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TRIANGULATIONS OF SEIFERT FIBERED 3-MANIFOLDS

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ABSTRACT. For an oriented compact, connected Seifert fibred 3-manifold M with nonempty boundary, we construct a simplicial complex using the equivalence classes of marked annulus systems and show that it is contractible.

1. Triangulations of Seifert fibered 3-manifolds

Let S be a compact connected surface, possibly with boundary, and let $V = \{v_1, \dots, v_n\}$ be an arbitrary finite subset of S. By an *essential arc* in (S, V) we mean an embedded arc $\alpha \subset S$ meeting $\partial S \cup V$ only in the endpoints of α, which lie in V and are permitted to coincide, with the condition that if $S \setminus \alpha$ has two components, neither component of the surface obtained by cutting S along α is a disk meeting V only in the endpoints of α.

A collection $\{\alpha_0, \dots, \alpha_k\}$ of essential arcs in S which are disjoint except perhaps for the endpoints of the arcs, and such that no two α_i's are ambient isotopic fixing V, we call a *curve system*. The ambient isotopic classes (rel V) of curve systems form the k-simplex $[\alpha_0, \dots, \alpha_k]$ being obtained by passing to subcollections of $\{\alpha_0, \dots, \alpha_k\}$.

A 3-manifold M is a Seifert fibered space if M is the union of a collection $\{C_\alpha\}$ of pairwise disjoint simple closed curves (called fibers) such that for each α, there is a closed neighborhood U of C_α which is a solid torus and a covering map $q : B^2 \times S^1 \longrightarrow U$ satisfying (i) q maps each $x \times S^1 (x \in B^2)$

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to some C_β (hence U is a union of fibers), (ii) $q^{-1}(C_\alpha)$ is connected, and (iii) the group of covering translations is generated by $\tau_{n,m}$ of relatively prime integer where

$$\tau_{n,m}(re^{i\theta}, e^{i\phi}) = (re^{i(\theta+2\frac{m}{n}\pi)}, e^{i(\phi+\frac{2\pi}{n})}).$$

If $|n| = 1$, then q is an embedding and we say that C_α is a regular fiber. If $|n| > 1$, then by (ii), $C_\alpha = q(0 \times S^1)$, and for $x \neq 0$, q maps $x \times S^1$ homeomorphically to a fiber which crosses the meridional disk $q(B^2 \times 1)$ n times and wraps m times meridionally about C_α. That is, there are standard generators μ, λ for $\pi_1(\partial U)$ with μ nullhomotopic in U and λ generating $\pi_1(U)$ such that for $\beta \neq \alpha$, C_β represents the element $\mu^m\lambda^n$ of $\pi_1(U - C_\alpha) \cong \pi_1(\partial U)$. In this case we say C_α is a singular fiber of type (n, m). We will always assume that $n > 0$, and since $\pm m$ may be changed modulo n, that $0 \leq m \leq n/2$. Since every other fiber in some neighborhood of a singular fiber is regular, it follows that if M is compact, there can be but finitely many singular fibers. Furthermore, by invariance of domain, ∂M is a union of regular fibers.

