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LINKS IN SPECTRUM OF THE COORDINATE RING OF QUANTUM AFFINE SPACE

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ABSTRACT. We generalize the Artin-Rees Theorem about the AR-property and study linking diagram in the set of prime ideals of the coordinate ring of quantum affine space.

The aim of this short note is to generalize the Artin-Rees Theorem about the AR-property (see Theorem 2 and Corollary 3) and to study linking diagram in spec $\mathcal{O}_q(k^n)$, the set of prime ideals of the coordinate ring of quantum affine space, using this generalized Theorem.

We say that an ideal I in a ring R has the right AR-property if for every right ideal K of R, there is a positive integer n such that $K \cap I^n \leq KI$. The left AR-property is defined similarly, and we say that I has the AR-property if it has both the right and left AR-property. Note that the Rees ring of an ideal I of ring R is the subring $\mathcal{R}(I)$ of the polynomial ring $R[x]$ defined by

$$\mathcal{R}(I) = R + Ix + I^2x^2 + ... + I^ix^i + ...$$

LEMMA 1. *If I is an ideal in a ring R, and the Rees ring $\mathcal{R}(I)$ is right (left) noetherian, then I satisfies the right (left) AR-property.*

PROOF. [2, Lemma 11.12]. □

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THEOREM 2. *Let R be a noetherian ring with automorphisms $\alpha_1, ..., \alpha_n$ commuting each other and let I be an ideal of R generated by $a_1, ..., a_n$ such that*

$$a_i r = \alpha_i(r) a_i, \alpha_i(a_j) = \lambda_{ij} a_j$$

for $r \in R$, $i, j = 1, ..., n$ and for some invertible central elements $\lambda_{ij} \in R$. Then the Rees ring $\mathcal{R}(I)$ is noetherian and I has AR-property.

PROOF. By induction on $i = 1, ..., n$, there is an extension of α_i, denoted by α_i', to $S_{i-1} = R[x_1; \alpha_1'] \cdots [x_{i-1}; \alpha_{i-1}']$ by setting $\alpha_i'(x_j) = \lambda_{ij} x_j, j < i$ and thus there is an iterated skew polynomial ring $S = R[x_1; \alpha_1'] \cdots [x_n; \alpha_n']$. Moreover a map from S into the Rees ring $\mathcal{R}(I)$ given by $r \mapsto r, r \in R, x_i \mapsto a_i x$ for each $i = 1, ..., n$, is an epimorphism. Hence $\mathcal{R}(I)$ is noetherian since S is noetherian and I has the AR-property by Lemma 1. □

COROLLARY 3. (Artin-Rees Theorem [2, Theorem 11.13]) *If R is a noetherian ring and I is an ideal of R generated by central elements, then $\mathcal{R}(I)$ is noetherian and hence I satisfies the AR-property.*

PROOF. Since R is noetherian, I can be generated by finite central elements $a_1, ..., a_n$. Applying Theorem 2 by replacing each automorphism α_i to the identity map and each λ_{ij} to 1, we get the conclusion. □

Let k be a field and $0 \neq q \in k$. The coordinate ring of quantum affine n-space, denoted by $\mathcal{O}_q(k^n)$, is a k-algebra generated by n-generators $x_1, ..., x_n$ subject to the relations

$$x_i x_j = q x_j x_i, \quad i < j.$$

It is well-known that $\mathcal{O}_q(k^n)$ is an iterated skew polynomial ring (see [1] or [3]), and hence it is noetherian by [2, 1.12]. In fact, under the same notation of the paper [5], $\mathcal{O}_q(k^n)$ is contained in the class of iterated skew polynomial rings $R_m^{C,\Lambda,U}(k)$; If $n = 2m$ then $\mathcal{O}_q(k^{2m}) = R_m^{C,\Lambda,U}(k)$, where $C = (1, ..., 1, 1) \in (k^*)^{m+1}$, $\Lambda = (\lambda_{ji})_{1 \leq i < j \leq m}, \lambda_{ji} = q^{-1}$, $U = (0, ..., 0)$. If $n = 2m + 1$ then $\mathcal{O}_q(k^{2m+1}) = R_{m+1}^{C,\Lambda,U}(k)/\langle x_1 \rangle$, which is an iterated skew polynomial ring as in [5, Example 5].

