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AN ISOMORPHISM FOR INFINITE DIMENSIONAL CALCULUS

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Abstract. We show that the foundational isomorphism exists in the category of filter convergence spaces which contains the category of Banach spaces as a replete subcategory.

1. Introduction

It was shown recently (see [2], [5], [6]) that the basic properties of real differential calculus arise as purely categorical consequences of a certain natural isomorphism ed in the category of Banach spaces. The typical component at the Banach space E,

$$\mathsf{ed}_E : \mathsf{ad}\mathcal{C}(I \times I, E) \to \mathcal{C}(I, E), \quad \mathsf{ed}_E(\mathfrak{A})(\lambda) = \mathfrak{A}(\lambda, \lambda)$$

provides an isometric representation of the familiar space $\mathcal{C}(I, E)$ of continuous curves $I \to E$ on the nondegenerate compact interval I. The maps ed_E carry the 'germ' of differentiation, their inverses the 'germ' of integration. In [2, 6] this isomorphism evolved to form a more general setting of infinite dimensional differential calculus as a Foundational Isomorphism, which generates the categorical differentiation theory.

In this paper we show that the foundational isomorphism exists in the category of filter convergence spaces which contains the category of Banach spaces as a replete subcategory.

For general categorical differential calculus we refer to L. D. Nel [3, 4] and for the convergence space to E. Binz [1].

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2. Preliminaries

From now on I denotes the nondegenerate compact interval in $\mathbb{R}$. Top denotes the category of topological spaces and continuous maps. Ban denotes the category of Banach spaces and linear continuous maps.

Recall that (X, ξ) is a *filter convergence space* or a *$\mathcal{C}_c$-space* if X is a set and ξ is a function which assigns to every $x \in X$ a set $\xi(x)$ of filters on X subject to the following single axiom:

For every $x \in X$, the filter generated by $\{x\}$ belongs to $\xi(x)$.

A map $f : (X, \xi) \to (Y, \eta)$ is called a *continuous map* or *$\mathcal{C}_c$-map* if $f(\mathcal{G}) \in \eta(f(x))$ whenever $\mathcal{G} \in \xi(x)$. The category of $\mathcal{C}_c$-spaces and $\mathcal{C}_c$-maps is denoted by $\mathcal{C}_c$.

We already know that $\mathcal{C}_c$ is a *toponome* (see [3]) in which the real field $\mathbb{R}$ is structured as $\mathcal{C}_c$-space so that the arithmetical operations become $\mathcal{C}_c$-maps. A *linear $\mathcal{C}_c$-space* is a $\mathcal{C}_c$-space E on which addition $E \times E \to E$ and scalar multiplication $\mathbb{R} \times E \to E$ have been defined so as to be $\mathcal{C}_c$-maps, subject to the usual linear space axioms. Such spaces, together with linear continuous maps between them, build the category $\mathcal{LC}_c$. It contains Ban as a replete subcategory.

Let X be a $\mathcal{C}_c$-space, let E and F be $\mathcal{LC}_c$-spaces. Then $\mathcal{C}_c(X, F)$ denotes the $\mathcal{LC}_c$-space of all continuous maps $X \to F$, equipped with the canonical mapping space structure and with pointwise linear operations. The $\mathcal{LC}_c$-subspace $[E, F]$ of $\mathcal{C}_c(E, F)$ consists of all $\mathcal{LC}_c$-maps $E \to F$. The functors $\mathcal{C}_c(X, -)$ and $[E, -]$ are right adjoints $\mathcal{LC}_c \to \mathcal{LC}_c$, even $\mathcal{C}_c$-enriched and $\mathcal{LC}_c$-enriched, respectively.

DEFINITION. [4] An *analyte* in $\mathcal{LC}_c$ is a subcategory $\mathsf{a}\mathcal{LC}_c$ such that:

(1) $\mathsf{a}\mathcal{LC}_c$ is replete and reflective,
(2) $\mathsf{a}\mathcal{LC}_c$ is preserved by all toponomial functors $\mathcal{C}_c(X, -)$,
(3) $\mathsf{a}\mathcal{LC}_c$ has $\mathbb{R}$ among its spaces.

Let $\mathsf{m}\mathcal{LC}_c$ be the replete reflective subcategory of $\mathcal{LC}_c$ induced by the outer class of all monomorphisms and $\mathsf{c}\mathcal{LC}_c$ the subcategory of closed embeddable $\mathcal{LC}_c$-subspaces. Then $\mathsf{c}\mathcal{LC}_c \subseteq \mathsf{m}\mathcal{LC}_c \subseteq \mathsf{a}\mathcal{LC}_c$ (see [4]).

