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ON THE CLOSURE OF DOMINANT OPERATORS

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ABSTRACT. Let $\mathfrak{D}^-$ denote the closure of the set $\mathfrak{D}$ of dominant operators in the norm topology. We show that the Weyl spectrum of an operator $T \in \mathfrak{D}^-$ satisfies the spectral mapping theorem for analytic functions, which is an extension of [5, Theorem 1]. Also we show that an operator approximately equivalent to an operator of class $\mathfrak{D}^-$ is of class $\mathfrak{D}^-$.

1. Introduction

Throughout this paper H will denote an infinite dimensional Hilbert space and $B(H)$ the space of all bounded linear operators on H. If $T \in B(H)$, we write $\sigma(T)$ for the spectrum of T. An operator $T \in B(H)$ is said to be *Fredholm* if its range ran T is closed and both the null spaces ker T and ker T^* are finite dimensional. $T \in B(H)$ is said to be *semi-Fredholm* if its range ran T is closed and either ker T or ker T^* is finite dimensional. The *index* of a semi-Fredholm operator T, denoted by $i(T)$, is defined by

$$i(T) = \dim\ker T - \dim\ker T^*.$$

The *essential spectrum* of T, denoted by $\sigma_e(T)$, is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Fredholm}\}.$$

A Fredholm operator of index zero is called *Weyl.* The *Weyl spectrum* of T, denoted by $\omega(T)$, is defined by

$$\omega(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Weyl}\}.$$

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It was shown ([1]) that $w(T)$ is a nonempty compact subset of $\sigma(T)$, and that $\omega(T)$ satisfies the one-way spectral mapping theorem for analytic functions: if f is analytic on a neighborhood of $\sigma(T)$ then

$$\omega(f(T)) \subseteq f(\omega(T)). \tag{1}$$

The inclusion (1) may be proper(see [1, Example 3.3]). If T is normal then $\sigma_e(T)$ and $\omega(T)$ coincide. Thus if T is normal since $f(T)$ is also normal, it follows that $\omega(T)$ satisfies the spectral mapping theorem for analytic functions. W. Y. Lee and S. H. Lee ([5]) showed that the Weyl spectrum of a hyponormal operator satisfies the spectral mapping theorem for analytic functions.

An operator $T \in B(H)$ is said to be *dominant* if for every $z \in \sigma(T)$ there exists $M_z > 0$ such that

$$(T-z)(T-z)^* \leq M_z(T-z)^*(T-z) \tag{2}$$

In this case, if $\sup_{z\in\sigma(T)} M_z = M < \infty$, T is said to be $M-$hyponormal. If $M = 1$, T is hyponormal. Evidently,

$$T \text{ is hyponormal} \Longrightarrow T \text{ is } M\text{-hyponormal} \Longrightarrow T \text{ is dominant.}$$

Let $\mathfrak{D}$ denote the class of dominant operators in $B(H)$ and $\mathfrak{D}^-$ denote the closure of $\mathfrak{D}$ in the norm topology. First note that $\mathfrak{D} \neq \mathfrak{D}^-$ (unlike the classes of normal or hyponormal operators). For example, let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis for H. Define $T_n \in B(H)$ as follows

$$T_n e_j = n^{(1-j)} e_{j+1} \quad \text{for} \quad j, n = 1, 2, \cdots.$$

It is easy to see that each $T_n \in \mathfrak{D}$, since each one is quasi-nilpotent. Note that $T_n \to T$ in norm where

$$Te_j = \begin{cases} e_{j+1} & \text{for } j = 1 \\ 0 & \text{for } j \neq 1 \end{cases} \tag{3}$$

and T is certainly not in $\mathfrak{D}$ [8]. We also note that T is not Fredholm since $\ker T$ and $\ker T^*$ are infinite dimensional.

In this paper we show that the Weyl spectrum of an operator $T \in \mathfrak{D}^-$ satisfies the spectral mapping theorem for analytic functions, which is an improvement of [5, Theorem 1]. Also we show that an operator approximately equivalent to an operator of class $\mathfrak{D}^-$ is of class $\mathfrak{D}^-$.

2. Spectral mapping theorem

The operator T in (3) is in $\mathfrak{D}^-$ and $\ker T \not\subseteq \ker T^*$. Also $\mathfrak{D}^- \neq B(H)$, as can be seen by the following result.

