Comm. Korean Math. Soc. **13** (1998), No. 3, pp. 445–459

COMBINATORIAL PROOF FOR THE GENERALIZED SCHUR IDENTITY

JAEJIN LEE

ABSTRACT. Let λ be a partition with all distinct parts. In this paper we give a bijection between the set $\Gamma_\lambda(X)$ of pairs $(P^{1}_{2}, \bar{\sigma})$ satisfying a certain condition and the set $\pi_\lambda(X)$ of circled permutation tableaux of shape λ on the set X, where P^{1}_{2} is a tail circled shifted rim hook tableaux of shape λ and $\bar{\sigma}$ is a barred permutation on X.

Specializing to the partition λ with one part, this bijection gives a combinatorial proof of the Schur identity:

$$\sum 2^{\ell(\mathrm{type}(\sigma))} = 2n!$$

summed over all permutation $\sigma \in S_n$ with type$(\sigma) \in OP_n$.

0. Introduction

There has been a recent surge of interest in the projective representations of symmetric groups and shifted tableaux. Morris[1] constructed a projective analog of the Murnaghan–Nakayama character recurrence and Stembridge[2] found a Frobenius-type characteristic map and an analog of the Littlewood–Richardson rule. Sagan[3] and Worley[4] has developed independently a combinatorial theory of shifted tableaux parallel to the theory of ordinary tableaux. This theory includes shifted versions of the Robinson-Schensted-Knuth correspondence, Green's invariants, Knuth relation, and Schützenberger's jeu de taquin.

Received August 16, 1996. Revised April 30, 1998.
1991 Mathematics Subject Classification: 05A19.
Key words and phrases: shifted rim hook tableaux, circled permutation tableaux.
This work was supported under a grant from Hallym University, 1995.

In this paper we give a bijection ϕ between the set $\Gamma_\lambda(X)$ of pairs $(P^1_2, \bar{\sigma})$ satisfying a certain condition and the set $\pi_\lambda(X)$ of circled permutation tableaux of shape λ on the set X, where λ is a partition with all distinct parts and where P^1_2 is a tail circled shifted rim hook tableaux of shape λ and $\bar{\sigma}$ is a barred permutation on X.

Specializing to the partition λ with one part in the above, we get a combinatorial proof of the Schur identity:

$$\sum 2^{\ell(\text{type}(\sigma))} = 2n!$$

summed over all permutation $\sigma \in S_n$ with type$(\sigma) \in OP_n$.

Furthermore, using the recurrence formula for the irreducible spin characters of the symmetric group S_n, the above bijection gives us a bijective proof for the identity

$$\sum_{\sigma \in \tilde{S_n}} \varphi(\sigma)\varphi(\sigma^{-1}) = 2n!,$$

where φ is an irreducible character of $\tilde{S_n}$. (See Corollary 2.10 for the definition of $\tilde{S_n}$)

In section 1, we outline the definitions and notation used in this paper. In section 2, we construct a bijection ϕ and explain some properties obtained from the bijection ϕ.

1. Definitions

In this section we introduce the most basic unit in this paper.

DEFINITION 1.1. A *partition* λ of a positive integer n is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that

(1) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$,

(2) $\sum_{i=1}^{\ell} \lambda_i = n$.

We write $\lambda \vdash n$, or $|\lambda| = n$. We say each term λ_i is a *part* of λ. The number of parts is called the *length* of λ and is written $\ell = \ell(\lambda)$. We sometimes abbreviate the partition λ with the notation $1^{j_1}2^{j_2}\ldots$, where j_i is the number of parts of size i. Sizes which do not appear are omitted and, if $j_i = 1$ then it is not written. Thus, a partition $(5,3,2,2,2,1) \vdash 15$ can be written 12^335.

NOTATION 1.2. We denote

$$\begin{aligned} \mathcal{P}_n &= \{ \mu \mid \mu \text{ is a partition of } n \} \\ OP_n &= \{ \mu \in \mathcal{P}_n \mid \text{every part of } \mu \text{ is odd} \} \\ DP_n &= \{ \mu \in \mathcal{P}_n \mid \mu \text{ has all distinct parts} \} \\ DP &= \{ \mu \in DP_n \mid n \in \mathbb{N} \}. \end{aligned}$$

DEFINITION 1.3. For each $\lambda \in DP$, a *shifted diagram* D'_λ of shape λ is defined by

$$D'_\lambda = \{ (i,j) \in \mathbb{Z}^2 \mid i \leq j \leq \lambda_j + i - 1, 1 \leq i \leq \ell(\lambda) \}.$$

And for $\lambda, \mu \in DP$ with $\mu \subseteq \lambda$, a *shifted skew diagram* $D'_{\lambda/\mu}$ is defined as the set-theoretic difference $D'_\lambda \setminus D'_\mu$.

FIGURE 1.1 shows D'_λ and $D'_{\lambda/\mu}$ when $\lambda = (9,7,4,2)$ and $\mu = (5,3,1)$.