Take an $\mathsf{a}\mathcal{LC}_c$-space E and let $\mathsf{ad}\mathcal{C}_c(I \times I, E)$ denote the $\mathcal{LC}_c$ subspace of $\mathcal{C}_c(I \times I, E)$ formed by all members $\mathfrak{A}$ satisfying the following additivity law:

$$(\beta-\alpha)\mathfrak{A}(\alpha,\beta)+(\gamma-\beta)\mathfrak{A}(\beta,\gamma)+(\alpha-\gamma)\mathfrak{A}(\gamma,\alpha)=0,(\gamma,\beta,\gamma\in I).$$

3. Construction of an isomorphism

Let E and F be $\mathsf{c}\mathcal{LC}_c$-spaces. Then any $\mathcal{LC}_c$-map $u : E \to F$ induces another $\mathcal{LC}_c$-map

$$\mathcal{C}_c(I,u) : \mathcal{C}_c(I,E) \to \mathcal{C}_c(I,F), \quad f \mapsto u \circ f.$$

Thus we have a functor $\mathcal{C}_c(I,-) : \mathsf{c}\mathcal{LC}_c \to \mathsf{c}\mathcal{LC}_c$. Similarly we have a functor $\mathsf{ad}\mathcal{C}_c(I \times I,-) : \mathsf{c}\mathcal{LC}_c \to \mathsf{c}\mathcal{LC}_c$.

For $\mathsf{c}\mathcal{LC}_c$-space E, consider the map

$$\mathsf{ed}_E : \mathsf{ad}\mathcal{C}_c(I\times I,E) \to \mathcal{C}_c(I,E), \quad \mathsf{ed}_E(\mathfrak{A})(\lambda)=\mathfrak{A}(\lambda,\lambda).$$

Since the map $\mathsf{diag} : I \to I \times I$, $\mathsf{diag}(\lambda) = (\lambda,\lambda)$ is an embedding as a Top-map, it becomes a $\mathcal{C}_c$-map. Since $\mathsf{ed}_E(\mathfrak{A}) = \mathfrak{A} \circ \mathsf{diag}$, $\mathsf{ed}_E(\mathfrak{A})$ is a $\mathcal{C}_c$-map. Thus ed_E is well -defined. It is easy to show that ed_E is a linear $\mathcal{C}_c$-map for all $E \in \mathsf{c}\mathcal{LC}_c$. In fact, ed_E becomes a component of a natural transformation in $\mathsf{c}\mathcal{LC}_c$ as follows.

THEOREM 1. *For a $\mathsf{c}\mathcal{LC}_c$-space E and a compact interval I, the map*

$$\mathsf{ed}_E : \mathsf{ad}\mathcal{C}_c(I\times I,E) \to \mathcal{C}_c(I,E), \quad \mathsf{ed}_E(\mathfrak{A})(\lambda)=\mathfrak{A}(\lambda,\lambda)$$

is a component of a natural transformation in the category $\mathsf{c}\mathcal{LC}_c$.

PROOF. For any $\mathcal{LC}_c$-map $u : E \to F$, consider the following diagram.

$$\begin{array}{ccc}
\mathsf{ad}\mathcal{C}_c(I\times I,E) & \xrightarrow{\ \mathsf{ed}_E\ } & \mathcal{C}_c(I,E) \\
{\scriptstyle \mathsf{ad}\mathcal{C}_c(I\times I,u)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{C}_c(I,u)} \\
\mathsf{ad}\mathcal{C}_c(I\times I,F) & \xrightarrow{\ \mathsf{ed}_F\ } & \mathcal{C}_c(I,F)
\end{array}$$

Since $u \circ \mathsf{ed}_E(\mathfrak{A})(\lambda) = u(\mathfrak{A}(\lambda,\lambda)) = (u\circ\mathfrak{A})(\lambda,\lambda) = \mathsf{ed}_F(u\circ\mathfrak{A})(\lambda)$ for all $\lambda \in I$, we have $\mathcal{C}_c(I,u)\circ\mathsf{ed}_E = u\circ\mathsf{ed}_E(\mathfrak{A}) = \mathsf{ed}_F(u\circ\mathfrak{A}) = \mathsf{ed}_F\circ\mathsf{ad}\mathcal{C}_c(I\times I,u)$. Hence the diagram commutes. □

Moreover we have the following result.

THEOREM 2. *The map*

$$\mathsf{ed}_E : \mathsf{ad}\mathcal{C}_c(I \times I, E) \to \mathcal{C}_c(I, E), \quad \mathsf{ed}_E(\mathfrak{A})(\lambda) = \mathfrak{A}(\lambda, \lambda)$$

is injective for any c$\mathcal{LC}_c$*-space* E.