Let M be an orientable compact, connected Seifert fibered 3-manifold with nonempty boundary ∂M. Let B the orbit space obtained from M by identifying each fiber C_α to a point and let $p : (M, \partial M) \longrightarrow (B, \partial B)$ be the identification map. Then B is a compact surface, perhaps non-orientable, and ∂B is a union of boundary circles of B. The images of the singular fibers form a finite subset $F \subset B \setminus \partial B$. Let $B' = B \setminus F$. Let $T \subset B'$ be a finite set consisting of one point in each component of ∂B. Consider a pair $(p^{-1}(\alpha), \tilde{\alpha})$, where α is an essential arc in B' with endpoints in T, $\tilde{\alpha}$ is an arc in $p^{-1}(\alpha)$ with $p(\tilde{\alpha}) = \alpha$ and the endpoints of $\tilde{\alpha}$ are those of α's. We call the pair $(p^{-1}(\alpha), \tilde{\alpha})$ a *marked annulus* in (M', T), where $M' = M - p^{-1}(F)$. We define $(p^{-1}(\alpha), \tilde{\alpha})$ is equivalent to $(p^{-1}(\beta), \tilde{\beta})$ if α is isotopic to β (rel T) and $\tilde{\alpha}$ is isotopic to $\tilde{\beta}$ (rel T). A collection $\{(p^{-1}(\alpha_0), \tilde{\alpha_0}), \dots, (p^{-1}(\alpha_k), \tilde{\alpha_k})\}$ of marked annuli is called a *marked annulus system* if either $\{\alpha_0, \dots, \alpha_k\}$ forms a curve system in (B', T) or a subcollection $\{\tilde{\alpha_{i_j}}\}$ of $\{\tilde{\alpha_i}\}$ forms a curve system in $(p^{-1}(\alpha_k), \alpha_k \cap T)$ for some k when each element of the subcollection $\{\alpha_{i_j}\}$ of $\{\alpha_i\}$ is isotopic to α_k for some k. We may consider the curves in $\{\tilde{\alpha_{i_j}}\}$ are in the same $p^{-1}(\alpha_k)$ by ambient isotopies. The equivalence classes of marked annulus systems $\{(p^{-1}(\alpha_0), \tilde{\alpha_0}), \dots, (p^{-1}(\alpha_k), \tilde{\alpha_k})\}$ form the k-simplices $[(p^{-1}(\alpha_0), \tilde{\alpha_0}), \dots, (p^{-1}(\alpha_k), \tilde{\alpha_k})]$ of a simplicial complex

$\mathcal{A} = \mathcal{A}(M', T)$ and the faces of $[(p^{-1}(\alpha_0), \tilde{\alpha_0}), \dots, (p^{-1}(\alpha_k), \tilde{\alpha_k})]$ can be obtained by passing to subcollections.

THEOREM 1.1. *The complex $\mathcal{A}(M', T)$ is contractible.*

PROOF OF 1.1. Let $(p^{-1}(\beta), \tilde{\beta})$ be a marked annulus in (M', T). We will construct a deformation retraction (in fact, a flow which is linear on simplices) of $\mathcal{A}$ onto the star of the vertex $[(p^{-1}(\beta), \tilde{\beta})]$. This star consists of the simplices $[(p^{-1}(\alpha_0), \tilde{\alpha_0}), \dots, (p^{-1}(\alpha_k), \tilde{\alpha_k})]$ in $\mathcal{A}$ coming from either curve systems $\alpha_0 \cup \cdots \cup \alpha_k$ disjoint from β or curve system $\tilde{\alpha_{i_1}} \cup \cdots \cup \tilde{\alpha_{i_l}}$ disjoint from $\tilde{\beta}$ (one $\tilde{\alpha_{i_j}}$ can be isotopic to $\tilde{\beta}$) for a subcollection $\{\alpha_{i_j} \mid \alpha_{i_j} \sim \beta\,(\mathrm{rel}\,T)\}$ of $\{\alpha_i\}$. Since stars are always contractible, the theorem will be proved.

Let $[(p^{-1}(\alpha_0), \tilde{\alpha_0}), \dots, (p^{-1}(\alpha_k), \tilde{\alpha_k})]$ be a simplex of $\mathcal{A}$. We may apply the same method as in pp. 192 of [1].

Case I: $\{\alpha_0, \dots, \alpha_k\}$ forms a curve system in (B', T).

We may assume $\alpha_0 \cup \cdots \cup \alpha_k$ has been isotoped to intersect β minimally. We shall use the well-known fact that any two such minimal positions for $\alpha_0 \cup \cdots \cup \alpha_k$ are isotopic through a family of minimum position curve system. So we may assume $p^{-1}(\alpha_0) \cup \cdots \cup p^{-1}(\alpha_k)$ has been isotopic to intersect $p^{-1}(\beta)$ minimally.

