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PSEUDOSPECTRAL METHOD FOR THE DAMPED BOUSSINESQ EQUATION

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Abstract. Numerical approximations by pseudospectral method are obtained for the damped Boussinesq equation which is a modification of the good Boussinesq equation. The consistency and stability of the method are obtained using the extended Lax-Richtmyer equivalence theorem, which imply the convergence of the method. We obtain error estimates of $O(h^s + k^2)$ for a fully discrete pseudospectral method.

1. Introduction

The good Boussinesq equation

$$u_{tt} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}, \quad x \in R, \quad t > 0, \tag{1.1}$$

has been studied extensively in recent years (see [2]-[3], [5]-[8]). Here $u(x,t)$ denotes an elevation of the free surface of shallow water and constants α, β depend on the depth of the fluid as well as the characteristic speed of long waves.

Existence and uniqueness of solution of (1.1) on a periodic domain have been shown by Liu and Russell [3] with some feedback control terms u_t and u for sufficiently small initial data. De Frutos, Ortega and Sanz-Serna [2] have examined the nonlinear stability and convergence using the pseudospectral method for (1.1) with $\alpha = \beta = 1$. Finite element analysis has been studied by Pani and Saranga [8] and finite difference approximate solutions have been obtained by Ortega and Sanz-Serna [7].

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In many real applications, viscosity plays an important role. Counting the viscosity term, the good Boussinesq equation has been modified as the damped Boussinesq equation

$$(1.2) \qquad u_{tt} - 2bu_{xxt} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}, \quad x \in R, \quad t > 0,$$

where constants α and b are positive and β is unrestricted in sign.

Existence and uniqueness of local solution of (1.2) has been discussed based on spectral and perturbation theories by Varlamov [10] for the Cauchy problem. Varlamov [11] has also studied constructive solution of (1.2) on a periodic domain with sufficiently small initial data. Choo and Chung [1] have obtained the order of convergence for finite difference scheme for the damped Boussinesq equation.

In this paper, the solution of pseudospectral scheme for the damped Boussinesq equation and the corresponding error estimates are studied. The results show the improved convergence order compared to those in [1]. In section 2, we introduce some preliminaries. In section 3, the pseudospectral discretization for (1.2) is analyzed and the error estimates for the fully discrete solution are discussed.

2. Preliminaries

We will consider the damped Boussinesq equation

(2.1)

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}, \quad x \in R, \quad 0 < t < T < \infty,$$

with 1-periodic boundary conditions

$$(2.2) \qquad u(x,t) = u(x+1,t), \quad x \in R, \quad 0 < t < T < \infty,$$

and initial conditions

$$(2.3) \qquad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in R.$$

Here u_0 and u_1 are given 1-periodic functions. The constants α and b are positive and β is unrestricted in sign.

Let L^2 be the usual space of square-integrable 1-periodic functions, $\partial_x^2 = \frac{\partial^2}{\partial x^2}$ and $l(v) = \int_0^1 v(x)dx$. Then for $v \in L^2$, Fourier analysis shows that there is a unique function $\partial_x^{-2} v \in L^2$ such that

$$\partial_x^2(\partial_x^{-2}v) = v - l(v), \qquad l(\partial_x^{-2}v) = l(v). \tag{2.4}$$

Let $h = \frac{1}{2J}$ with a positive integer J. Let $x_i = ih$ for $i = 0, \pm1, \pm2, \cdots$, and $\mathbb{Z}_h$ the set of all 1-periodic functions defined on $S_h = \{ih | i = 0, \pm1, \pm2, \cdots\}$. Then each element $V = (V_i) \in \mathbb{Z}_h$ is satisfies $V_i = V_{i+2J}$ for $i = 0, \pm1, \pm2, \cdots$. We define the discrete L^2-norm $\| \cdot \|$ as

$$\|V\| = (h \sum_{j=0}^{2J}{}'' V_j^2)^{\frac{1}{2}},$$

where the double prime in the summation means that the first and last term are halved. The inner product $(\cdot, \cdot)$ is associated with this norm. The standard maximum norm in $\mathbb{Z}_h$ is denoted by $\| \cdot \|_\infty$. The notation $[V]_p^\wedge$ refers to the pth discrete Fourier coefficient of V, i.e.,

$$[V]_p^\wedge = \frac{1}{2J} \sum_{j=0}^{2J}{}'' V_j exp(-2\pi ipjh), \quad -J \le p \le J.$$