LEMMA 2.1. ([8]) *If T is Fredholm and in $\mathfrak{D}^-$, then $i(T) \leq 0$.*

PROOF. First assume that S is Fredholm and $S \in \mathfrak{D}$. Since ran $S \subseteq$ ran S^*, it follows that ker $S^* \supseteq$ ker S and thus $i(S) \leq 0$. Now let $T \in \mathfrak{D}^-$ be Fredholm. For $\|T - S\|$ sufficiently small, S is Fredholm and $i(S) = i(T)$, which completes the proof. □

REMARK. It is possible for the product of non-Weyl operators to be Weyl. For example, consider the unilateral shift U on l_2. Then U and U^* are Fredholm operators of index -1 and 1 respectively and so U and U^* are not Weyl operators. But UU^* is Fredholm and $i(UU^*) = i(U) + i(U^*) = -1 + 1 = 0$ by the index product theorem. Thus UU^* is Weyl.

THEOREM 2.2. *Let S and T be operators in $B(H)$. Suppose the indices of S and T are either both nonnegative or both nonpositive. Then*

$$S, T \text{ Weyl} \iff ST \text{ Weyl}. \tag{4}$$

PROOF. If S, T are Weyl, then S, T are Fredholm and $i(S) = i(T) = 0$. By [2], ST is Fredholm and by the index product theorem, $i(ST) = i(S) + i(T) = 0$. Hence ST is Weyl.

Conversely, suppose that ST is Weyl and each index is nonpositive. Then ST is Fredholm and $i(ST) = 0$. Since $\ker S^* \subseteq \ker(ST)^*$ and $i(S) \leq 0$, $\dim\ker S \leq \dim\ker S^* \leq \dim\ker(ST)^* < \infty$, and so $\ker S$ and $\ker S^*$ are finite dimensional. Also ran S is closed. Thus S is Fredholm. By [7, Theorem 5.3.5], S and T are Fredholm. Since each index is nonpositive and $0 = i(ST) = i(S) + i(T)$, $i(S) = i(T) = 0$. Hence S and T are Weyl.

Suppose that ST is Weyl and each index is nonnegative. By the similar argument, S and T are Weyl. □

From Lemma 2.1 and Theorem 2.2, we have the following result.

COROLLARY 2.3. *If S and T are operators in $\mathfrak{D}^-$, then*

$$S, T \text{ Weyl} \iff ST \quad \text{Weyl.}$$

The following result is an improvement of [5, Theorem 1], since Theorem 1 in [5] holds even though the "commuting " condition is dropped.

COROLLARY 2.4. *If S and T are hyponormal operators, then*

$$S, T \text{ Weyl} \iff ST \quad \text{Weyl.}$$

The following theorem is an improvement of [5, Theorem 1].

THEOREM 2.5. *If T is in $\mathfrak{D}^-$ and f is analytic on a neighborhood of $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$.*

PROOF. Suppose that p is any polynomial. Let

$$p(T) - \lambda I = a_0(T - \mu_1 I) \cdots (T - \mu_n I).$$

Since T is an operator in $\mathfrak{D}^-$, $T - \mu_i I$ are commuting operators in $\mathfrak{D}^-$ for each $i = 1, 2, \cdots, n$. It thus follows from Corollary 2.3 that

$$\begin{aligned} \lambda \notin \omega(p(T)) &\iff p(T) - \lambda I \text{ is Weyl} \\ &\iff a_0(T - \mu_1 I) \cdots (T - \mu_n I) \text{ is Weyl} \\ &\iff T - \mu_i I \text{ is Weyl for each } i = 1, 2, \cdots, n \\ &\iff \mu_i \notin \omega(T) \text{ for each } i = 1, 2, \cdots, n \\ &\iff \lambda \notin p(\omega(T)) \end{aligned}$$

which says that $\omega(p(T)) = p(\omega(T))$.

Next suppose r is any rational function with no poles in $\sigma(T)$. Write $r = p/q$, where p and q are polynomials and q has no zeros in $\sigma(T)$. Then

$$r(T) - \lambda I = (p - \lambda q)(T)(q(T))^{-1}.$$

By the first argument,

$$(p - \lambda q)(T) \text{ Weyl} \iff p - \lambda q \text{ has no zeros in } \omega(T).$$