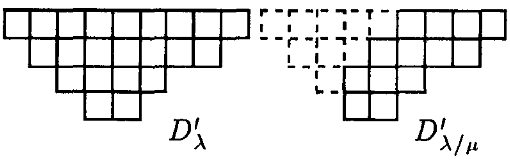

FIGURE 1.1

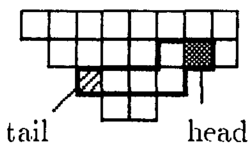

FIGURE 1.2

DEFINITION 1.4. A shifted skew diagram θ is called a *single rim hook* if θ is connected and contains no 2×2 block of cells. If θ is a single rim hook, then its *head* is the upper rightmost cell in θ and its *tail* is the lower leftmost cell in θ. See FIGURE 1.2.

DEFINITION 1.5. *double rim hook* is a shifted skew diagram θ formed by the union of two single rim hooks both of whose tails are on the main diagonal. If θ is a double rim hook, we denote by $\mathcal{A}[\theta]$ (resp., $\alpha_1[\theta]$) the set of diagonals of length two (resp., one). Also let $\beta_1[\theta]$ (resp., $\gamma_1[\theta]$) be a single rim hook in θ which starts on the upper (resp., lower) of the two main diagonal cells and ends at the head of $\alpha_1[\theta]$. The tail of $\beta_1[\theta]$ (resp., $\gamma_1[\theta]$) is called the *first tail* (resp., *second tail*) of θ. Hence we have the following descriptions for a double rim hook θ: $\theta = \mathcal{A}[\theta] \cup \alpha_1[\theta] = \beta_1[\theta] \cup \beta_2[\theta] = \gamma_1[\theta] \cup \gamma_2[\theta]$.

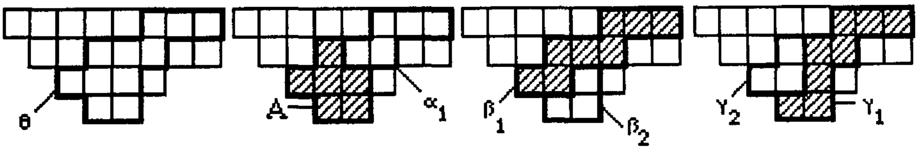

FIGURE 1.3

Definition 1.5 is illustrated in FIGURE 1.3.

We will use the term *rim hook* to mean a single rim hook or a double rim hook.

DEFINITION 1.6. A *shifted rim hook tableau of shape* $\lambda \in DP_n$ *and content* $\rho = (\rho_1, \ldots, \rho_m)$ is defined recursively. If $m = 1$, a rim hook with all 1's and shape λ is a shifted rim hook tableau. Suppose P of shape λ has content $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$ and the cells containing the m's form a rim hook inside λ. If the removal of the m's leaves a shifted rim hook tableau, then P is a shifted rim hook tableau.

DEFINITION 1.7. If θ is a single rim hook then the *rank* $r(\theta)$ is one less than the number of rows it occupies and the *weight* $w(\theta) = (-1)^{r(\theta)}$; if θ is a double rim hook then the *rank* $r(\theta)$ is $|\mathcal{A}[\theta]|/2 + r(\alpha_1[\theta])$ and the *weight* $w(\theta)$ is $2(-1)^{r(\theta)}$.

The *weight* of a shifted rim hook tableau P, $w(P)$, is the product of the weights of its rim hooks.

Let P be a shifted rim hook tableau. We write $\kappa_P\langle r\rangle$ (or just $\kappa\langle r\rangle$) for a rim hook of P containing r. FIGURE 1.4 shows an example of a shifted rim hook tableau P of shape $(6,5,3,2)$ and content $(6,4,2,3,1)$. Here $r(\kappa\langle 1\rangle) = 2$, $r(\kappa\langle 2\rangle) = 1$, $r(\kappa\langle 3\rangle) = 0$, $r(\kappa\langle 4\rangle) = 1$ and $r(\kappa\langle 5\rangle) = 0$. Also $w(\kappa\langle 1\rangle) = 2$, $w(\kappa\langle 2\rangle) = -1$, $w(\kappa\langle 3\rangle) = 1$, $w(\kappa\langle 4\rangle) = -1$ and $w(\kappa\langle 5\rangle) = 1$. Hence $w(P) = (2)\cdot(-1)\cdot(1)\cdot(-1)\cdot(1) = 2$.

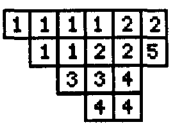

FIGURE 1.4

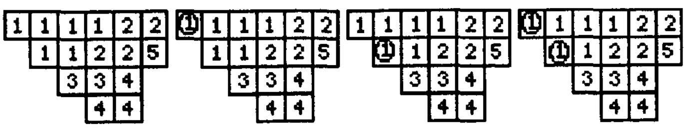

FIGURE 1.5

DEFINITION 1.8. Suppose P is a shifted rim hook tableau. P_2^1 is obtained from P by circling or not circling the first tail and the second tail of each double rim hook in P and is called a *tail circled rim hook tableau.* We use the notation $|\cdot|$ to refer to the uncircled version; e.g., $|P_2^1| = P$.