Note that $[V]_0^\wedge$ is the mean value of V and $[V]_J^\wedge = [V]_{-J}^\wedge$. The recovery of V from its Fourier coefficients is achieved by the inverse discrete Fourier transformation

$$V_j = V^*(x_j), \qquad j = 0, \pm1, \pm2, \cdots,$$

where $V^*(x)$ is the trigonometric interpolant of V given by

$$V^*(x) = \sum_{p=-J}^{J}{}'' [V]_p^\wedge exp(2\pi ipx), \quad x \in R,$$

so that

$$V_j = \sum_{p=-J}^{J}{}'' [V]_p^\wedge exp(2\pi ipjh).$$

Differentiating this interpolant and evaluating the results at grid points, we obtain the following definition of pseudospectral difference operators D and D^2 mapping $\mathbb{Z}_h$ into itself:

$$(2.5)\qquad \begin{aligned}(DV)_j &= \sum_{p=-J}^{J}{}'' [V]\hat{}_p (2\pi ip) exp(2\pi ipjh),\\ (D^2V)_j &= \sum_{p=-J}^{J}{}'' [V]\hat{}_p (2\pi ip)^2 exp(2\pi ipjh).\end{aligned}$$

The expression V^2 of the elements in $\mathbb{Z}_h$ is to be interpreted componentwise. The operators D^2 and $D^4 = D^2D^2$ are clearly self-adjoint with respect to $(\cdot,\cdot)$. We now introduce the grid function I which takes, at each grid point, the values

$$I_j = 1, \quad j = 0, \pm 1, \pm 2, \cdots .$$

Then the operator D^{-2} mapping $\mathbb{Z}_h$ into itself is defined by

$$(2.6)\qquad D^2(D^{-2}V) = V - [V]\hat{}_0 I, \qquad [D^{-2}V]\hat{}_0 = [V]\hat{}_0.$$

Note that D^{-2} and D^2 commute so that $D^2(D^{-2}V) = V - [V]\hat{}_0 I$. Let $k = T/N$ for a positive integer N and $t_n = nk, n = 0, 1, \cdots, N$. Define a discrete energy norm by

$$(2.7)\qquad \|(V,W)\|_E = \{\|D^{-2}(V-W)/k\|^2 + \frac{\alpha}{2}\|V\|^2 + \frac{\alpha}{2}\|W\|^2\}^{\frac{1}{2}}.$$

Then we obtain

LEMMA 2.1. *For $V, W \in \mathbb{Z}_h$, we have*

(1) $\|[V]\hat{}_0 I\| \le \|V\| \le (\frac{2}{\alpha})^{\frac{1}{2}} \|(V,W)\|_E$.

(2) $\|V\|_\infty \le h^{-\frac{1}{2}} \|V\|$.

PROOF. Using the fact that

$$\begin{aligned}\|[V]_0^{\widehat{}} I\| &= |[V]_0^{\widehat{}}| = |h\sum_{j=0}^{2J}{}'' V_j| \\ &= |\sum_{j=0}^{2J}{}'' h^{\frac{1}{2}} h^{\frac{1}{2}} V_j| \le |\sum_{j=0}^{2J}{}'' h|^{\frac{1}{2}} \; |h\sum_{j=0}^{2J}{}'' V_j^2|^{\frac{1}{2}} = \|V\|,\end{aligned}$$

we obtain the desired results. □

3. Convergence of the pseudospectral scheme

The pseudospectral scheme for (2.1)–(2.3) is defined as

$$\begin{aligned}&\frac{U^{n+1} - 2U^n + U^{n-1}}{k^2} - 2b\frac{1}{2k}(D^2U^{n+1} - D^2U^{n-1}) \\ &= -\frac{\alpha}{2}(D^4U^{n+1} + D^4U^{n-1}) + \frac{1}{2}(D^2U^{n+1} + D^2U^{n-1}) \\ &\quad + \frac{\beta}{2}\{D^2[(U^{n+1})^2] + D^2[(U^{n-1})^2]\}, \qquad n \ge 2,\end{aligned} \tag{3.1}$$

with periodic boundary conditions

$$U^n \in \mathbb{Z}_h, \quad n \ge 0, \tag{3.2}$$

and initial conditions

$$\begin{aligned}&U^0 = u_0^h, \\ &U^1 = u_0^h + ku_1^h + \frac{k^2}{2}\{2bu_{1,xx}^h - \alpha u_{0,xxxx}^h + u_{0,xx}^h + \beta[(u_0^h)^2]_{xx}\},\end{aligned} \tag{3.3}$$

where u_0^h and u_1^h have the values of u_0 and u_1 on S_h, respectively.

