(t)$ in Theorem 3.3 can be weakened to the weakly asymptotic regularity. We improve the Theorem 3.3 when the space X is assumed to be satisfying the uniform Opial condition.

THEOREM 3.4. *Let C be a nonempty weakly compact convex subset of a Banach space X satisfying the uniform Opial condition, G a right reversible semitopological semigroup, $\mathcal{S} = \{S(t) : t \in G\}$ an asymptotically nonexpansive semigroup on C, and $\{u(t)\}$ an almost-orbit of $\mathcal{S}$. If $\mathcal{W}(u) \neq \emptyset$ and $u(t)$ is weakly asymptotically regular, that is, $\lim_{t \in G}\{u(st) - u(t)\} = 0$ weakly for all $s \in G$, then $\{u(t)\}$ converges weakly to a point of $\mathcal{F}(\mathcal{S})$.*

PROOF. In order to prove Theorem 3.4, we have to show that $\mathcal{W}(u) \subseteq \mathcal{F}(\mathcal{S})$. Let y be a weak limit of subnet $\{u(t_\alpha)\}$ of $\{u(t)\}$. Since $u(t)$ is weakly asymptotically regular, $\{u(st_\alpha)\}$ weakly converges to y for all $s \in G$.
Letting

$$r(s) = \overline{\lim_{\alpha \in G}} \|u(st_\alpha) - y\|.$$

Then, by Opial's condition, we have

$$\begin{aligned} r(ts) &= \overline{\lim_{\alpha \in G}} \|u(tst_\alpha) - y\| \\ &\leq \overline{\lim_{\alpha \in G}} \|u(tst_\alpha) - S(t)y\| \\ &= \overline{\lim_{\alpha \in G}} \|u(tst_\alpha) - S(t)u(st_\alpha) + S(t)u(st_\alpha) - S(t)y\| \\ &\leq \overline{\lim_{\alpha \in G}} \Big\{ \phi(st_\alpha) + (1 + k_t)\|u(st_\alpha) - y\| \Big\}. \end{aligned}$$

Therefore,

$$\overline{\lim_{t \in G}} r(ts) \leq \overline{\lim_{\alpha \in G}} \|u(st_\alpha) - y\| = r(s)$$

for all $s \in G$. On the other hand, since G is right reversible, it is obvious that $a \succcurlyeq ts$ if and only if there is a b with $b \succcurlyeq t$ and $a = bs$. Hence, we obtain

$$\overline{\lim_{t \in G}} r(t) \leq \sup_{a \succcurlyeq ts} r(a) = \sup_{b \succcurlyeq t} r(bs)$$

for all $t \in G$, which gives $\overline{\lim}_{t \in G} r(t) \leq \overline{\lim}_{t \in G} r(ts)$. Hence, we have

$$\overline{\lim_{t \in G}} r(t) \leq \overline{\lim_{t \in G}} r(ts) \leq r(s)$$

for all $s \in G$. Therefore, we have

$$\lim_{s \in G} r(s) = r \quad (\equiv \inf_{s \in G} r(s))$$

and $r \leq r(s)$ for all $s \in G$. Further, we know that if $\lim_{s \in G} r(s)$ exists, then $\lim_{s \in G} r(s) = \lim_{s \in G} r(ts)$ for each $t \in G$.

First, if $r = 0$, then since

$$\begin{aligned}||S(t)y - y|| &\leq ||u(st_\alpha) - y|| + ||S(t)u(st_\alpha) - S(t)y|| \\ &\quad + ||u(tst_\alpha) - S(t)u(st_\alpha)|| + ||u(tst_\alpha) - y|| + ||u(st_\alpha) - y|| \\ &\leq r(s) + (1 + k_t)r(s) + \phi(st_\alpha) + r(ts) + r(s) \\ &= (3 + k_t)r(s) + \phi(st_\alpha) + r(ts)\end{aligned}$$

for each $s, t \in G$, we know that the net $\{S(t)y\}$ converges strongly to y by taking $\overline{\lim}_{t\in G}$ first and next $\overline{\lim}_{s\in G}$. Hence we have, from the continuity of $S(s)$,

$$S(s)y = \lim_{t\in G} S(s)S(t)y = \lim_{t\in G} S(st)y = \lim_{t\in G} S(t)y = y.$$

This implies that $\mathcal{W}(u) \subseteq \mathcal{F}(\mathcal{S})$.