We now define a new weight function w' for tail circled rim hook tableaux. If τ is a rim hook of P_2^1, we define $w'(\tau) = 1$. The weight $w'(P_2^1)$ is the product of the weights of rim hooks in P_2^1. Hence we have $w'(P_2^1) = 1$ for any tail circled tableau P_2^1. If P is a shifted rim hook tableau in FIGURE 1.4, FIGURE 1.5 shows tail circled rim hook tableaux P_2^1.

2. Combinatorial proof for the generalized Schur identity

DEFINITION 2.1. A rim hook γ is called *the rim hook inside* λ if γ is contained in λ and its removal from λ leaves another legal shape. The shape created by the removal of γ is denoted by $\lambda - \gamma$. If γ is disjoint from λ but its addition to λ creates a new shape, then γ is *a rim hook outside* λ and the new shape formed by its addition to λ is denoted by $\lambda + \gamma$. In FIGURE 2.1, σ is the rim hook inside λ and τ is the rim hook outside λ, where $\lambda = (7, 4, 2, 1)$.

DEFINITION 2.2. Let $\lambda \in DP$ and D'_λ be the shifted shape. The *shifted hook* h_α of the cell $\alpha = (i, j) \in \lambda$ is

$$h_\alpha = \{(i,j)\} \cup \{(i,j') \mid j' > j\} \cup \{(i',j) \mid i' > i\} \cup \{(j+1,j') \mid j' > j\}$$

with hook length $h(\alpha) = h(i,j) = |h_\alpha|$. We call the sets $\{(i,j') \mid j' > j\}$ and $\{(j+1,j') \mid j' > j\}$ *the first arm* and *the second arm* of the hook, respectively. The set $\{(i',j) \mid i' > i\}$ is called the *leg* of the hook. In FIGURE 2.2, the shaded part shows the hook h_α of the $\alpha = (2,3)$ in the partition $\lambda = (6,5,3,2,1)$ which has hook length $h(\alpha) = h(2,3) = 7$.

It is frequently necessary in discussions involving tableaux and shapes to refer to the directions within the shape. Generally speaking, x will be SE of y if the row of x is the same as or below the row of y and the column of x is the same as or to the right of the column of y. Also, x

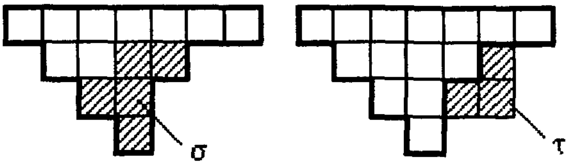

FIGURE 2.1

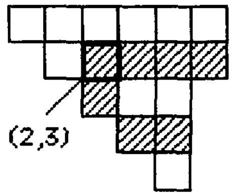

FIGURE 2.2

will be *strictly* SE of y if x is SE of y but not in the same row or same column.

PROPOSITION 2.3. *If $\lambda \in DP_n$, there is a one-to-one correspondence between the set D'_λ of cells in λ and the set of rim hooks inside λ.*

PROOF. If $\alpha \in D'_\lambda$, remove the associated hook h_α from λ and then push every entry strictly SE of α diagonally NW. Then we get another shape λ', and λ/λ' is a *rim hook inside* λ. In fact, if h_α has a second arm, then λ/λ' is a double rim hook with the head of β_1 in the same row as the first arm of h_α and the head of β_2 in the same row as the second arm of h_α. If h_α does not have a second arm, then λ/λ' is a single rim hook whose head is in the same row as the arm of h_α and whose tail is in the same column as the leg of h_α.

This procedure is easily reversed. If τ is a double rim hook, then the heads of β_1 and β_2 determine the first and second arms of h_α. These determine α. If τ is a single rim hook, then the head and tail of τ determine the row and column of α. □

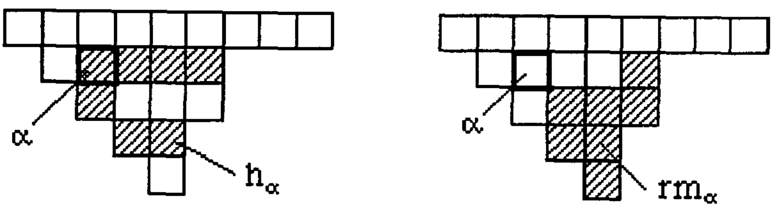

FIGURE 2.3

The rim hook associated with the cell α in this manner will be denoted rm_α. Note that the size of rm_α is the same as the size of h_α. FIGURE 2.3 shows us the hook and the rim hook rm_α associated with α.

If we use the induction on the number of rows of D'_λ, we can prove the following lemma.