PROOF. Since E is a c$\mathcal{LC}_c$-space, it is an m$\mathcal{LC}_c$-space. Thus the family $\{u_\lambda |\, u_\lambda : E \to \mathbb{R} \text{ is a } \mathcal{LC}_c \text{ map}\}_{\lambda\in\Lambda}$ forms a monofamily, and hence an injective family. Thus the family

$$\{\mathsf{ad}\mathcal{C}_c(I \times I, u_\lambda) : \mathsf{ad}\mathcal{C}_c(I \times I, E) \to \mathsf{ad}\mathcal{C}_c(I \times I, \mathbb{R})\}_{\lambda\in\Lambda}$$

is also a monofamily. Since ed_E is a natural transformation for E, the diagram

$$\begin{array}{ccc} \mathsf{ad}\mathcal{C}_c(I \times I, E) & \xrightarrow{\mathsf{ad}\mathcal{C}_c(I\times I, u_\lambda)} & \mathsf{ad}\mathcal{C}_c(I \times I, \mathbb{R}) \\ \mathsf{ed}_E \big\downarrow & & \big\downarrow \mathsf{ed}_{\mathbb{R}} \\ \mathcal{C}_c(I, E) & \xrightarrow{\mathcal{C}_c(I, u_\lambda)} & \mathcal{C}_c(I, \mathbb{R}) \end{array}$$

commutes. That is $\mathsf{ed}_{\mathbb{R}} \circ \mathsf{ad}\mathcal{C}_c(I \times I, u_\lambda) = \mathcal{C}_c(I, u_\lambda) \circ \mathsf{ed}_E$. Note that $\mathsf{ed}_{\mathbb{R}} : \mathsf{ad}\mathcal{C}_c(I\times I, \mathbb{R}) \to \mathcal{C}_c(I, \mathbb{R})$ is injective. Since $\mathsf{ed}_{\mathbb{R}} \circ \mathsf{ad}\mathcal{C}_c(I\times I, u_\lambda)$ is a monofamily, ed_E is a monofamily and hence a monomorphism. Therefore ed_E is injective. □

If $E = \mathcal{C}_c(X, \mathbb{R})$, the map $\mathsf{ed}_{\mathcal{C}_c(X,\mathbb{R})}$ is an isomorphism as follows.

THEOREM 3. *The map*

$$\mathsf{ed}_{\mathcal{C}_c(X,\mathbb{R})} : \mathsf{ad}\mathcal{C}_c(I \times I, \mathcal{C}_c(X, \mathbb{R})) \to \mathcal{C}_c(I, \mathcal{C}_c(X, \mathbb{R})),\ \mathsf{ed}(\mathfrak{A})(\lambda) = \mathfrak{A}(\lambda, \lambda)$$

is an isomorphism.

PROOF. Note that the map $\S : \mathcal{C}_c(I, \mathcal{C}_c(X, \mathbb{R})) \to \mathcal{C}_c(X, \mathcal{C}_c(I, \mathbb{R}))$, $\S(f)(x)(w) = f(w)(x)$ is an isomorphism (see [3]). Now, consider the following diagram

$$\begin{array}{ccc} \mathsf{ad}\mathcal{C}_c(I \times I, \mathcal{C}_c(X, \mathbb{R})) & \xrightarrow{\S} & \mathcal{C}_c(X, \mathsf{ad}\mathcal{C}_c(I \times I, \mathbb{R})) \\ \mathsf{ed}_{\mathcal{C}_c(X,\mathbb{R})} \big\downarrow & & \big\downarrow \mathcal{C}_c(X, \mathsf{ed}_{\mathbb{R}}) \\ \mathcal{C}_c(I, \mathcal{C}_c(X, \mathbb{R})) & \xrightarrow{\S} & \mathcal{C}_c(X, \mathcal{C}_c(I, \mathbb{R})). \end{array}$$

For $\mathfrak{A} \in \mathsf{ad}\mathcal{C}_c(I \times I, \mathcal{C}_c(X, \mathbb{R}))$,

$$\begin{aligned}
&(\beta-\alpha)\S(\mathfrak{A})(x)(\alpha,\beta) + (\gamma-\beta)\S(\mathfrak{A})(x)(\beta,\gamma) + (\alpha-\gamma)\S(\mathfrak{A})(x)(\gamma,\ \alpha) \\
&= (\beta-\alpha)(\mathfrak{A})(\alpha,\beta)(x) + (\gamma-\beta)\mathfrak{A}(\beta,\gamma)(x) + (\alpha-\gamma)\mathfrak{A}(\gamma,\ \alpha)(x) \\
&= [(\beta-\alpha)(\mathfrak{A})(\alpha,\beta) + (\gamma-\beta)\mathfrak{A}(\beta,\gamma) + (\alpha-\gamma)\mathfrak{A}(\gamma,\ \alpha)](x) \\
&= 0(x).
\end{aligned}$$