In order to investigate the existence, stability and convergence of (3.1)–(3.3), we recall extension of Lax-Richtmyer equivalence theorem, which avoids the establishment of a priori bounds for convergence in Lopez-Marcos and Sanz-Serna [4];

Let u be a solution of (2.1)–(2.3) and u_h is a discrete evaluation of u on S_h. Let U_h be an approximate solution of u, which is obtained by solving a discrete equation

$$\Phi(U_h) = 0, \tag{3.4}$$

where $\Phi : X_h \to Y_h$ is a continuous mapping and X_h, Y_h are normed spaces having the same dimension. The local discretization error is given by $L_h = \Phi(u_h)$. The finite difference scheme (3.4) is said to be convergent if (3.4) has a solution U_h such that $\lim_{h\to 0} \|U_h - u_h\| = 0$. The discretization (3.4) is said to be consistent if $\lim_{h\to 0} \|\Phi(u_h)\| = 0$. The scheme (3.4) is said to be stable in R_h if there exists a positive constant C such that for an open ball $B(u_h, R_h) \subset X_h$

$$\|V_h - W_h\| \leq C\|\Phi(V_h) - \Phi(W_h)\|, \quad \forall\, V_h, W_h \in B(u_h, R_h). \tag{3.5}$$

Then stability and consistency imply convergence.

The following theorem is an extended Lax-Richtmyer equivalence theorem, which gives existence and convergence of approximate solutions. For a proof, we refer [4].

THEOREM 3.1. *Assume that* (3.4) *is consistent and stable in* R_h. *If* Φ *is continuous in* $B(u_h, R_h)$ *and* $\|L_h\| = o(R_h)$ *as* $h \to 0$, *then*

(1) *For sufficiently small* h, (3.4) *has a unique solution in* $B(u_h, R_h)$.
(2) *The solution in* (1) *converges with an order of convergence not smaller than the order of consistency.*

According to Theorem 3.1, we have only to show that (3.1)–(3.3) is consistent and stable. We take $X_h = Y_h = \mathbb{Z}_h^{N+1}$ with norms $\|\cdot\|_{X_h}$ and $\|\cdot\|_{Y_h}$ such that

$$\|W\|_{X_h} = \max\{\|(W^{n+1}, W^n)\|_E : 0 \leq n \leq N-1\},$$
$$\|G\|_{Y_h} = \|(G^1, G^0)\|_E + \sum_{n=2}^{N} k\|D^{-2}G^n\|,$$

for $W = [W^0, W^1, \cdots, W^N] \in X_h$ and $G = [G^0, G^1, \cdots, G^N] \in Y_h$.

Define a mapping $\Phi_h : X_h \to Y_h$ by $\Phi_h(W) = G$, where

(3.6)
$$G^{n+1} = \frac{(W^{n+1} - 2W^n + W^{n-1})}{k^2} - 2b\frac{1}{2k}\{D^2(W^{n+1} - W^{n-1})\} + \frac{\alpha}{2}(D^4W^{n+1} + D^4W^{n-1}) - \frac{1}{2}(D^2W^{n+1} + D^2W^{n-1}) - \frac{\beta}{2}\{D^2[(W^{n+1})^2] + D^2[(W^{n-1})^2]\}, \qquad 1 \le n \le N-1,$$

(3.7)
$$G^1 = W^1 - u_0^h - ku_1^h - \frac{k^2}{2}\{2bu_{1,xx}^h - \alpha u_{0,xxxx}^h + u_{0,xx}^h + \beta[(u_0^h)^2]_{xx}\},$$

(3.8)
$$G^0 = W^0 - u_0^h.$$

Then we obtain the consistency of (3.6)–(3.8).