LEMMA 2.4. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in DP_n$ and let R_k^λ be the set of cells in the kth row of D'_λ. Let*

$$E_k^\lambda = \{x \in R_k^\lambda \mid h(x) \text{ is even}\} \quad \text{and}$$
$$O_k^\lambda = \{x \in R_k^\lambda \mid h(x) \text{ is odd}, \ h(x) \neq \lambda_k\}.$$

Then we have

$$|E_k^\lambda| = |O_k^\lambda| = \lfloor \lambda_k/2 \rfloor.$$

DEFINITION 2.5. Let $\lambda \in DP_n$ and let $r_k = \lfloor \lambda_k/2 \rfloor$. Let O_k^λ and E_k^λ be as in Lemma 2.4. Suppose we have

$$O_k^\lambda = \{x_{k,1}, x_{k,2}, \ldots, x_{k,r_k}\} \text{ with } \ h(x_{k,1}) < h(x_{k,2}) < \cdots < h(x_{k,r_k}),$$
$$E_k^\lambda = \{y_{k,1}, y_{k,2}, \ldots, y_{k,r_k}\} \text{ with } \ h(y_{k,1}) > h(y_{k,2}) > \cdots > h(y_{k,r_k}).$$

We define a function v_k from $O_k^\lambda \cup E_k^\lambda$ to itself via $x_{k,j} \mapsto y_{k,j}$ and $y_{k,j} \mapsto x_{k,j}$. Then v_k is an involution on $O_k^\lambda \cup E_k^\lambda$.

Now define $v : E \cup O \to E \cup O$ by $v(x) = v_k(x)$ if $x \in O_k^\lambda \cup E_k^\lambda$, where

$$E = \cup_{k=1}^{\ell} E_k^\lambda \qquad \text{and}$$
$$O = \cup_{k=1}^{\ell} O_k^\lambda.$$

Then v is also an involution on $E \cup O$.

DEFINITION 2.6. Suppose X is a set of positive integers. A *permutation tableau* on X is a tableau where each number of X appears exactly once. Clearly the number of permutation tableaux of shape λ on X is $|X|!$.

Let S_X be the set of permutations on X (if $X = \{1, 2, \ldots, n\}$, $S_X = S_n$). Let $\sigma \in S_X$ and write σ in cycle form, $\sigma = \sigma_1\sigma_2 \ldots \sigma_m$, where the cycles σ_i are written in increasing order of the largest in the cycle. Content (σ) of σ is defined as the sequence $\rho = (\rho_1, \rho_2, \dots, \rho_m)$, where $\rho_i = |\sigma_i|$ =length of the cycle σ_i. And type (σ) of σ is defined as a partition $1^{m_1}2^{m_2}\dots$, where m_i is the number of i-cycles in σ. If $\sigma \in S_X$, then let $\overline{\sigma}$ be a permutation obtained from σ in which each cycle of σ is either barred or unbarred. If $\sigma = (42)(8371)$, $\overline{\sigma}$ is one of $(42)(8371)$, $\overline{(42)}(8371)$, $(42)\overline{(8371)}$ and $\overline{(42)(8371)}$. We use the notation $|\sigma|$ to refer to the unbarred version of any $\sigma \in S_X$; e.g., $|\overline{(21)}(43)| = |\overline{(21)(43)}| = (21)(43)$.

Definition 2.7. Let P be a shifted rim hook tableau of content $\rho = (\rho_1, \dots, \rho_m)$. If $\rho' = (\rho_1, \dots, \rho_k), k < m$, then the *restriction* of P to ρ', denoted by $P|_{\rho'}$, is the shifted rim hook tableau obtained from P by removing all entries greater than k. If P and Q are shifted rim hook tableaux, let $P \cap Q$ denote the largest shifted rim hook tableau R which has the property that $R = P|_{\rho'} = Q|_{\rho'}$.

Now let $X = \{a_1 < a_2 < \cdots < a_n\}$ be a set of positive integers and let $\lambda \in DP_n$. Let

$$\Gamma_\lambda(X) = \{(P_2^1, \overline{\sigma}) \mid P \in \mathcal{A}_\lambda, \sigma \in S_X \text{ with } \operatorname{content}(P) = \operatorname{content}(\sigma)\},$$

where $\mathcal{A}_\lambda$ is the set of shifted rim hook tableaux of shape λ and S_X is the set of permutations on X with type$(\sigma) \in OP_n$.

Let $\pi_\lambda(X)$ be the set of permutation tableaux of shape λ on X with each main diagonal entry either circled or uncircled. Then we have the following bijection from $\Gamma_\lambda(X)$ to $\pi_\lambda(X)$.

Theorem 2.8. *Let $\lambda \in DP_n$. Then there is a bijection ϕ from $\Gamma_\lambda(X)$ to $\pi_\lambda(X)$.*

Proof. We describe the bijection recursively. Suppose $T \in \pi_\lambda(X)$. Find the largest entry a_n in T and let $\alpha = (i,j)$ denote its cell. Some of the steps will split into three cases. These cases are:

Case I : $h(\alpha)$ is odd and $h(\alpha) \neq \lambda_i$.

Case II : $h(\alpha)$ is even.

Case III : $h(\alpha)$ is odd and $h(\alpha) = \lambda_i$.

Step a.1. Modify T and determine the hook to remove. In Case II, modify T by exchanging a_n with the entry in $v(\alpha)$. Let $\beta = v(\alpha)$. In Cases I and III, no change in T is necessary; simply let $\beta = \alpha$.