Now suppose that $r > 0$. In order to get the desired result, it suffices to show that $\{S(t)y\}$ converges strongly to y, in view of above case $r = 0$. If not, there exist an $\varepsilon > 0$ and subnet $\{t_\beta\}$ in G such that $||S(t_\beta)y - y|| \geq \varepsilon$. Since $\lim_{s\in G} r(s) = r(\equiv \inf r(s))$, there exists a $s_0 \in G$ such that $r(s_0) < r(1 + r_X(c))$, where $r_X(c)$ is an Opial's modulus of X and $c = \frac{\varepsilon}{r}(> 0)$. And also, we know that, for each $\beta \in G$,

$$\frac{u(t_\beta s_0 t_\alpha) - y}{r} \rightharpoonup 0$$

as $\alpha \in G$ and

$$\overline{\lim_{\alpha\in G}} \left|\left| \frac{u(t_\beta s_0 t_\alpha) - y}{r} \right|\right| = \frac{r(t_\beta s_0)}{r} \geq 1,$$

with $\frac{||y - S(t_\beta)y||}{r} \geq c$. Since $r_X(c) > 0$, we have

$$1 + r_X(c) \leq \overline{\lim_{\alpha\in G}} \left|\left| \frac{u(t_\beta s_0 t_\alpha) - y}{r} + \frac{y - S(t_\beta)y}{r} \right|\right|.$$

On the other hand, from

$$\begin{aligned}\overline{\lim_{\alpha\in G}} ||u(t_\beta s_0 t_\alpha) - S(t_\beta)y|| &\leq \overline{\lim_{\alpha\in G}} \left(\sup_{\beta\in G} ||u(t_\beta s_0 t_\alpha) - S(t_\beta)u(s_0 t_\alpha) \right) \\ &\quad + \overline{\lim_{\alpha\in G}} (1 + k_{t_\beta}) ||u(s_0 t_\alpha) - y||,\end{aligned}$$

we have

$$1 + r_X(c) > \frac{r(s_0)}{r} \geq \overline{\lim_{\beta \in G}}\ \overline{\lim_{\alpha \in G}} \left\| \frac{u(t_\beta s_0 t_\alpha) - y}{r} + \frac{y - S(t_\beta)y}{r} \right\|.$$

This is a contradiction which implies that $\{S(t)y\}$ converges strongly to y. This completes the proof. □

An interesting problem, which is open so far, is whether the conclusion of Theorem 3.4 is still true if the uniform Opial condition is weakened to Opial's condition, but we have the following partial answer. We know the following beautiful proposition in [9]

PROPOSITION 3.5. ([9]) *If X is a k-uniformly rotund Banach space for some $k \geq 1$ and satisfies the Opial's condition, then it satisfies the uniform Opial condition.*

We are now in a position to prove Theorem 3.6.

THEOREM 3.6. *Let X be a k - uniformly rotund Banach space for some $k \geq 1$ with the Opial's condition and let $C, G, \mathcal{S}$, and $\{u(t)\}$ be as in* Theorem 3.4. *If $u(t)$ is weakly asymptotically regular, then $\{u(t)\}$ converges weakly to a point of $\mathcal{F}(\mathcal{S})$.*

PROOF. We know that X satisfies the uniform Opial condition from the Proposition 3.5. And hence, by Theorem 3.4, $\{u(t)\}$ converges weakly to a point of $\mathcal{F}(\mathcal{S})$. □

REMARK. Since the k - uniformly rotund Banach space is reflexive ([27]), we need not the assumption $\mathcal{W}(u) \neq \emptyset$ in Theorem 3.6.

Clearly, $V(x_1, x_2) = \| x_1 - x_2 \|$ and thus the 1 - UR space simply the uniformly convex Banach space. So, we have the next corollary.

COROLLARY 3.7. (cf. [18], [28] and [30]) *Let X be a uniformly convex Banach space with the Opial's condition, and let $C, G, \mathcal{S}$ and $\{u(t)\}$ be as in* Theorem 3.4. *Then the conclusion of* Theorem 3.6 *holds.*

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