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ANOTHER PROOF OF KUMMER'S SECOND THEOREM

ARJUN K. RATHIE AND JUNESANG CHOI

ABSTRACT. We aim at giving another method of proving the well-known and useful Kummer's second theorem without changing its original form.

1. Introduction

From the theory of differential equations, Kummer [3] derived the following very interesting useful result which is known in the literature of hypergeometric series as Kummer's Second Theorem:

If 2α is not an odd integer< 0, then

$$e^x \, {}_0F_1\left(-\,;\alpha+\frac{1}{2}\,\middle|\,\frac{x^2}{4}\right) = {}_1F_1(\alpha\,;\,2\alpha\,|\,2x). \tag{1.1}$$

In 1928, Bailey [1] derived this formula in an equivalent form

$$e^{-x/2}\,{}_1F_1(\alpha\,;\,2\alpha\,|\,x) = {}_0F_1\left(-\,;\alpha+\frac{1}{2}\,\middle|\,\frac{x^2}{16}\right) \tag{1.2}$$

by using the well-known Gauss's second summation theorem (cf., e.g., [4, p. 69]):

$$ {}_2F_1\left(a,\, b\,;\, \frac{1}{2}(a+b+1)\,\middle|\,\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} \tag{1.3}$$

provided $a+b+1 \neq 0, -2, -4, \ldots$; Γ denotes the well-known Gamma function.

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By the way, we also show that Kummer's theorem (1.1) can be derived by using the well-known Gauss's summation theorem (see [3]) which is a more useful and convenient form than (1.3): For $\mathrm{Re}(c-a-b)>0$ and c being neither zero or a negative integer,

$$ {}_2F_1\,(a,\,b;\,c\,|\,1) = \frac{\Gamma\,(c)\,\Gamma\,(c-a-b)}{\Gamma\,(c-a)\,\Gamma\,(c-b)} \tag{1.4} $$

without changing the original Kummer's theorem (1.1).

2. Proof

We first introduce the Pochhammer symbol $(\alpha)_n$ defined by

$$ (\alpha)_n := \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1) & \text{if } n \in \mathbf{N} := \{1,\,2,\,3\,\ldots\}; \\ 1 & \text{if } n=0, \end{cases} \tag{2.1} $$

which is also written in terms of Gamma function

$$ (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}. \tag{2.2} $$

We give some known identities involving the Pochhammer symbol required in this note:

$$ (\lambda)_{n-k} = \frac{(-1)^k(\lambda)_n}{(1-\lambda-n)_k} \quad (0 \le k \le n;\; n \in \mathbf{N}\cup\{0\}), \tag{2.3} $$

$$ (\lambda)_{2n} = 2^{2n}\left(\frac{\lambda}{2}\right)_n \left(\frac{\lambda+1}{2}\right)_n \quad (n \in \mathbf{N}\cup\{0\}). \tag{2.4} $$

The special case $\lambda=1$ of (2.3) yields a useful identity

$$ (n-k)! = \frac{(-1)^k\,n!}{(-n)_k} \quad (0 \le k \le n). \tag{2.5} $$

We also give a known formal series manipulation:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A(k,n-2k), \tag{2.6}$$

where $[x]$ denotes the greatest integer $\leq x$, and $A(k,n)$ is a function of variables k and n.

Now we are ready to prove (1.1). Let

$$e^x\, {}_0F_1\left(-\,;\alpha+\frac{1}{2}\,\middle|\,\frac{x^2}{4}\right) = \sum_{n=0}^{\infty} a_n\, x^n. \tag{2.7}$$

Expressing the left-hand side of (2.7) in term of the product of two power series, and using (2.6), we have

$$\begin{aligned}&\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\left(\sum_{k=0}^{\infty}\frac{x^{2k}}{k!\left(\alpha+\frac{1}{2}\right)_k 2^{2k}}\right)\\ &=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{[n/2]}\frac{1}{(n-2k)!\,k!\,\left(\alpha+\frac{1}{2}\right)_k 2^{2k}}\right)x^n,\end{aligned}$$

with which, considering (2.7), we have

$$a_n = \sum_{k=0}^{[n/2]}\frac{1}{(n-2k)!\,k!\,\left(\alpha+\frac{1}{2}\right)_k 2^{2k}},$$

to which applying (2.5) and (2.4), we obtain

$$a_n = \frac{1}{n!}{}_2F_1\left(-\frac{n}{2},-\frac{n}{2}+\frac{1}{2}\,;\alpha+\frac{1}{2}\,\middle|\,1\right). \tag{2.8}$$

Finally, applying Gauss's theorem (1.4) and Legendre duplication formula for Gamma function to (2.8), we get

$$a_n = \frac{1}{n!}\cdot\frac{(\alpha)_n}{(\alpha)_{2n}}\cdot 2^n. \tag{2.9}$$

Substituting (2.9) in (2.7) arrives immediately at our desired result, that is, the right-hand side of (1.1).

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Arjun K. Rathie
Department of Mathematics
Government Girls College (MDS University)
SRIGANGANAGAR, Rajasthan State
INDIA

Junesang Choi
Department of Mathematics
College of Natural Sciences
Dongguk University
Kyongju 780-714, Korea
E-mail: junesang@email.dongguk.ac.kr