Step a.2. Removal of hook. Remove h_β from T and push every entry strictly SE of β diagonally NW. Call this new permutation tableau T'. Let $b_1, b_2, \ldots, b_r$ be the entries in T in the first arm of h_β (read left to right); let $c_1, c_2, \ldots, c_s$ be the entries in T in the leg (if it exists) of h_β (read top to bottom); let $d_1, d_2, \ldots, d_t$ be the entries in T in the second arm (if it exists) of h_β (read left to right). Then $T' \in \pi_{\lambda'}(X - \{a_n, b_1, \ldots, b_r, c_1, \ldots, c_s, d_1, \ldots, d_t\}) = \pi_{\lambda'}(Y)$. Finally, let τ be the cycle $(a_n b_1 \ldots b_r c_1 \ldots c_s d_1 \ldots d_t)$.

Step a.3. Recursive step. Recursively construct $({P'}^1_2, \overline{\sigma'}) \in \Gamma_{\lambda'}(Y)$ from T'.

Step a.4. Determine a new rim hook. Let γ be the rim hook inside λ associated with β. That is, $\lambda' = \lambda - \gamma$. Let $(\rho_1, \ldots, \rho_{k-1}) = \rho'$ be the content of ${P'}^1_2$. Define P^1_2 so that $P^1_2|_{\rho'} = {P'}^1_2$ and $\kappa_{P^1_2}\langle k\rangle = \gamma$.

Step a.5. Determine circles. In cases I and II, place a circle on the first tail (resp., second tail) of γ if the cell β or the leg (resp., second arm) of h_β has a circle. No circles are drawn in Case III.

Step a.6. Determine a new permutation. In Case I, let $\overline{\sigma} = \overline{\sigma'}\tau$. In Case II, let $\overline{\sigma} = \overline{\sigma'}\overline{\tau}$. In Case III, let $\overline{\sigma} = \overline{\sigma'}\tau$ if h_β has no circle and let $\overline{\sigma} = \overline{\sigma'}\overline{\tau}$ if h_β has a circle.

Then $(P^1_2, \overline{\sigma}) \in \Gamma_\lambda(X)$.

This construction can be reversed easily. Suppose we were given $(P^1_2, \overline{\sigma}) \in \Gamma_\lambda(X)$. Suppose content$(P^1_2) = (\rho_1, \rho_2, \ldots, \rho_k)$. Let $\gamma = \kappa\langle k\rangle$ in P^1_2 and let $\alpha = (i,j)$ be the cell in λ with $\gamma = rm_\alpha$. Let r be the length of the first arm of h_α; let s be the length of the leg (if it exists) of h_α; let t be the length of the second arm (if it exists) of h_α. Write the last cycle in $|\overline{\sigma}|$ as $(a_n b_1 \ldots b_r c_1 \ldots c_s d_1 \ldots d_t)$. (Recall that a_n is the largest in the cycle.) Note that $r + s + t + 1 = |h_\alpha| = \rho_k$.

Step b.1. Removal of the last rim hook and the last cycle. Remove the last rim hook γ from P^1_2 to get ${P'}^1_2 = P^1_2|_{(\rho_1, \ldots, \rho_{k-1})}$. Let λ' denote the shape of ${P'}^1_2$ and $\rho' = (\rho_1, \ldots, \rho_{k-1})$ its content. Then $\lambda' = \lambda - \gamma$. Let $\overline{\sigma'}$ denote the permutation on $X - \{a_n, b_1, \ldots, b_r, c_1, \ldots, c_s, d_1, \ldots, d_t\} = Y$ obtained by removing the last cycle from $\overline{\sigma}$. Thus, $({P'}^1_2, \overline{\sigma'}) \in \Gamma_{\lambda'}(Y)$.

Step b.2. Recursive step. Recursively construct $T' \in \pi_{\lambda'}(Y)$ from $({P'}^1_2, \overline{\sigma'})$.

Step b.3. Construction of a new permutation tableau T'' from T'. Shift every entry in T' of α SE. Place a_n into α ; $b_1, \ldots, b_r$ into the first arm of h_α (left to right); $c_1, \ldots, c_s$ into the leg of h_α (top to bottom); and $d_1, \ldots, d_t$ into the second arm of h_α (left to right). Circle the cell α or the main diagonal of the leg (resp., second arm) of h_α if γ has a circle on its first tail (resp., second tail). The resulting tableau $T'' \in \pi_\lambda(X)$.

Step b.4. Modify T'' to get T. Finally define T as follows:

(1) If σ_k has no bar on it, let $T = T''$.
(2) If $h(\alpha) \neq \lambda_i$ and σ_k has a bar on it, T is obtained from T'' by interchanging a_n and the entry in $v(\alpha)$.
(3) If $h(\alpha) = \lambda_i$ and σ_k has a bar on it, T is obtained from T'' by circling the tail of h_α.