THEOREM 3.2. *Let u be the solution of (2.1)–(2.3). If $u \in H^{s+2}$ with $s > \frac{1}{2}$, $u_{tttt}, u_{xxtt}, u_{tt}^2$ are bounded and u^2, u_t, u_{tt} are in H^s uniformly in t, then*

$$\|\Phi(u_h)\|_{Y_h} \le C(k^2 + h^s),$$

where C is a constant depending only on u and T.

PROOF. Let $u^n = u_h(\cdot, t_n)$ and $\Phi(u_h) = E$, where

$$E^{n+1} = \frac{(u^{n+1} - 2u^n + u^{n-1})}{k^2} - 2b\frac{1}{2k}\{D^2(u^{n+1} - u^{n-1})\} + \frac{\alpha}{2}(D^4u^{n+1} + D^4u^{n-1}) - \frac{1}{2}D^2(u^{n+1} + u^{n-1}) - \beta\frac{1}{2}D^2\{(u^{n+1})^2 + (u^{n-1})^2\}, \quad 1 \le n \le N-1,$$

$$E^1 = u^1 - u_0^h - ku_1^h - \frac{k^2}{2}\{2bu_{1,xx}^h - \alpha u_{0,xxxx}^h + u_{0,xx}^h + \beta[(u_0^h)^2]_{xx}\},$$

$$E^0 = u^0 - u_0^h.$$

It follows from the above relation that

$$\begin{aligned} D^{-2}E^{n+1} &= \frac{D^{-2}(u^{n+1}-2u^n+u^{n-1})}{k^2} - 2b\Big(\frac{u^{n+1}-u^{n-1}}{2k}\Big) \\ &+ 2b\Big[\frac{u^{n+1}-u^{n-1}}{2k}\Big]_0^{\widehat{}} I + \frac{\alpha}{2}(D^2u^{n+1}+D^2u^{n-1}) \\ &- \frac{1}{2}(u^{n+1}+u^{n-1}) + \frac{1}{2}[u^{n+1}+u^{n-1}]_0^{\widehat{}} I \\ &- \beta\frac{1}{2}\{(u^{n+1})^2+(u^{n-1})^2\} + \beta\frac{1}{2}[(u^{n+1})^2+(u^{n-1})^2]_0^{\widehat{}} I. \end{aligned}$$

On the other hand, since we obtain from (2.1) that

$$0 = -\partial_x^{-2}u_{tt} + 2bu_t - 2bl(u_t) - \alpha\partial_x^2 u + u - l(u) + \beta u^2 - \beta l(u^2),$$

$$\begin{aligned} &D^{-2}E^{n+1} \\ &= D^{-2}\{k^{-2}(u^{n+1}-2u^n+u^{n-1}) - u_{tt}^n\} + (D^{-2}u_{tt}^n - \partial_x^{-2}u_{tt}^n) \\ &- 2b\Big(\frac{u^{n+1}-u^{n-1}}{2k} - u_t^n\Big) + 2b\Big\{\Big[\frac{u^{n+1}-u^{n-1}}{2k}\Big]_0^{\widehat{}} - l(u_t^n)\Big\} I \\ &+ \frac{\alpha}{2}\{(D^2u^{n+1} - \partial_x^2 u^{n+1}) + (D^2u^{n-1} - \partial_x^2 u^{n-1})\} \\ &+ \frac{\alpha}{2}\partial_x^2(u^{n+1}+u^{n-1}-2u^n) - \Big(\frac{u^{n+1}+u^{n-1}}{2} - u^n\Big) \\ &+ \Big\{\Big[\frac{u^{n+1}+u^{n-1}}{2}\Big]_0^{\widehat{}} - l(u^n)\Big\} I - \beta\Big\{\frac{(u^{n+1})^2+(u^{n-1})^2}{2} - (u^n)^2\Big\} \\ &+ \beta\Big\{\Big[\frac{(u^{n+1})^2+(u^{n-1})^2}{2}\Big]_0^{\widehat{}} - l((u^n)^2)\Big\} I. \end{aligned}$$

