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THE IMAGE OF DERIVATIONS ON CERTAIN BANACH ALGEBRAS

Byung-Do Kim

Abstract. Let A be the non-commutative Banach algebra with identity satisfying certain conditions. We show that if D is a derivation on A, then $D(A)$ is contained in the radical of A.

1. Introduction

Throughout, R will represent an associative ring. Let A be a complex Banach algebra. A linear map D from A to A is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in A$. An additive mapping D from R to R is called a Jordan derivation on R if $D(x^2) = (Dx)x + x(Dx)$ holds for all $x \in R$. A linear map D from A to A is called a linear Jordan derivation on A if $D(x^2) = (Dx)x + x(Dx)$ holds for all $x \in A$. We will denote by $Q(A)$ the set of all quasinilpotent elements in a Banach algebra A.

In 1955, Singer and Wermer [6] proved that a continuous derivation on a commutative Banach algebra maps into the (Jacobson) radical, and they conjectured that this result holds even if the derivation is discontinuous. In 1988, Thomas [7] solved the long standing problem by showing that the conjecture is true.

In 1991, Kim and Jun [3] proved that if $D : A \longrightarrow A$ is a derivation on a noncommutative Banach algebra A satisfying the condition $[[A, A], A] = \{0\}$ then $D(A) \subseteq Q(A)$.

In 1992, Vukman [8] proved that if $D : A \longrightarrow A$ is a linear Jordan derivation on a noncommutative Banach algebra such that the map $F(x) = [[Dx, x], x]$ is commuting on A, that is $[F(x), x] = 0$ for all

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$x \in A$, then $D = 0$.
In 1992, Mathieu and Runde [5] proved that if D is a centralizing derivation on a Banach algebra A, then $D(A) \subseteq \text{rad}(A)$.
In 1994, Brešar [2] showed that if D is a bounded derivation of a Banach algebra such that $[D(x), x] \in Q(A)$ for every $x \in A$, then $D(A) \subseteq \text{rad}(A)$ where $\text{rad}(A)$ denotes the Jacobson radical of A. In this paper, we are to prove that if $D : A \longrightarrow A$ is a linear Jordan derivation on A where A denotes the noncommutative Banach algebra satisfying the following: for some $u, v \in A$, and $\lambda \in \mathbb{C}$,
$u - v = \lambda 1$, $x(u+v)y - y(u+v)x = 0$, $[xu, zv]y - y[vx, uz] = 0$ for all x, y, and $z \in A$, then $D(A) \subseteq Q(A)$.
And also, we show that if $D : A \longrightarrow A$ is a derivation on A where A denotes the noncommutative Banach algebra satisfying the following: for some $u, v \in A$, and $\lambda \in \mathbb{C}$,
$u - v = \lambda 1$, $x(u+v)y - y(u+v)x = 0$, $[xu, zv]y - y[vx, uz] = 0$ for all x, y, and $z \in A$, then $D(A) \subseteq \text{rad}(A)$.

2. Preliminaries

We need the following theorem, which is due to Thomas [7].

THEOREM 2.1. *Let D be any derivation on a commutative Banach algebra A. Then $D(A)$ is in the radical of A.*

The following theorem is due to Mathieu and Runde [5].

THEOREM 2.2. *Let D be a centralizing derivation on a Banach algebra A. Then $D(A)$ is in the radical of A.*

Let r denote the spectral radius in a normed algebra $(A, ||\cdot||)$. An operator T on A is said to be spectrally bounded if there exists an $M \geq 0$ such that $r(T(x)) \leq M||x||$ for all $x \in A$. In particular, if $M = 0$, then we say that T is spectrally infinitesimal.
Mathieu [4] proved that the following two results hold.

THEOREM 2.3. *Each spectrally bounded derivation fixes the radical.*

THEOREM 2.4. *Let D be a derivation on a Banach algebra A. Then the following three conditions are equivalent:*

(i) $[x, d(x)] \in rad(A)$ *for all* $x \in A$.

(ii) *D is spectrally bounded;*
(iii) $D(A) \subseteq rad(A)$.

3. Main results

THEOREM 3.1. *Let A be a noncommutative Banach algebra with identity* 1 *and* $rad(A)$, *and let* $D : A \longrightarrow A$ *be a linear Jordan derivation. Suppose there exist a nonzero* $\lambda \in \mathbb{C}$ *and the elements* $u, v \in A$ *such that*

$$u - v = \lambda 1, \quad x(u+v)y - y(u+v)x = 0, \quad [xu, zv]y - y[vx, uz] = 0$$

for all x, y, and $z \in A$. *Then we have* $D(A) \subseteq Q(A)$.