It is easy to see that these two constructions are inverses of one other. □

In FIGURE 2.4–FIGURE 2.10, we give examples of each case in the above description. In these figures, we use the alphabet $1 < 2 < \cdots < 9 < a < b < c$. In FIGURE 2.4, T is given with α marked. Since $h(\alpha)$ is odd and $h(\alpha) \neq \lambda_1$, no change in T was made (Step a.1). Next remove h_α from T to get T' (Step a.2). The recursive Step a.3 produces ${P'}_2^1$ and $\overline{\sigma'}$. The h_α corresponds to the rim hook γ which is attached to ${P'}_2^1$ to form P_2^1 (Step a.4 and Step a.5). Finally, the cycle formed from the entries in h_α together with $\overline{\sigma'}$ yields $\overline{\sigma}$ (Step a.6).

In FIGURE 2.5, note that $h(\alpha)$ is even. Hence we switch the entry "c" in the cell α with the entry "9" in $v(\alpha)$ to get the T''. Now we do the same steps as in FIGURE 2.4 except $\overline{\sigma}$ has a bar on its last cycle (Step a.6).

In FIGURE 2.6 and FIGURE 2.7, depending on the circle of the tail of h_α, we place a bar on the last cycle. In FIGURE 2.8, P_2^1 and $\overline{\sigma}$ are given. ${P'}_2^1$ is obtained by deleting the last rim hook γ; $\overline{\sigma'}$ is obtained by deleting the last cycle (Step b.1). Then recursive Step b.2 produces T'. Next T' is adjusted to make room for the last cycle (Step b.3). The last cycle is inserted into the hook h_α corresponding to γ to give T'' (Step b.3). Since there is no bar on the last cycle in $\overline{\sigma}$, we have $T = T''$ (Step b.4). FIGURE 2.9 and FIGURE 2.10 show examples of (2) and (3) of the Step b.4, respectively.