It is known that if $u_{ttttt}, u_{xxtt}, u_{tt}^2$ are bounded in $0 \le x \le 1$, $0 < t < T$, then the first, third, sixth, seventh, and ninth terms on the right hand side of the above equation are $O(k^2)$. Further if $u \in H^{s+2}$ with $s > \frac{1}{2}$ and u^2, u_t, u_{tt} are in H^s uniformly in t, then the remaining terms are $O(h^s)$ (see [9]). Therefore

$$\max_{1\le n\le N-1} \|D^{-2}E^{n+1}\| = O(k^2 + h^s).$$

From the definition of the norm $\|\cdot\|_{Y_h}$,

$$\|\Phi_h(u_h)\|_{Y_h} \leq \|(E^1, E^0)\|_E + C(k^2 + h^s).$$

Since $E_j^0 = (u^0 - u_0^h)_j = 0$ and $E_j^1 = O(k^3)$, we obtain

$$\begin{aligned}\|(E^1, E^0)\|_E &= \{\|\frac{D^{-2}(E^1 - E^0)}{k}\|^2 + \frac{\alpha}{2}\|D^2E^1\|^2 + \frac{\alpha}{2}\|D^2E^0\|^2\}^{\frac{1}{2}} \\ &= \{\|\frac{D^{-2}E^1}{k}\|^2 + \frac{\alpha}{2}\|D^2E^1\|^2\}^{\frac{1}{2}} \\ &\leq Ck^2.\end{aligned}$$

This implies the desired error estimate $\|\Phi_h(u_h)\|_{Y_h} \leq C(k^2 + h^s)$. □

We now investigate the stability of (3.1)–(3.3). Let $(V^0, \cdots, V^N)$ and $(W^0, \cdots, W^N)$ be two sequences of elements in $\mathbb{Z}_h$. Let $\Phi(V) = F$ and $\Phi(W) = G$. Then $\{V^n\}$ and $\{W^n\}$ can be viewed as perturbed solutions of (3.1)–(3.3) with the corresponding perturbation $\{F^n\}$ and $\{G^n\}$, respectively, where

(3.9)
$$\begin{aligned}F^{n+1} = &\frac{(V^{n+1} - 2V^n + V^{n-1})}{k^2} - 2b\frac{1}{2k}D^2(V^{n+1} - V^{n-1}) \\ &+ \frac{\alpha}{2}(D^4V^{n+1} + D^4V^{n-1}) - D^2(\frac{V^{n+1} + V^{n-1}}{2}) \\ &- \frac{\beta}{2}\{D^2[(V^{n+1})^2] + D^2[(V^{n-1})^2]\}, \qquad 1 \leq n \leq N-1,\end{aligned}$$

(3.10) $$F^1 = V^1 - u_0^h - ku_1^h + \frac{k^2}{2}\{2bu_{1,xx}^h - \alpha u_{0,xxxx}^h + u_{0,xx}^h + \beta[(u_0^h)^2]_{xx}\},$$

(3.11) $$F^0 = V^0 - u_0^h.$$

The stability analysis consists of estimating the size of $V^n - W^n$ in terms of $F^n - G^n$.

THEOREM 3.3. *Let $\{V^n\}, \{W^n\}, \{F^n\}, \{G^n\}$ are defined as above. If*

$$\max_{0\le n\le N-1} \|(V^{n+1}-u^{n+1}, V^n - u^n)\|_E \le h^{\frac{1}{2}}$$

and

$$\max_{0\le n\le N-1} \|(W^{n+1}-u^{n+1}, W^n - u^n)\|_E \le h^{\frac{1}{2}},$$

then for some constant S

$$\max_{0\le n\le N-1} \|(V^{n+1}-W^{n+1}, V^n - W^n)\|_E$$
$$\le exp(ST)\left\{ \|(F^1-G^1, F^0-G^0)\|_E + k\sum_{2\le n\le N} \|D^{-2}(F^n-G^n)\| \right\}.$$

PROOF. Let $e^n = V^n - W^n$, $\bar{e}^n = \frac{e^{n+1}+e^{n-1}}{2}$, $K^n = F^n - G^n$. Then we obtain, for $n = 1, 2, \cdots, N-1$,

(3.12)
$$D^{-2}\frac{e^{n+1}-2e^n+e^{n-1}}{k^2} - 2bD^{-2}D^2\frac{e^{n+1}-e^{n-1}}{2k} + \frac{\alpha}{2}D^2(e^{n+1}+e^{n-1})$$
$$= \bar{e}^n - [\bar{e}^n]_0^\frown I + \beta\{(\bar{V}^n)^2 - (\bar{W}^n)^2\} - \beta[(\bar{V}^n)^2 - (\bar{W}^n)^2]_0^\frown I - D^{-2}K^{n+1}.$$