Thus we have

$$\begin{aligned}\lambda \notin \omega(r(T)) &\Longleftrightarrow (p-\lambda q)(T) \text{ is Weyl}\\ &\Longleftrightarrow p-\lambda q \ \text{ has no zeros in } \omega(T)\\ &\Longleftrightarrow ((p-\lambda q)(x))q(x)^{-1} \neq 0 \ \text{ for any } \ x \in \omega(T)\\ &\Longleftrightarrow \lambda \notin r(\omega(T))\end{aligned}$$

which says that $\omega(r(T)) = r(\omega(T))$. If f is analytic on a neighborhood of $\sigma(T)$, then by Runge's theorem([2]), there is a sequence $\{r_n(t)\}$ of rational functions with no poles in $\sigma(T)$ such that $\{r_n\}$ converges to f uniformly on a neighborhood of $\sigma(T)$. Since each $r_n(T)$ commutes with $f(T)$, by [8]

$$f(\omega(T)) = \lim r_n(\omega(T)) = \lim \omega(r_n(T)) = \omega(f(T)). \qquad \square$$

COROLLARY 2.6. *If T is dominant and f is analytic on a neighborhood of $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$.*

COROLLARY 2.7. ([5]) *If T is hyponormal and f is analytic on a neighborhood of $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$.*

DEFINITION 2.8. An operator $T \in B(H)$ is called *analytic* if there exists a nonzero analytic function f on an open neighborhood Ω of $\sigma(T)$ such that $f(T) = 0$.

As a natural extension of algebraicity, Halmos([3], Problem 97) introduced the concept of analyticity (only for a quasinilpotent operator). Evidently, we have

$$(5) \qquad T \text{ is algebraic} \Longrightarrow T \text{ is analytic.}$$

However the converse of (5) is not true in general: for example, consider a Riesz operator whose spectrum is infinite.

Analyticity guarantees the existence of an isolated point of Weyl spectrum of $T \in \mathfrak{D}^-$.

THEOREM 2.9. *If $T \in \mathfrak{D}^-$ is analytic, then $\omega(T)$ has an isolated point.*

PROOF. Suppose that $T \in \mathfrak{D}^-$ is analytic. Then there exists a nonzero analytic function f on an open neighborhood Ω of $\sigma(T)$ such that $f(T) = 0$. By Theorem 2.5, $f(\omega(T)) = \omega(f(T)) = \omega(0) = \{0\}$, and so all values of $\omega(T)$ are zeros of f. Thus if all values of $\omega(T)$ are accumulation points of $\omega(T)$, then it follows from the Identity Theorem in the elementary complex analysis that $f \equiv 0$ on $\omega(T)$, which leads a contradiction. □

LEMMA 2.10. *An operator unitarily equivalent to a dominant operator is dominant.*

PROOF. Suppose that $S = U^*TU$, T dominant and U unitary. Since T is dominant, for any $z \in \sigma(T)$ there exists a constant $M_z > 0$ such that $\|(T-z)^*x\| \leq M_z\|(T-z)x\|$ for any $x \in H$. Since $\sigma(T) = \sigma(S)$ by [3, Problem 75], for any $x \in H$ and any $z \in \sigma(S)$,

$$\begin{aligned}\|(S-z)^*x\| &= \|U^*(T^*-\bar{z})Ux\| = \|(T^*-\bar{z})Ux\| \\ &\leq M_z\|(T-z)Ux\| = M_z\|U^*(T-z)Ux\| \\ &= M_z\|(S-z)x\|.\end{aligned}$$

Thus $S = U^*TU$ is dominant. □

Two operators S, T are said to be approximately equivalent if there exists a sequence $\{U_n\}$ of unitary operators such that $\|U_n^*TU_n - S\| \to 0$.

THEOREM 2.11. *An operator approximately equivalent to an operator of class $\mathfrak{D}^-$ is of class $\mathfrak{D}^-$.*

PROOF. Suppose that $P = U^*QU$, where Q is of class $\mathfrak{D}^-$ and U is unitary. Then there exists a sequence $\{Q_n\}$ of dominant operators such that $Q_n \to Q$. Let $P_n = U^*Q_nU$. Then by Lemma 2.10, P_n is dominant. Since $P = U^*QU = \lim U^*Q_nU = \lim P_n$, P is of class $\mathfrak{D}^-$.

Finally suppose that T is of class $\mathfrak{D}^-$ and T is approximately equivalent to S. Then there exists a sequence $\{U_n\}$ of unitary operators such that $\|U_n^*TU_n - S\| \to 0$. By the first argument, $U_n^*TU_n \in \mathfrak{D}^-$ for each n. Since $\mathfrak{D}^-$ is closed, $S \in \mathfrak{D}^-$. □

COROLLARY 2.12. *An operator unitarily equivalent to an operator of class $\mathfrak{D}^-$ is of class $\mathfrak{D}^-$.*

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