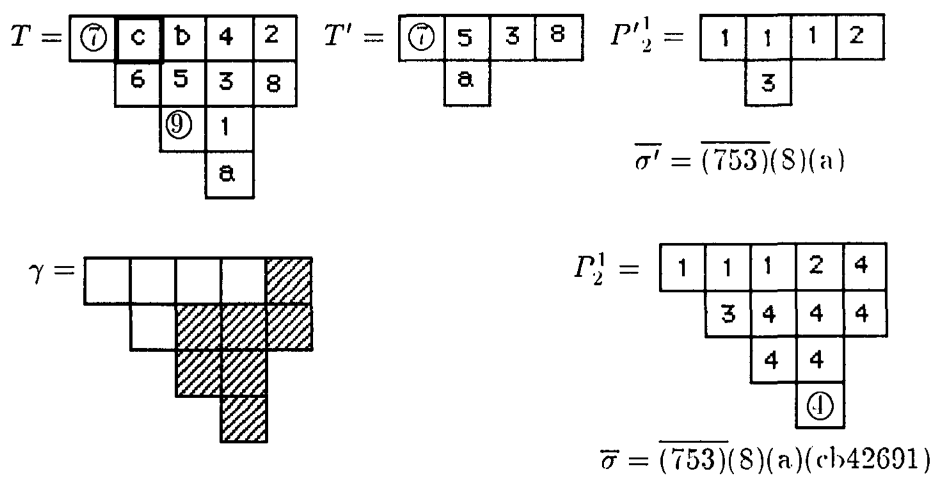

FIGURE 2.4

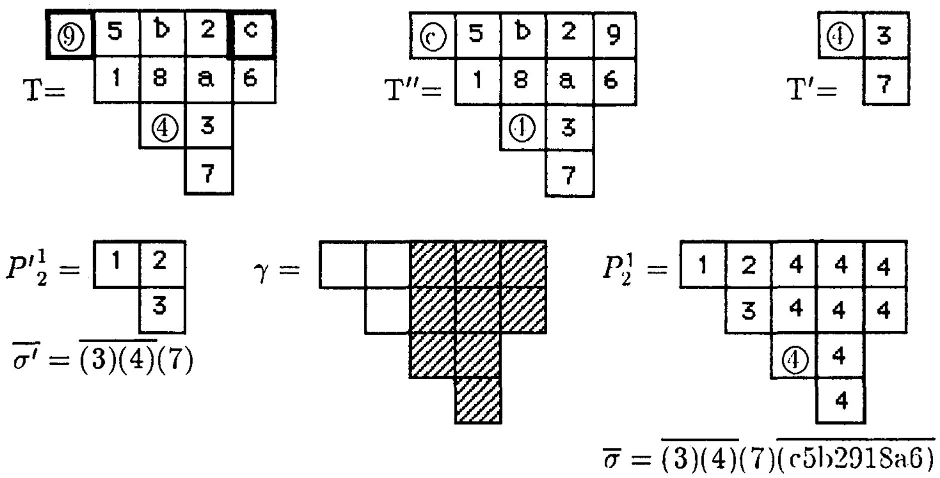

FIGURE 2.5

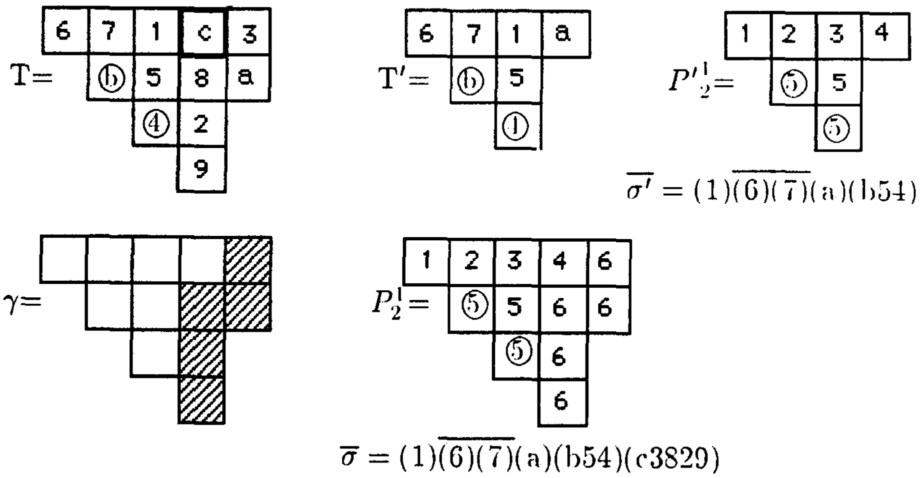

FIGURE 2.6

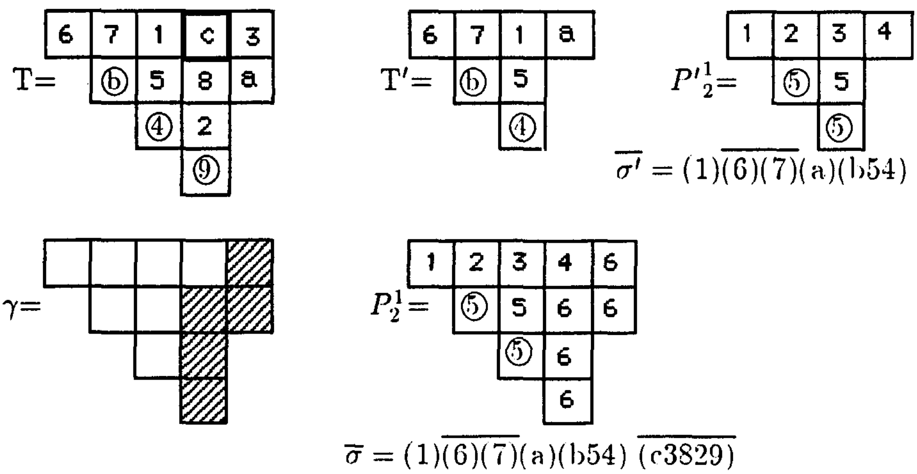

FIGURE 2.7

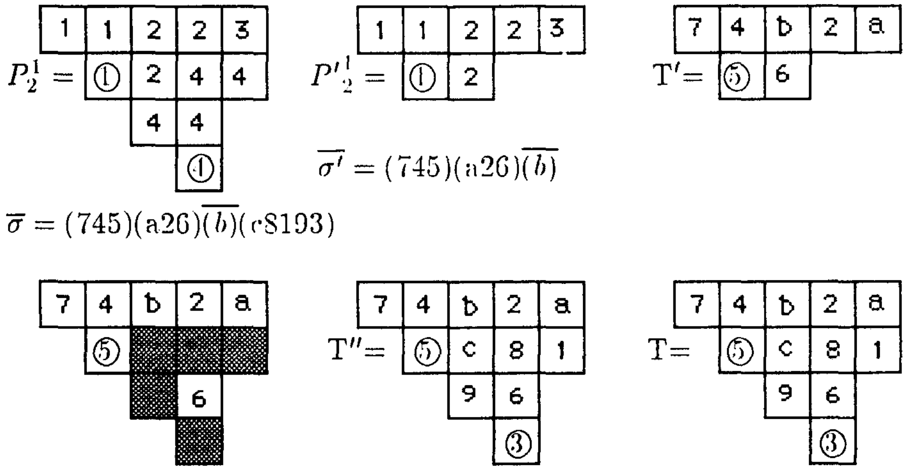

FIGURE 2.8

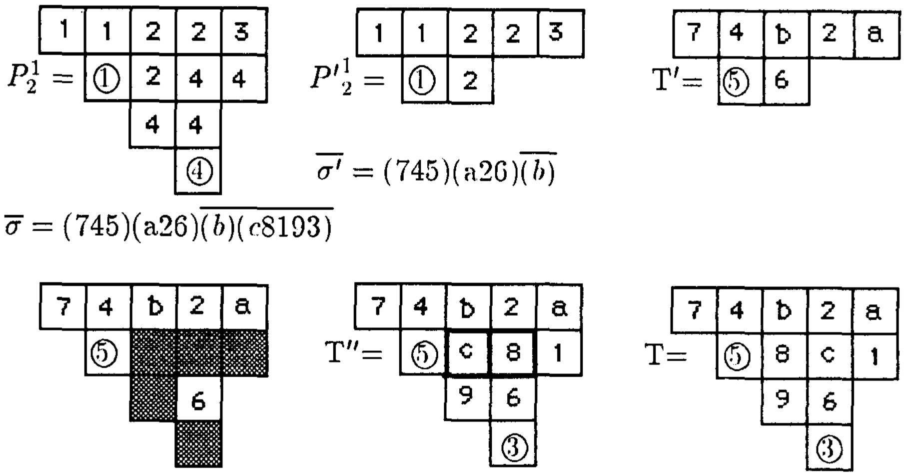

FIGURE 2.9

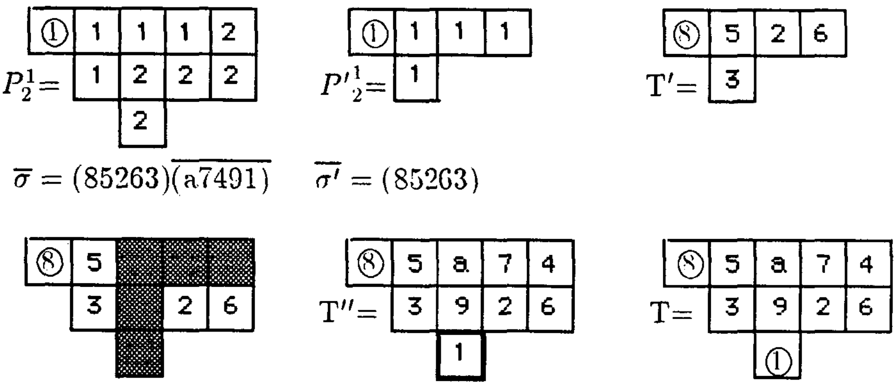

FIGURE 2.10

THEOREM 2.9. *Let $\lambda \in DP_n$. Then*

$$\sum 2^{\ell(type(\sigma))}(w(P))^2 = 2^{\ell(\lambda)}n!,$$

where the sum is over shifted rim hook tableaux P of shape λ, and $\sigma \in S_n$ with type(σ) $\in OP_n$, content(P)=content(σ).

PROOF. Let P be a shifted rim hook tableau of shape λ. For each double rim hook τ in P, we have four tail circled tableaux depending on circling the first tail and second tail or not. Since $|w(\tau)| = 2$ and $w'(\tau) = 1$, we have

$$\sum 2^{\ell(type(\sigma))}(w(P))^2 = \sum 2^{\ell(type(\sigma))}w'(P_2^1)$$
$$= 2^{\ell(\lambda)}n!. \qquad \square$$

If we use the recurrence formula for the irreducible spin characters of the symmetric group S_n(see [2]), Theorem 2.9 gives us the following identity:

COROLLARY 2.10. *For a positive integer n, let $\tilde{S}_n$ be the group generated by $t_1, t_2, \ldots, t_{n-1}, -1$ subject to relations*

$$t_i^2 = -1 \quad \text{for } i = 1, 2, \ldots, n-1,$$
$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \quad \text{for } i = 1, 2, \ldots, n-2,$$
