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# A CHARACTERIZATION OF MINIMAL SEMIPOSITIVITY OF SIGN PATTERN MATRICES

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Abstract. A real $m \times n$ matrix $A$ is *semipositive* (*SP*) if there is a vector $x \geq 0$ such that $Ax > 0$, inequalities being entrywise. $A$ is *minimally semipositive* (*MSP*) if $A$ is semipositive and no column deleted submatrix of $A$ is semipositive. We give a necessary and sufficient condition for the sign pattern matrix with $n$ positive entries to be minimally semipositive.

## 1. Introduction

The property of qualitative semipositivity has been examined in [3]. And we are going to give a necessary and sufficient condition for the sign pattern matrix with $n$ positive entries to be minimally semipositive. The concept of semipositivity has been found useful in a number of settings as followings:

(1) In the class of $Z$-matrices; the semipositivity of matrices characterizes the non-singular $M$-matrices.

(2) Given a finite set $\mathcal{D}$ of diagonal matrices, semipositivity of a matrix constructed from $\mathcal{D}$ determines convergence of $\mathcal{D}$.

(3) The $m \times n$ matrix $A$ is semipositive only if the interior of the cone generated by the columns of $A$ interests the positive orthant.

A real $m \times n$ matrix $A$ is called *semipositive* (*SP*) if there is a real vector $x \geq 0$ such that $Ax > 0$, inequalities being entrywise. $A$ is *minimally semipositive* (*MSP*) if $A$ is semipositive and no column deleted submatrix of $A$ (i.e. matrix obtained from $A$ by deleting a column) is

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semipositive. A characterization of the sign patterns that allow minimal semipositivity for full sign pattern matrices was given in [4]. Later a question was raised in [4] for $\{+,-,0\}$-sign patterns. It is the purpose of this paper to generalize the results and give an answer to the open question.

In this paper we discuss $\{+,-,0\}$-sign patterns that allow minimal semipositivity with exactly $n$ positive entries. We note that this restriction is the least necessary condition of semipositivity. By a *sign pattern matrix* we mean an $m \times n$ array $\boldsymbol{B} = (b_{ij})$ each of whose entries $b_{ij}$ is an element of the set $\{+,-,0\}$. The *sign pattern class* $Q(\boldsymbol{B})$ associated with $\boldsymbol{B}$ consists of all $m \times n$ real matrices $A = (a_{ij})$ such that $a_{ij}$ is positive (respectively, negative, zero) if and only if $b_{ij}$ is $+$ (respectively, $-, 0$).

If P is a property that a matrix may have, then the sign pattern $\boldsymbol{B}$ *allows* P if there is $A \in Q(\boldsymbol{B})$ that enjoys property P, and $\boldsymbol{B}$ *requires* P if each $A \in Q(\boldsymbol{B})$ enjoys P. A real square matrix is *inverse nonnegative (inverse positive)* provided it is nonsingular and its inverse is entrywise nonnegative (positive). The following theorem gives an equivalent condition for minimal semipositivity.

THEOREM 1.1 [3]. *A real square matrix is minimally semipositive if and only if it is inverse nonnegative.*

The qualitative properties of semipositivity were examined in [3]. In this paper, we give a necessary and sufficient condition for the sign pattern matrix with $n$ positive entries to allow minimal semipositivity. As in [4], a $\{+,-,0\}$-sign pattern is said to have *form G* provided each row has a nonzero entry, the rightmost one being a $+$, and in each row after the first the rightmost nonzero entry occurs in a position not to the left of the rightmost nonzero entry in the preceding row. The rightmost entry in each row of a sign pattern having form $G$ is called a *frontal plus* of that row, and of the sign pattern. A *generalized positive column* is a sign pattern $\boldsymbol{S}$ which is permutationally equivalent to a sign pattern having form $G$; that is, $\boldsymbol{S} = P\boldsymbol{S}'Q$ for some sign pattern $\boldsymbol{S}'$ having form $G$ and some permutation matrices $P$ and $Q$.

We conclude this introductory section with a summary of the results, each proved in [2]. Let $S$ be a $\{+,-,0\}$-sign pattern matrix.