Let $P = \sum t_i(p^{-1}(\alpha_i), \tilde{\alpha}_i) = \sum(t_i p^{-1}(\alpha_i), t_i\tilde{\alpha}_i)$ be a point in the simplex $[(p^{-1}(\alpha_0), \tilde{\alpha_0}), \dots, (p^{-1}(\alpha_k), \tilde{\alpha_k})]$, expressed in terms of barycentric coordinates t_i with $\sum t_i = 1$. We may think of P as a weighted sum of the pairs of the annuli $p^{-1}(\alpha_i)$ and curves $\tilde{\alpha}_i$, and we can interpret this weighted sum geometrically by replacing each $t_i(p^{-1}(\alpha_i), \tilde{\alpha}_i)$ with a pair of families of nearby parallel annuli and nearby parallel curves of each total thickness t_i with the end curves of annuli (the fibers of endpoints of the base curves) and the endpoints of arcs within a family pinched together at $p^{-1}(T)$ and T, respectively. For convenience, pinch all the families of parallel annuli in P into a single family, where they cross $p^{-1}(\beta)$, with total thickness θ. Now for $t \in [0, 1]$ let P_t be obtained from P by first cutting part of the way through the thickness θ bundle of annuli crossing $p^{-1}(\beta)$, starting from a chosen end of $p^{-1}(\beta)$ (the fiber of a chosen end of β) and cutting in to a thickness $t\theta$, then redirecting the two ends of the resulting cut part to the given end of $p^{-1}(\beta)$.

In terms of the marked annulus system $c_1 = (p^{-1}(\alpha_0), \tilde{\alpha_0}) \cup \cdots \cup (p^{-1}(\alpha_k), \tilde{\alpha_k})$ this procedure can be described as follows. Let $s_1, \dots, s_m$ be the fibers of intersection of β with $\alpha_0 \cup \cdots \cup \alpha_k$, ordered as they are ordered along β starting at the chosen end of β. Sliding the intersection curve s_1 (simple closed curve) to the end of $p^{-1}(\beta)$ and sliding the intersection point of $\tilde{\alpha_1} \cap s_1$ to the end of β through the isotopic move converts c_1 to a new marked annulus system c_2 (discard parallel and ∂-parallel components) meeting $p^{-1}(\beta)$ in $s_2, \ldots, s_m$. The union $c_1 \cup c_2$ is a marked annulus system determining a simplex σ_1 containing $[(p^{-1}(\alpha_0), \tilde{\alpha_0}), \ldots, (p^{-1}(\alpha_k), \tilde{\alpha_k})]$ as a face. Initial segments of the paths P_t move linearly along fibers of a linear projection of σ_1 onto the face corresponding to c_2, stopping when they reach this face. Now repeat the process, sliding s_2 to the end of $p^{-1}(\beta)$ to convert c_2 to c_3, etc., until all the s_i's have been eliminated. We see from this the paths P_t starting at points of $[(p^{-1}(\alpha_0), \tilde{\alpha_0}), \ldots, (p^{-1}(\alpha_k), \tilde{\alpha_k})]$ flow linearly across a finite sequence of simplices $\sigma_1, \ldots, \sigma_m$. So the flow is continuous on $[(p^{-1}(\alpha_0), \tilde{\alpha_0}), \ldots, (p^{-1}(\alpha_k), \tilde{\alpha_k})]$. Furthermore, it is clear that this flow restricted to a face of $[(p^{-1}(\alpha_0), \tilde{\alpha_0}), \ldots, (p^{-1}(\alpha_k), \tilde{\alpha_k})]$ is the flow associated to that face. σ_m is the simplex containing $[(p^{-1}(\beta), \tilde{\beta})]$ as a vertex. So we construct a flow which is linear on the simplex $[(p^{-1}(\alpha_0), \tilde{\alpha_0}), \ldots, (p^{-1}(\alpha_k), \tilde{\alpha_k})]$ onto the star of $[(p^{-1}(\beta), \tilde{\beta})]$

Case II: Some α_i's are isotopic to β.