PROOF. We define a new algebra multiplication by $x * y = xuy - yvx$ for all $x, y \in A$. Then since $[xu, zv]y - y[vx, uz] = 0$ for all x, y and $z \in A$, we have

$$\begin{aligned}(x * y) * z - x * (y * z) &= (xuy - yvx)uz - zv(xuy - yvx)\\ &- (xu(yuz - zvy) - (yuz - zvy)vx)\\ &= xuyuz - yvxuz - zvxuy + zvyvx\\ &- xuyuz + xuzvy + yuzvx - zvyvx\\ &= -yvxuz - zvxuy + xuzvy + yuzvx\\ &= y[uz, vx] - [zv, xu]y\\ &= 0\end{aligned}$$

for every x, y, and $z \in A$. Thus $(A, *)$ is associative. And also, we get $x * y - y * x = xuy - yvx - (yux - xvy) = 2(x(u+v)y - y(u+v)x) = 0$ for every $x, y \in A$. Consequently, $(A, *)$ is an associative commutative Banach algebra under the new algebra multiplication $*$ and a new norm on $(A, *)$ is $||| \cdot |||| = (||u|| + ||v||)|| \cdot ||$. On the other hand, since D is a linear Jordan derivation and $u - v = \lambda 1$,

$$\begin{aligned} D(x * x) &= D(xux - xvx) \\ &= D(x(u-v)x) \\ &= \lambda(D(x^2)) \\ &= \lambda(D(x)x + xD(x)) \\ &= D(x)(\lambda 1)x + x(\lambda 1)D(x) \\ &= D(x)(u-v)x + x(u-v)D(x) \\ &= D(x)ux - D(x)vx + xuD(x) - xvD(x) \\ &= (D(x)ux - xvD(x)) + (xuD(x) - D(x)vx) \\ &= D(x) * x + x * D(x). \end{aligned}$$

Hence $D : (A, *) \longrightarrow (A, *)$ becomes a linear Jordan derivation on a commutative Banach algebra. And also, a simple calculation shows that $D(x * y) = D(x) * y + x * D(y)$ for all $x, y \in A$. It is obvious that $D : (A, *) \longrightarrow (A, *)$ is a derivation on a commutative Banach algebra. Hence by Theorem 2.1 we have $D(A, *) \subseteq \text{rad}(A, *)$. But since $r_*(x) = \lim_{n\to\infty} |||\overbrace{x * \cdots * x}^{n}|||^{\frac{1}{n}} = \lim_{n\to\infty} (||u|| + ||v||)^{\frac{1}{n}} |\lambda|^{\frac{n-1}{n}} ||x^n||^{\frac{1}{n}}$ $= |\lambda| r(x)$, we have

$$\begin{aligned} \text{rad}(A, *, |||\cdot|||) &= Q(A, *, |||\cdot|||) \\ &= \{x \in A \,|r_*(x) = 0\} \\ &= \{x \in A \,|r(x) = 0\} = Q(A) \end{aligned}$$

where r_* denotes the spectral radius in $(A, *, |||\cdot|||)$. And clearly, $D(A, *)$ $= D(A)$. Therefore $D(A) \subseteq Q(A)$. □

THEOREM 3.2. *Let A be a noncommutative Banach algebra with the identity 1 and the radical rad(A), and let $D : A \longrightarrow A$ be a derivation. Suppose there exist a nonzero $\lambda \in \mathbb{C}$ and the elements $u, v \in A$ such that*

$$u - v = \lambda 1, \quad x(u+v)y - y(u+v)x = 0, \quad [xu, zv]y - y[vx, uz] = 0$$

for all x, y, and $z \in A$. Then we have $D(A) \subseteq$ rad(A).

PROOF. As in Theorem 3.1, $(A,*)$ is an associative commutative Banach algebra under the algebra multiplication $*$ and the norm on $(A,*)$ is $|||\cdot|||| = (||u||+||v||)||\cdot||$. On the other hand, since D is a derivation, and by the assumption $u-v=\lambda 1$, and $x(u+v)y-y(u+v)x=0$ for all $x,y\in A$, we have the following: $D(u)-D(v)=0$, and

$$\begin{aligned}0 &= D(x(u+v)y-y(u+v)x)\\ &= D(x)(u+v)y+xD(u+v)y+x(u+v)D(y)-D(y)(u+v)x\\ &\quad -yD(u+v)x-y(u+v)D(x)\\ &= D(x)(u+v)y-y(u+v)D(x)+x(u+v)D(y)-D(y)(u+v)x\\ &\quad +xD(u+v)y-yD(u+v)x\\ &= xD(u+v)y-yD(u+v)x\\ &= x(D(u)+D(v))y-y(D(u)+D(v))x\\ &= 2(xD(u)y-yD(v)x).\end{aligned}$$