LEMMA 1.1. *A sign pattern matrix $S$ allows semipositivity if and only if each row of $S$ contains a $+$.*

LEMMA 1.2. *A sign pattern matrix $S$ requires semipositivity if and only if $S$ is a generalized positive column.*

LEMMA 1.3. *A sign pattern matrix $S$ requires minimal semipositivity if and only if $S$ is a permutationally equivalent to a sign pattern $S'$, with $S'$ having form G and each column of $S'$ containing a frontal plus that is the only $+$ in its row.*

## 2. Sign patterns that allow minimal semipositivity

In this section, we shall characterize the $\{+,-,0\}$-sign patterns that allow minimal semipositivity. It will be shown that they have a certain form, we now introduce $D^+$-form of sign pattern matrix. A square sign pattern matrix $B$ is said to have *$D^+$-form* if $B$ can be written (i.e. permutationally equivalent to) as the following form

$$PBQ = \begin{pmatrix} + & & & & \\ & + & & * & \\ & & \ddots & & \\ & * & & + & \\ & & & & + \end{pmatrix}$$

for some permutation matrices $P$ and $Q$, and all off diagonal entries of $PBQ$ are nonpositive. And a square sign pattern matrix $B$ is said to have *$T^+$-form* if $B$ can be written as the following upper triangular block matrix form

$$PBQ = \begin{pmatrix} B_1 & & & \\ & B_2 & & * \\ & & \ddots & \\ & 0 & & \\ & & & B_k \end{pmatrix}$$

for some permutation matrices $P$ and $Q$, and all off diagonal entries of $PBQ$ are nonpositive.

An $n \times n$ matrix $A$ is said to be *fully indecomposable* if $A$ cannot be expressed in the form

$$P\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}Q$$

where $A_{11}, A_{22}$ are squares of order at least one and $P, Q$ are permutation matrices. An $n \times n$ matrix $A$ is said to be *partly decomposable* if it is not fully indecomposable.

First of all, we characterize a fully indecomposable sign pattern matrix which is minimally semipositive.

Let $\boldsymbol{B}$ be an $n \times n$ fully indecomposable sign pattern matrix with $n \geq 2$. If $\boldsymbol{B}$ requires minimal semipositivity, then $\boldsymbol{B}$ is permutationally equivalent to a sign pattern matrix $\boldsymbol{B}'$, with $\boldsymbol{B}'$ having form G. But this is impossible since $\boldsymbol{B}$ is fully indecomposable. Therefore we can have the following.

LEMMA 2.1. *Let $\boldsymbol{B}$ be an $n \times n$ fully indecomposable sign pattern matrix. Then $\boldsymbol{B}$ requires minimal semipositivity if and only if*

$$\boldsymbol{B} = (+)_{1\times 1}.$$

PROOF. Suppose that $n \geq 2$ and $\boldsymbol{B}$ requires minimal semipositivity. Then, by Lemma 1.3, $\boldsymbol{B}$ is permutationally equivalent to a sign pattern $\boldsymbol{B}'$, with $\boldsymbol{B}'$ having form G and each column of $\boldsymbol{B}'$ containing a frontal plus that is the only + in its row. But this is impossible since $\boldsymbol{B}$ is fully indecomposable. So $\boldsymbol{B}$ should be an $1 \times 1$ matrix

$$\boldsymbol{B} = (+)_{1\times 1}.$$

The converse is trivial. The proof is complete. □

LEMMA 2.2. *Let $\boldsymbol{B}$ be an $n\times n$ fully indecomposable sign pattern matrix with exactly $n$ positive entries. If $\boldsymbol{B}$ allows minimal semipositivity, then $\boldsymbol{B}$ has $D^+$-form.*

PROOF. Since $\boldsymbol{B}$ allows minimal semipositivity, $\boldsymbol{B}$ allows semipositivity. So, each row of $\boldsymbol{B}$ contains a $+$. If $\boldsymbol{B}$ has a column without $+$ (say, $i$-th column), then the $i$-th column deleted submatrix of $\boldsymbol{B}$ is semipositive. This violates the minimality. So each column of $\boldsymbol{B}$ contains a $+$. Hence there are permutation matrices $P$ and $Q$ so that

$$PBQ = \begin{pmatrix} + & & & & \\ & + & & * & \\ & & \ddots & & \\ & * & & + & \\ & & & & + \end{pmatrix}.$$

Therefore $\boldsymbol{B}$ has $D^{+}$-form. □

THEOREM 2.1. *Let $\boldsymbol{B}$ be an $n \times n$ fully indecomposable sign pattern matrix with $n \geq 2$. If $\boldsymbol{B}$ has $D^{+}$-form, then $\boldsymbol{B}$ allows minimal semipositivity.*