We may assume that $\{\alpha_1, \ldots, \alpha_m\}_{m \leq k}$ are isotopic to β by reindexing if it is necessary. From the definition of marked annulus system, $\{\tilde{\alpha_1}, \ldots, \tilde{\alpha_m}\}$ forms a curve system in $(p^{-1}(\beta), \beta \cap T)$. We may assume that $\tilde{\alpha_0} \cup \cdots \cup \tilde{\alpha_k}$ has been isotoped to intersect $\tilde{\beta}$ minimally. Let $c = \tilde{\alpha_1} \cup \cdots \cup \tilde{\alpha_k}$ and let $x_1, \cdots, x_n$ be the points of intersection of $\tilde{\beta}$ with c, ordered as they are ordered along $\tilde{\beta}$ starting at the chosen end point of $\tilde{\beta}$. Sliding the intersection point x_1 to the chosen endpoint of $\tilde{\beta}$. Now repeat the process until all the x_i's have been eliminated. (For more details, see pp. 192 of [1].) In this way, we can construct a flow we need.

Hence the flows in both cases are continuous on all of $\mathcal{A}$ (it is the identity on the star of $[(p^{-1}(\beta), \tilde{\beta})]$), defining a deformation retraction of $\mathcal{A}$ onto the star of $[(p^{-1}(\beta), \tilde{\beta})]$. □

The barycentric subdivision $\mathcal{A}'$ of $\mathcal{A}$ is the simplicial complex associated to the partially ordered set of equivalent classes of marked annulus systems, with the partial ordering given by inclusion of systems. We are interested in the subcomplex $\mathcal{B} \subset \mathcal{A}'$ associated to the partially ordered set of systems whose complements are either solid tori or solid tori minus their cores, the core being a singular fiber in M. The proof in Theorem 1.1 that $\mathcal{A}$ is contractible restricts to a proof that $\mathcal{B}$ is contractible.

2. Geometric finiteness in Seifert fibered 3-manifold mapping class group

For a compact connected 3-manifold M, let $\mathrm{Diff}(M, \mathrm{rel}\,\partial M)$ denote the group of self-diffeomorphisms of M restricting to the identity on ∂M, the boundary of M. We give $\mathrm{Diff}(M, \mathrm{rel}\,\partial M)$ the C^∞ topology, as usual.

Let M be an orientable compact connected Seifert fibered 3-manifold with ∂M a nonempty boundary of M. Let $\mathrm{Diff}_f(M, \mathrm{rel}\,\partial M)$ be the subgroup of $\mathrm{Diff}(M, \mathrm{rel}\,\partial M)$ consisting of diffeomorphisms taking fibers to fibers. The inclusion of $\mathrm{Diff}_f(M, \mathrm{rel}\,\partial M)$ into $\mathrm{Diff}(M, \mathrm{rel}\,\partial M)$ induces an isomorphism on π_0. A proof of this without the "rel V" is indicated on pp. 85-86 of [5], and the same proof works for rel V. The group $\pi_0\mathrm{Diff}(M', \mathrm{rel}\,\partial M)$ is isomorphic to $\pi_0\mathrm{Diff}_f(M', \mathrm{rel}\,\partial M)$ and a finite extension of $\pi_0\mathrm{Diff}(M, \mathrm{rel}\,\partial M)$ where $M' = M - p^{-1}(F)$. There is a natural homomorphism

$$\Phi : \mathrm{Diff}_f(M', \mathrm{rel}\,\partial M) \longrightarrow \mathrm{Diff}(B', \mathrm{rel}\,\partial B)$$

since elements of $\mathrm{Diff}_f(M', \mathrm{rel}\,\partial M)$ take singular fibers to singular fibers. Then $\pi_0\mathrm{Diff}(M', \mathrm{rel}\,\partial M)$ acts on $\mathcal{A}$, as follows. For each element $[f] \in \pi_0\mathrm{Diff}(M', \mathrm{rel}\,\partial M)$,

$$\begin{aligned} &[f] \quad [(p^{-1}(\alpha_0), \tilde{\alpha_0}), \dots, (p^{-1}(\alpha_k), \tilde{\alpha_k})] \\ &= \quad [(p^{-1}(\Phi f(\alpha_0)), f(\tilde{\alpha_0})), \dots, (p^{-1}(\Phi f(\alpha_k)), f(\tilde{\alpha_k}))]. \end{aligned}$$