Taking an inner product of (3.12) with $D^{-2}(e^{n+1}-e^n) + D^{-2}(e^n - e^{n-1})$, we obtain

$$\begin{aligned} &k^{-2}(D^{-2}(e^{n+1}-e^n) - D^{-2}(e^n-e^{n-1}),\\ &\quad D^{-2}(e^{n+1}-e^n) + D^{-2}(e^n-e^{n-1}))\\ &\quad - k^{-1}b(D^{-2}D^2(e^{n+1}-e^{n-1}), D^{-2}(e^{n+1}-e^{n-1}))\\ &\quad + \frac{\alpha}{2}(D^2e^{n+1} + D^2e^{n-1}, D^{-2}(e^{n+1}-e^{n-1}))\\ &= (\bar{e}^n - [\bar{e}^n]_0^\frown I + \beta\{(\bar{V}^n)^2 - (\bar{W}^n)^2\} - \beta[(\bar{V}^n)^2 - (\bar{W}^n)^2]_0^\frown I\\ &\quad - D^{-2}K^{n+1}, D^{-2}(e^{n+1}-e^n) + D^{-2}(e^n-e^{n-1})). \end{aligned} \tag{3.13}$$

Since

$$(D^2e^{n+1}+D^2e^{n-1},D^{-2}(e^{n+1}-e^{n-1}))$$
$$=(\|e^{n+1}\|^2+\|e^n\|^2)-(\|e^n\|^2+\|e^{n-1}\|^2)$$
$$-(e^{n+1}+e^{n-1},[e^{n+1}-e^{n-1}]_0^{\wedge}I),$$

it follows from (3.13) that

(3.14)

$$\begin{aligned}&(k^{-2}\|D^{-2}(e^{n+1}-e^n)\|^2+\frac{\alpha}{2}\|e^{n+1}\|^2+\frac{\alpha}{2}\|e^n\|^2)\\&\quad-(k^{-2}\|D^{-2}(e^n-e^{n-1})\|^2+\frac{\alpha}{2}\|e^n\|^2+\frac{\alpha}{2}\|e^{n-1}\|^2)\\&\quad+k^{-1}b(D^2D^{-2}(e^{n+1}-e^{n-1}),D^2D^{-2}(e^{n+1}-e^{n-1}))\\&=\frac{\alpha}{2}(e^{n+1}+e^{n-1},[e^{n+1}-e^{n-1}]_0^{\wedge}I)\\&\quad+(\bar{e}^n-[\bar{e}^n]_0^{\wedge}I+\beta\{(\bar{V}^n)^2-(\bar{W}^n)^2\}-\beta[(\bar{V}^n)^2-(\bar{W}^n)^2]_0^{\wedge}I\\&\quad-D^{-2}K^{n+1},D^{-2}(e^{n+1}-e^n)+D^{-2}(e^n-e^{n-1})).\end{aligned}$$

From (2.6), Lemma 2.1 and the definition of $\|(\cdot,\cdot)\|_E$, it follows

$$\|[e^{n+1}-e^{n-1}]_0^{\wedge}I\|\le\{\|(e^{n+1},e^n)\|_E+\|(e^n,e^{n-1})\|_E\}$$

and

$$\|(D^{-2}(e^{n+1}-e^n)+D^{-2}(e^n-e^{n-1})\|\le k\{\|(e^{n+1},e^n)\|_E+\|(e^n,e^{n-1})\|_E\}.$$

The equation (3.14) gives

$$\begin{aligned}&\|(e^{n+1},e^n)\|_E-\|(e^n,e^{n-1})\|_E\\&\qquad\le\frac{\alpha}{2}k\|e^{n+1}+e^{n-1}\|+k\|\bar{e}^n-[\bar{e}^n]_0^{\wedge}I+\beta\{(\bar{V}^n)^2-(\bar{W}^n)^2\}\\&\qquad-\beta[(\bar{V}^n)^2-(\bar{W}^n)^2]_0^{\wedge}I-D^{-2}K^{n+1}\|.\end{aligned}$$