PROOF. Let $\tilde{B} \in \boldsymbol{B}$ with $\tilde{B} = [\tilde{b}_{ij}]$ $(\tilde{b}_{ii} > 0)$. Then

$$\tilde{B}e = \begin{pmatrix} \sum_{j=1}^{n} \tilde{b}_{1j} \\ \sum_{j=1}^{n} \tilde{b}_{2j} \\ \vdots \\ \sum_{j=1}^{n} \tilde{b}_{nj} \end{pmatrix}$$

where $e = (1, 1, \cdots, 1)^T$. So we can choose $b_{ii}$ are sufficiently large with $b_{ii} > \sum_{k=1, k \neq i}^{n} b_{ik}$. Then $\tilde{B}e > 0$. Hence $\boldsymbol{B}$ allows minimal semipositivity. The proof is complete. □

Let $A$ be an $m \times n$ real matrix, then the sign pattern $\boldsymbol{s}(A) = (s_{ij})$ of $A$ is defined by

$$s_{ij} = \begin{cases} + \ , & \text{if} \quad a_{ij} > 0 \\ - \ , & \text{if} \quad a_{ij} < 0 \\ 0 \ , & \text{if} \quad a_{ij} = 0 \end{cases}.$$

For a fully indecomposable matrix $A$ which $\boldsymbol{s}(A)$ has $D^{+}$-form, $A$ has exactly $n$ positive entries which consists of an $n$-cycle. Then the following theorem is an immediate consequence.

THEOREM 2.2. *Let $\boldsymbol{B}$ be an $n \times n$ fully indecomposable sign pattern matrix with exactly $n$ positive entries ($n \geq 2$). Then $\boldsymbol{B}$ allows minimal semipositivity if and only if $\boldsymbol{B}$ has an $n$-cycle with exactly $n$ positive entries.*

We have given a necessary and sufficient condition for a fully indecomposable sign pattern matrix with exactly $n$ positive entries to be minimally semipositive. We now consider an $n \times n$ partly decomposable sign pattern matrix. Since a partly decomposable matrix $\boldsymbol{B}$ is permutationally equivalent to the form

$$\begin{pmatrix} \boldsymbol{B}_1 & & & \\ & \boldsymbol{B}_2 & & * \\ & & \ddots & \\ & 0 & & \\ & & & \boldsymbol{B}_k \end{pmatrix}$$

where each block $\boldsymbol{B}_i$ is a fully indecomposable square matrix ($i = 1, 2, \ldots, k$).

THEOREM 2.3. *Let $\boldsymbol{B}$ be an $n \times n$ partly decomposable sign pattern matrix with exactly $n$ positive entries. Then $\boldsymbol{B}$ requires minimal semipositivity if and only if $\boldsymbol{B}$ is permutationally equivalent to an upper triangular matrix which has positive main diagonal and all off diagonal entries are nonpositive.*

PROOF. Suppose that $\boldsymbol{B}$ is permutationally equivalent to the block upper triangular matrix

$$\begin{pmatrix} \boldsymbol{B}_1 & & & \\ & \boldsymbol{B}_2 & & * \\ & & \ddots & \\ & 0 & & \\ & & & \boldsymbol{B}_k \end{pmatrix}$$

where each block $\boldsymbol{B}_i$ is a fully indecomposable square matrix ($i = 1, 2, \ldots, k$). If there is a block $\boldsymbol{B}_j$ whose size is not $1 \times 1$, then $\boldsymbol{B}$ does not require minimal semipositivity by Lemma 2.1. So

$$\boldsymbol{B}_i = (+)_{1\times 1}$$

for all $i = 1, 2, \ldots, k$ and $k = n$. That is, there are permutation matrices $P$ and $Q$ so that

$$PBQ = \begin{pmatrix} + & & & & \\ & + & & * & \\ & & \ddots & & \\ & 0 & & + & \\ & & & & + \end{pmatrix}$$

and super diagonal entries $(*)$ are all nonpositive. Conversely, suppose that $PBQ$ has the above form. For each $\tilde{B} \in Q(PBQ)$, consider the equation $\tilde{B}x = y$, that is,

$$\begin{pmatrix} a_1 & & & & \\ & a_2 & & * & \\ & & \ddots & & \\ & 0 & & a_{n-1} & \\ & & & & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

For $i < j$, we choose $x_i$ which is sufficiently larger than $x_j$. Then we have $y$ which is entrywise positive. Hence $B$ requires minimal semipositivity. The proof is complete. □

For the partly decomposable sign pattern matrix, we had characterized the sign patterns with $n$ positive entries that require minimal semipositivity. From now on,. we only need to consider the case of it allows but not require.