Thus $\S(\mathfrak{A})(x) \in \mathsf{ad}\mathcal{C}_c(I \times I, \mathbb{R})$. Hence $\S(\mathsf{ad}\mathcal{C}_c(I \times I, \mathcal{C}_c(X, \mathbb{R})) \subseteq \mathcal{C}_c(X, \mathsf{ad}\mathcal{C}_c(I \times I, \mathbb{R}))$. Conversely, for $\mathfrak{B} \in \mathcal{C}_c(X, \mathsf{ad}\mathcal{C}_c(I \times I, \mathbb{R}))$,

$$\begin{aligned}
&[(\beta-\alpha)\S^{-1}(\mathfrak{B})(\alpha,\beta) + (\gamma-\beta)\S^{-1}(\mathfrak{B})(\beta,\gamma) + (\alpha-\gamma)\S^{-1}(\mathfrak{B})(\gamma,\ \alpha)](x) \\
&= (\beta-\alpha)\S^{-1}(\mathfrak{B})(\alpha,\beta)(x) + (\gamma-\beta)\S^{-1}(\mathfrak{B})(\beta,\gamma)(x) \\
&\quad + (\alpha-\gamma)\S^{-1}(\mathfrak{B})(\gamma,\ \alpha)(x) \\
&= (\beta-\alpha)\mathfrak{B}(x)(\alpha,\beta) + (\gamma-\beta)\mathfrak{B}(x)(\beta,\gamma) + (\alpha-\gamma)\mathfrak{B}(x)(\gamma,\ \alpha) \\
&= 0,
\end{aligned}$$

because $\mathfrak{B}(x) \in \mathsf{ad}\mathcal{C}_c(I \times I, \mathbb{R})$. Thus $\S^{-1}(\mathfrak{B}) \in \mathsf{ad}\mathcal{C}_c(I \times I, \mathcal{C}_c(X, \mathbb{R}))$. Hence $\S^{-1}(\mathcal{C}_c(X, \mathsf{ad}\mathcal{C}_c(I \times I, \mathbb{R}))) \subseteq \mathsf{ad}\mathcal{C}_c(I \times I, \mathcal{C}_c(X, \mathbb{R}))$. Therefore $\S : \mathsf{ad}\mathcal{C}_c(I \times I, \mathcal{C}_c(X, \mathbb{R})) \to \mathcal{C}_c(X, \mathsf{ad}\mathcal{C}_c(I \times I, \mathbb{R}))$ is an isomorphism. Since $\mathsf{ed}_{\mathbb{R}}$ is an isomorphism, so is $\mathcal{C}_c(X, \mathsf{ed}_{\mathbb{R}})$. Moreover, for $\mathfrak{A} \in \mathsf{ad}\mathcal{C}_c(I \times I, \mathcal{C}_c(X, \mathbb{R}))$,

$$\begin{aligned}
(\mathcal{C}_c(X, \mathsf{ed}_{\mathbb{R}}) \circ \S)(\mathfrak{A})(x)(i) &= \mathcal{C}_c(X, \mathsf{ed}_{\mathbb{R}})\S(\mathfrak{A})(x)(i) \\
&= (\mathsf{ed}_{\mathbb{R}} \circ \S(\mathfrak{A}))(x)(i) = \mathsf{ed}_{\mathbb{R}}(\S(\mathfrak{A})(x))(i) \\
&= \S(\mathfrak{A})(x)(i,i) = \mathfrak{A}(i,i)(x)
\end{aligned}$$

and

$$\begin{aligned}
(\S \circ \mathsf{ed}_{\mathcal{C}_c(X,\mathbb{R})})(\mathfrak{A})(x)(i) &= \S(\mathsf{ed}_{\mathcal{C}_c(X,\mathbb{R})}(\mathfrak{A}))(x)(i) \\
&= \mathsf{ed}_{\mathcal{C}_c(X,\mathbb{R})}(\mathfrak{A})(i)(x) \\
&= \mathfrak{A}(i,i)(x).
\end{aligned}$$

Thus the diagram commutes. Hence $\mathsf{ed}_{\mathcal{C}_c(X,\mathbb{R})} : \mathsf{ad}\mathcal{C}_c(I \times I, \mathcal{C}_c(X, \mathbb{R})) \to \mathcal{C}_c(I, \mathcal{C}_c(X, \mathbb{R}))$ is an isomorphism. □

REMARK. All the above results can be extended to a toponome[3] i.e. a cartesian closed topological construct in which all single point spaces are discrete. Examples of toponome include the category of filter convergence spaces, the category of sequential convergence spaces and the category of compactly generated topological spaces.

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