Lemma 2.1 implies that

$$\|e^{n+1}+e^{n-1}\|\le\|(e^{n+1},e^n)\|_E+\|(e^n,e^{n-1})\|_E,\tag{3.15}$$

$$\|\bar{e}^n-[\bar{e}^n]_0^{\wedge}I\|\le\|(e^{n+1},e^n)\|_E+\|(e^n,e^{n-1})\|_E,\tag{3.16}$$

$$
(3.17)\qquad
\begin{aligned}
\|(\bar{V}^n)^2-(\bar{W}^n)^2\| &\le \|\bar{V}^n+\bar{W}^n\|_\infty\|\bar{e}^n\|\\
&\le\{\|\bar{V}^n-\bar{u}^n\|_\infty+\|\bar{W}^n-\bar{u}^n\|_\infty+2\|\bar{u}^n\|_\infty\}\|\bar{e}^n\|\\
&\le\{h^{-\frac{1}{2}}\|\bar{V}^n-\bar{u}^n\|+h^{-\frac{1}{2}}\|\bar{W}^n-\bar{u}^n\|+2\|\bar{u}^n\|_\infty\}\|\bar{e}^n\|\\
&\le C\|\bar{e}^n\|\le C\{\|(e^{n+1},e^n)\|_E+\|(e^n,e^{n-1})\|_E\},
\end{aligned}
$$

$$
(3.18)\qquad
\begin{aligned}
\|[(\bar{V}^n)^2-(\bar{W}^n)^2]_0^\frown I\| &\le \|(\bar{V}^n)^2-(\bar{W}^n)^2\|\\
&\le C\{\|(e^{n+1},e^n)\|_E+\|(e^n,e^{n-1})\|_E\},
\end{aligned}
$$

where C is independent of n. Using these estimates in (3.15)–(3.18), we obtain for $n = 1, 2, \cdots, N-1$,

$$(1-Ck)\|(e^{n+1},e^n)\|_E \le (1+Ck)\|(e^n,e^{n-1})\|_E + Ck\|D^{-2}K^{n+1}\|.$$

Applying the discrete Gronwall inequality, we obtain the desired results. □

It follows from Theorems 3.1–3.3, that we obtain the following error estimate for (3.1).

THEOREM 3.4. *Suppose that the hypotheses of* Theorem 3.2–3.3 *hold. Then*

$$\max_{0\le n\le N-1}\|(u^{n+1}-U^{n+1},u^n-U^n)\|_E \le C(k^2+h^s).$$

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References

[1] S. M. Choo and S. K. Chung, *Convergence of approximate solutions for the damped Boussinesq equation*, submitted.
[2] J. De Frutos, T. Ortega, and J. M. Sanz-Serna, *Pseudospectral method for the "good" Boussinesq equation*, Math. Comp. **57** (1991), 109–122.
[3] F. L. Liu and D. L. Russell, *Solution of the Boussinesq equation on a periodic domain*, J. Math. Anal. Appl. **192** (1995), 194–219.

[4] J. C. Lopez-Marcos and J. M. Sanz-Serna, *Stability and convergence in numerical analysis III: linear investigation of keynlinear stability*, IMA J. N. A. **8** (1988), 71–84.

[5] V. S. Manoranjan, A. R. Mitchell, and J. Ll. Morris, *Numerical solutions of the good Boussinesq equation*, SIAM Sci. Stat. Comp. **5** (1984), 946–957.

[6] V. S. Manoranjan, T. Ortega, and J. M. Sanz-Serna, *Soliton and antisoliton interactions in the "good" Boussinesq equation*, J. Math. Physics **29** (1988), 1964–1968.

[7] T. Ortega and J. M. Sanz-Serna, *Nonlinear stability and convergence of finite difference methods for the "good" Boussinesq equation*, Numer. Math. **58** (1990), 215–229.

[8] A. K. Pani and H. Saranga, *Finite element Galerkin method for the "good" Bousssinesq equation*, to appear in Nonlinear Anal. T. M. A..

[9] E. Tadmor, *The exponential accuracy of Fourier and Chebyshev differencing method*, SIAM J. N. Anal. **23** (1986), 1–10.

[10] V. Varlamov, *Existence and uniqueness of a solution to the Cauchy problem for the damped Boussinesq equation*, Math. Meth. Appl. Sci. **19** (1996), 639–649.

[11] ———, *On spatially periodic solutions of the damped Boussinesq equation*, to appear in Diff. and Int. Equations.

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