$$t_i t_j = -t_j t_i \quad \text{for } |i-j| > 1 \ (i, j = 1, 2, \ldots, n-1).$$

Then

$$\sum_{\sigma \in \tilde{S}_n} \varphi(\sigma)\varphi(\sigma^{-1}) = 2n!,$$

where φ is an irreducible character of $\tilde{S}_n$.

Specializing to $\ell(\lambda) = 1$ in Theorem 2.9 we have the Schur identity.

COROLLARY 2.11. (Schur identity)

$$\sum_{\substack{\mu \in OP_n \\ \mu = 1^{j_1} 3^{j_3} 5^{j_5} \ldots}} 2^{j_1 + j_3 + j_5 \cdots} \frac{n!}{1^{j_1} 3^{j_3} 5^{j_5} \ldots j_1! j_3! j_5! \ldots} = 2\, n!$$

PROOF. Set $\ell(\lambda) = 1$ in the above theorem. Then the left hand side of Theorem 2.9 is

$$\sum_{\substack{\sigma \in S_n \\ type(\sigma) \in OP_n}} 2^{\ell(type(\sigma))} = \sum_{\substack{\mu \in OP_n \\ \mu = 1^{j_1} 3^{j_3} \ldots}} 2^{j_1 + j_3 + \cdots} \frac{n!}{1^{j_1} 3^{j_3} \ldots j_1! j_3! \ldots},$$

while the right hand side is $2n!$. □

References

[1] A. O. Morris, *The spin representation of the symmetric group*, Proc. London Math Soc. (3) **12** (1962), 55–76.

[2] J. R. Stembridge, *Shifted tableaux and projective representations of symmetric groups*, Advances in Math. **74** (1989), 87–134.

[3] B. E. Sagan, *Shifted tableaux, Schur Q-functions and a conjecture of R. Stanley*, J. Combin. Theory (A) **45** (1987), 62–103.

[4] D. R. Worley, *A theory of Shifted Young Tableaux*, Ph. D. thesis, M.I.T., 1984.

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