The group $\pi_0\mathrm{Diff}(M', \mathrm{rel}\,\partial M)$ acts simplicially on $\mathcal{A}$ and $\mathcal{B}$. The action on $\mathcal{B}$ is without fixed points. For if a point x in $\mathcal{B}$ were fixed by an element $[f]$ in $\pi_0\mathrm{Diff}(M', \mathrm{rel}\,\partial M)$, $p(x)$ is a fixed point of $[\Phi(f)]$. Then $[\Phi(f)]$ is isotopic to the identity rel ∂B (see [2]). Therefore f must be vertical mapping class (see Proposition 25.3 in [4]). Let $\{(p^{-1}(\alpha_0), \tilde{\alpha_0}), \dots, (p^{-1}(\alpha_k), \tilde{\alpha_k})\}$ be the marked annulus system corresponding to x. The components of $B' - \{\alpha_0 \cup \dots \cup \alpha_k\}$ are either disks or once-punctured disks in B'. Suppose that f is not isotopic to the identity rel ∂M. Then there is an arc α_i such that $\tilde{\alpha_i}$ is not isotopic to $f(\tilde{\alpha_i})$. Then $[(p^{-1}(\alpha_i), \tilde{\alpha_i})] \neq [(p^{-1}(\Phi f(\alpha_i)), f(\tilde{\alpha_i})] = [(p^{-1}(\alpha_i), f(\tilde{\alpha_i})]$. Thus x is not fixed by the element $[f]$. Therefore the diffeomorphism would be isotopic to the identity rel ∂M. Thus quotient

$\mathcal{B}/\pi_0\mathrm{Diff}(M', \mathrm{rel}\,\partial M)$ is a $K(\pi_0\mathrm{Diff}(M', \mathrm{rel}\,\partial M), 1)$. This quotient is a finite complex since marked annulus systems fall into finitely many orbits under the action of $\mathrm{Diff}(M', \mathrm{rel}\,\partial M)$.

We say that a group G is *almost geometrically finite* if it acts simplicially and properly discontinuously on a contractible simplicial complex L such that L/G is compact and a group is called *geometrically finite* if it is isomorphic to the fundamental group of a finite aspherical (simplicial- or CW-) complex. It is clear that every geometrically finite group must be torsion-free. Since a properly discontinuous action of a torsion-free group is free, a group is geometrically finite if and only if it is almost geometrically finite and torsion-free.

THEOREM 2.1. *A group G is almost geometrically finite group. Then every torsion-free subgroup of finite index in G is geometrically finite.*

PROOF OF 2.1. Let Γ be a torsion-free subgroup of finite index subgroup of G and let G act properly discontinuously on the contractible complex L with compact quotient. Since Γ has finite index, L/Γ is compact, and so a finite simplicial complex. On the other hand, Γ acts freely on L, as Γ is torsion-free. Hence the fundamental group of L/Γ is Γ, as L is simply connected. It follows that Γ is geometrically finite. □

COROLLARY 2.2. *The group $\pi_0\mathrm{Diff}(M, \mathrm{rel}\,\partial M)$ is geometrically finite.*

PROOF OF 2.2. From the previous argument, the group $\pi_0\mathrm{Diff}(M', \mathrm{rel}\,\partial M)$ is almost geometrically finite. The group $\pi_0\mathrm{Diff}(M, \mathrm{rel}\,\partial M)$ is a subgroup of finite index in $\pi_0\mathrm{Diff}(M', \mathrm{rel}\,\partial M)$ and torsion-free (see Theorem 8.4 in [3]). Therefore $\pi_0\mathrm{Diff}(M, \mathrm{rel}\,\partial M)$ is geometrically finite by theorem 2.1. □

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