Let J be an ideal of $\mathcal{O}_q(k^n)$ generated by a subset of $\{x_1, ..., x_n\}$, say

$$x_{i_1}, ..., x_{i_r}, \quad i_1 < ... < i_r.$$

COROLLARY 4. *The Rees ring $\mathcal{R}(J) = \mathcal{O}_q(k^n) + Jx + J^2x^2 + \cdots$ is noetherian and hence J has the AR-property.*

PROOF. For each i_j, there is an automorphism α_{i_j} of $\mathcal{O}_q(k^n)$ defined by

$$\alpha_{i_j}(x_m) = \begin{cases} qx_m & i_j < m \\ x_m & i_j = m \\ q^{-1}x_m & i_j > m \end{cases}$$

Clearly, the automorphisms $\alpha_{i_j}, j = 1, ..., r$ satisfy the condition of Theorem 2. Hence the proof completes immediately from Theorem 2. □

Now we can study links in spec $\mathcal{O}_q(k^n)$, the set of prime ideals in $\mathcal{O}_q(k^n)$ using Corollary 4. Recall that if P and Q are prime ideals in a noetherian ring R then we say that there is a link from Q to P, denoted by $Q \rightsquigarrow P$, if there is an ideal A of R such that

$$Q \cap P > A \geq QP$$

and $(Q \cap P)/A$ is nonzero and torsion free as a right R/P-module and as a left R/Q-module. The graph of links of R is the directed graph whose vertices are the elements of spec R with an arrow from Q to P whenever $Q \rightsquigarrow P$ and the connected components of this graph are called cliques.

LEMMA 5. *Let I be an ideal in a noetherian ring R, and let P and Q be prime ideals of R with $Q \rightsquigarrow P$. If $I \leq P$ and I satisfies the right AR-property, then $I \leq Q$. Similarly, if $I \leq Q$ and I satisfies the left AR-property, then $I \leq P$.*

PROOF. [2, Proposition 11.16]. □

In $\mathcal{O}_q(k^n)$, put

$$\wp = \{x_1, ..., x_n\}.$$

One should observe that the set $\wp$ in $\mathcal{O}_q(k^n)$ is similar to a set of (almost) normal elements in algebras related to quantum groups. For example, z_i in $R_m^{C,\Lambda,U}(k)$ appeared in [5], x_i and Ω_i in $\mathcal{O}_q(sp\mathbb{C}^{2n})$ appeared in [4], and $\{a, b, c, \Omega\}$ in $\mathcal{O}_q(M_2(k))$ appeared in [6].

COROLLARY 6. *For prime ideals P, Q of $\mathcal{O}_q(k^n)$, if $P \rightsquigarrow Q$ then $P \cap \wp = Q \cap \wp$.*

PROOF. Note that $\mathcal{O}_q(k^n)$ is noetherian. The two ideals generated by $P \cap \wp$ and $Q \cap \wp$ have the AR-property by Corollary 4 and thus, by Lemma 5, $P \cap \wp$ and $Q \cap \wp$ are contained in Q and P, respectively. Therefore $P \cap \wp = P \cap \wp \cap Q = Q \cap \wp$. □

COROLLARY 7. *For each $S \subseteq \wp$, let $\mathrm{spec}_S\, \mathcal{O}_q(k^n)$ denote the set of all $P \in$ spec $\mathcal{O}_q(k^n)$ such that $S = P \cap \wp$. Then $\mathrm{clique}(P) \subseteq \mathrm{spec}_S\, \mathcal{O}_q(k^n)$, where $\mathrm{clique}(P)$ is the clique containing P.*

PROOF. It follows immediately from Corollary 6. □

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