Hence we get $xD(u)y-yD(v)x=0$ for all $x,y\in A$. And so, using the relations, we have

$$\begin{aligned}D(x*y) &= D(xuy-yvx)\\ &= D(xu)y+xuD(y)-D(yv)x-yvD(x)\\ &= D(x)uy+xD(u)y+xuD(y)-D(y)vx-yD(v)x-yvD(x)\\ &= D(x)uy-yvD(x)+xD(u)y-yD(v)x+xuD(y)-D(y)vx\\ &= D(x)*y+x*D(y)+xD(u)y-yD(v)x\\ &= D(x)*y+x*D(y).\end{aligned}$$

Hence $D:(A,*)\longrightarrow(A,*)$ becomes a derivation on a commutative Banach algebra. Hence by Theorem 2.1 we have $D(A,*)\subseteq \mathrm{rad}(A,*)$. But since $r_*(x)=\lim_{n\to\infty}|||\overbrace{x*\cdots*x}^{n}|||^{\frac{1}{n}}=\lim_{n\to\infty}(||u||+||v||)^{\frac{1}{n}}|\lambda|^{\frac{n-1}{n}}||x^n||^{\frac{1}{n}}$ $=|\lambda|r(x)$, we have

$$\begin{aligned}\mathrm{rad}(A,*,|||\cdot|||) &= Q(A,*,|||\cdot|||)\\ &= \{x\in A\ |r_*(x)=0\}\\ &= \{x\in A\ |r(x)=0\}=Q(A)\end{aligned}$$

where r_* denotes the spectral radius in $(A, *, |||\cdot|||)$. On the other hand, $D(A,*) = DA$. And so, $D(A) \subseteq Q(A)$. Then by Theorem 2.3 a derivation D is spectrally infinitesimal, so it is spectrally bounded, hence by Theorem 2.4 we have $D(A) \subseteq \text{rad}(A)$. □

COROLLARY 3.3. *Let A be the noncommutative Banach algebra satisfying the condition $[[A, A], A] = \{0\}$. Suppose that there exists a derivation on A. Then we have $D(A) \subseteq rad(A)$.*

EXAMPLE 3.3. Let

$$\mathfrak{A} = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & d & a \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\},$$

and let

$$U = \frac{\lambda}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad V = -\frac{\lambda}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$X = \begin{pmatrix} x_1 & 0 & 0 \\ y_1 & x_1 & 0 \\ z_1 & w_1 & x_1 \end{pmatrix}, \quad Y = \begin{pmatrix} x_2 & 0 & 0 \\ y_2 & x_2 & 0 \\ z_2 & w_2 & x_2 \end{pmatrix},$$
$$Z = \begin{pmatrix} x_3 & 0 & 0 \\ y_3 & x_3 & 0 \\ z_3 & w_3 & x_3 \end{pmatrix}, \quad E = \lambda\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

for some nonzero $\lambda \in \mathbb{C}$. And we define a norm $||\cdot|| : \mathfrak{A} \longrightarrow [0, \infty)$ by $||X|| = 3\max\{|w_1|, |x_1|, |y_1|, |z_1|\}$. Some calculations show that $||\cdot||$ is an algebra norm on the algebra $(\mathfrak{A}, \text{product of matrices}, ||\cdot||)$ over $\mathbb{C}$. By some calculations we can conclude that $(\mathfrak{A}, \text{product of matrices}, ||\cdot||)$ is an associative noncommutative Banach algebra with identity I where $||I|| = 3$. Now we define the multiplication $*$ by $X * Y = XUY - YVX$ for all $X, Y \in \mathfrak{A}$. And we need the following relations:

for all X, Y, and $Z \in \mathfrak{A}$,

$$U - V = \lambda I, \;\; XE = EX = x_1 E, \;\; YE = EY = x_2 E, \tag{1}$$
$$ZE = EZ = x_3 E,$$
$$XU = UX = \frac{\lambda}{2}X + \frac{x_1}{2}E, \;\; XV = VX = -\frac{\lambda}{2}X + \frac{x_1}{2}E,$$
$$YU = UY = \frac{\lambda}{2}Y + \frac{x_2}{2}E, \;\; YV = VY = -\frac{\lambda}{2}Y + \frac{x_2}{2}E,$$
$$ZU = UZ = \frac{\lambda}{2}Z + \frac{x_3}{2}E, \;\; ZV = VZ = \frac{\lambda}{2}Z + \frac{x_3}{2}E,$$
$$[X,Y] = \frac{(y_2 w_1 - y_1 w_2)}{\lambda}E, \;\; [X,Z] = \frac{(y_3 w_1 - y_1 w_3)}{\lambda}E,$$
$$[XU, ZV] = [VX, UZ] = -\frac{\lambda(y_3 w_1 - y_1 w_3)}{4}E.$$

Using (1), we have

$$\begin{aligned} X * Y - Y * X &= X(U+V)Y - Y(U+V)X \\ &= X(2U - \lambda I)Y - Y(2U - \lambda I)X \\ &= 2(XUY - YUX) - \lambda(XY - YX) \\ &= 2((\frac{\lambda}{2}X + \frac{x_1}{2}E)Y - Y(\frac{\lambda}{2}X + \frac{x_1}{2}E)) - \lambda(XY - YX) \\ &= \lambda(XY - YX) + x_1 EY - x_1 YE - \lambda(XY - YX) \\ &= 0 \end{aligned}$$

for all $X, Y \in \mathfrak{A}$. And also, according to (1), we get

$$\begin{aligned} &(X * Y) * Z - X * (Y * Z) \\ &= (XUY - YVX) * Z - X * (YUZ - ZVY) \\ &= XUYUZ - YVXUZ - ZVXUY + ZVYVX \\ &- XUYUZ + XUZVY + YUZVX - ZVYVX \\ &= [XU, ZV]Y - Y[VX, UZ] \\ &= -\frac{\lambda(y_3 w_1 - y_1 w_3)}{4}EY - Y(-\frac{\lambda(y_3 w_1 - y_1 w_3)}{4})E \\ &= \frac{\lambda(y_3 w_1 - y_1 w_3)}{4}[E, Y] = 0. \end{aligned}$$

Hence we see that $(\mathfrak{A}, *)$ is commutative and associative. Moreover, let $|||X||| := (||U|| + ||V||)||X|| = 3|\lambda|||X||$ for all $X \in \mathfrak{A}$ where $||U|| = ||V|| = \frac{3}{2}|\lambda|$. Then it is obvious that

$$\begin{aligned} |||X * Y||| &= (||U|| + ||V||)||X * Y|| \\ &= (||U|| + ||V||)||XUY - YVX|| \\ &\leq (||U|| + ||V||)(||X||||U||||Y|| + ||Y||||V||||X||) \\ &= (||U|| + ||V||)||X||(||U|| + ||V||)||Y|| \\ &= |||X||||||Y|||, X, Y \in \mathfrak{A}. \end{aligned}$$

Thus $||| \cdot |||$ is also an algebra norm on the algebra $(\mathfrak{A}, *)$ over $\mathbb{C}$.
In consequence, we can conclude that $(\mathfrak{A}, *, ||| \cdot |||)$ is an associative commutative Banach algebra with identity $\frac{1}{\lambda}I$ over $\mathbb{C}$ where $|||\frac{1}{\lambda}I||| = 9$. On the other hand, we see that

$$X = \begin{pmatrix} 0 & 0 & 0 \\ y_1 & 0 & 0 \\ z_1 & w_1 & 0 \end{pmatrix} + x_1 I$$

and

$$I \begin{pmatrix} 0 & 0 & 0 \\ y_1 & 0 & 0 \\ z_1 & w_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ y_1 & 0 & 0 \\ z_1 & w_1 & 0 \end{pmatrix} I.$$

And so, by the basic result that $r(x+y) \leq r(x) + r(y)$ whenever $xy = yx$ in [1, Corollary 3, p.19], and from the fact that there exist polynomials $P_{n,1}(w_1, x_1, y_1, z_1), P_{n,2}(w_1, x_1, y_1, z_1), P_{n,3}(w_1, x_1, y_1, z_1)$ such that

$$X^n = \begin{pmatrix} x_1^n & 0 & 0 \\ P_{n,1} & x_1^n & 0 \\ P_{n,2} & P_{n,3} & x_1^n \end{pmatrix},$$

it follows that

$$\begin{aligned}
|x_1| &= \lim_{n\to\infty} (3|x_1^n|)^{\frac{1}{n}} \\
&\leq \lim_{n\to\infty} (3\max\{|x_1^n|, |P_{n,1}|, |P_{n,2}|, |P_{n,3}|\})^{\frac{1}{n}} \\
&= \lim_{n\to\infty} ||X^n||^{\frac{1}{n}} = r(X) \\
&\leq r(\begin{pmatrix} 0 & 0 & 0 \\ y_1 & 0 & 0 \\ z_1 & w_1 & 0 \end{pmatrix}) + r(x_1 I) \\
&= |x_1|.
\end{aligned}$$