COROLLARY 2.1. *Let $B$ be an $n \times n$ partly decomposable sign pattern matrix with exactly $n (\geq 2)$ positive entries which does not require minimal semipositivity. Then $B$ allows minimal semipositivity if and only if $B$ is permutationally equivalent to the upper triangular block matrix,*

$$T^+ = \begin{pmatrix} B_1 & & & & \\ & B_2 & & * & \\ & & \ddots & & \\ & 0 & & & \\ & & & & B_k \end{pmatrix}$$

*with all the blocks $B_j$ are $D^+$-form and at least one is not $1 \times 1$.*

PROOF. Suppose that $\boldsymbol{B}$ allows minimal semipositivity and $\boldsymbol{B}$ is permutationally equivalent to $T^+$ whose diagonal blocks are all $1 \times 1$. Then $\boldsymbol{B}$ requires minimal semipositivity. This contradicts the assumption. Conversely, without loss of generality, we consider the $2 \times 2$ block sign pattern matrix which has $D^+$-form. That is,

$$\boldsymbol{B}_i = \begin{pmatrix} - & - \\ - & + \end{pmatrix}$$

Then $\boldsymbol{B}_i$ allows minimal semipositivity. □

Now we extend the characterization to rectangular sign pattern matrices, which was left to solve in [4]. Let $\boldsymbol{B}$ be an $m \times n$ $\{+, -, 0\}$-sign pattern matrix. If $\boldsymbol{B}$ has a row consisting entirely of $+$'s, then it does not affect the allowance of minimal semipositivity. So we may assume that there are no rows consisting entirely of $+$'s. Note also that, by [3. Corollary 3.3.], no $m \times n$ sign pattern with $m < n$ can allow minimal semipositivity. We now assume that $m \geq n > 1$.

THEOREM 2.4. *Let $\boldsymbol{B}$ be an $m \times n$ $\{+, -, 0\}$-sign pattern matrix with $m \geq n > 1$, and assume $\boldsymbol{B}$ has no row consisting entirely of $+$'s. Then $\boldsymbol{B}$ allows minimal semipositivity if $\boldsymbol{B}$ is permutationally equivalent to the following form of sign pattern matrix*

$$\begin{pmatrix} & \boldsymbol{B}_{11} & \\ \cdots & \cdots & \cdots \\ & \boldsymbol{B}_{21} & \end{pmatrix} \tag{1}$$

*where $B_{11}$ is an $n \times n$ submatrix with $D^+$-form or $T^+$-form, $B_{21}$ is an $(m - n) \times n$ submatrix whose each row has at least one positive entry.*

PROOF. Without loss of generality, we may assume that

$$\boldsymbol{B} = \begin{pmatrix} & \boldsymbol{B}_{11} & \\ \cdots & \cdots & \cdots \\ & \boldsymbol{B}_{21} & \end{pmatrix}$$

has the form in (1). Since $B_{11}$ has $D^+$-form or $T^+$-form, there is $\tilde{B}_{11} \in Q(\boldsymbol{B}_{11})$ such that $\tilde{B}_{11}$ is minimally semipositive. Let $x$ be a nonnegative real vector with $\tilde{B}_{11}x > 0$. If $\boldsymbol{s}(b_{ij}) = +$ for each $i > n$, then we can choose sufficiently large $\tilde{b}_{ij} \in Q(b_{ij})$ with $\sum_{j=i}^{n} \tilde{b}_{ij}x_j > 0$. Let $\tilde{B}_{21} \in Q(\boldsymbol{B}_{21})$ which contains $\tilde{b}_{ij}$ for each $i > n$. Then

$$\tilde{B} = \begin{pmatrix} & \tilde{\boldsymbol{B}}_{11} & \\ \cdots & \cdots & \cdots \\ & \tilde{\boldsymbol{B}}_{21} & \end{pmatrix}$$

is minimally semipositive. Therefore $\boldsymbol{B}$ allows minimal semipositivity. The proof is complete. □

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