Hence we get $r(X) = |x_1|$. But, for a product of matrices it follows that

$$\begin{aligned}
Q(\mathfrak{A}) &= \{X \in \mathfrak{A} | r(X) = 0\} \\
&= \{X \in \mathfrak{A} | |x_1| = 0\} \\
&= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ y_1 & 0 & 0 \\ z_1 & w_1 & 0 \end{pmatrix} | w_1, y_1, z_1 \in \mathbb{C} \right\}.
\end{aligned}$$

And also, a simple calculation shows that $\mathfrak{A}$ has a unique maximal modular left ideal of $\mathfrak{A}$. In fact, the ideal is the following:

$$\operatorname{rad}(\mathfrak{A}) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ y_1 & 0 & 0 \\ z_1 & w_1 & 0 \end{pmatrix} | w_1, y_2, z_1 \in \mathbb{C} \right\}.$$

Thus we can conclude that $\operatorname{rad}(\mathfrak{A}) = Q(\mathfrak{A})$. On the other hand, if r_* is the spectral radius in a Banach algebra $(\mathfrak{A}, *, |||\cdot|||)$, then it follows that

$$\begin{aligned} r_*(X) &= \lim_{n\to\infty} |||\overbrace{X*\cdots*X}^{2^n}|||^{\frac{1}{2^n}} \\ &= \lim_{n\to\infty} |||\overbrace{(X*X)*\cdots*(X*X)}^{2^{n-1}}|||^{\frac{1}{2^n}} \\ &= \lim_{n\to\infty} |||\lambda^{2^n-1}X^{2^n}|||^{\frac{1}{2^n}} \\ &= \lim_{n\to\infty} (3|\lambda|)^{\frac{1}{2^n}} |\lambda|^{\frac{2^n-1}{2^n}} ||X^{2^n}||^{\frac{1}{2^n}} \\ &= |\lambda| \lim_{n\to\infty} (3)^{\frac{1}{2^n}} \lim_{n\to\infty} ||X^{2^n}||^{\frac{1}{2^n}} \\ &= |\lambda| r(X). \end{aligned}$$

Hence, using the above relation and the fact that $\mathrm{rad}(A) = Q(A)$ in a commutative Banach algebra, we obtain

$$\begin{aligned} \mathrm{rad}(\mathfrak{A}, *, |||\cdot|||) &= Q(\mathfrak{A}, *, |||\cdot|||) \\ &= \{X \in \mathfrak{A} | r_*(X) = 0\} \\ &= \{X \in \mathfrak{A} | \lambda r(X) = 0\} \\ &= \{X \in \mathfrak{A} | r(X) = |x_1| = 0\} \\ &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ y_1 & 0 & 0 \\ z_1 & w_1 & 0 \end{pmatrix} | w_1, y_1, z_1 \in \mathbb{C} \right\} \\ &= Q(\mathfrak{A}) = \mathrm{rad}(\mathfrak{A}). \end{aligned}$$

In particular, we define a mapping D on a given noncommutative Banach algebra $\mathfrak{A} = (\mathfrak{A}, \text{product of matrices}, ||\cdot||)$ as follows:
Define $D : \mathfrak{A} \longrightarrow \mathfrak{A}$ by

$$D(\begin{pmatrix} x_1 & 0 & 0 \\ y_1 & x_1 & 0 \\ z_1 & w_1 & x_1 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{y_1}{2} & 0 & 0 \\ z_1 & \frac{w_1}{2} & 0 \end{pmatrix} (\in \mathrm{rad}(\mathfrak{A}))$$

for all $w_1, x_1, y_2, z_1 \in \mathbb{C}$. Then D is a derivation on $\mathfrak{A}$. Hence we see that all the conditions of Theorem 3.2 are fulfilled. Therefore we have $D(\mathfrak{A}) \subseteq \mathrm{rad}(\mathfrak{A})$.

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Department of Mathematics
Kangnung National University
Kangnung 210-702, Korea
E-mail: bdkim@knusun.kangnung.ac.kr