Comm. Korean Math. Soc. **13** (1998), No. 4, pp. 817–823

CERTAIN INFINITESIMAL TRANSFORMATIONS ON QUATERNIONIC KÄHLERIAN MANIFOLDS

JIN SUK PAK AND DAE WON YOON

ABSTRACT. In the present paper, we study conformal and projective Killing vector fields and infinitesimal Q-transformations on a quaternionic Kählerian manifold, and prove that an infinitesimal conformal or projective automorphism in a compact quaternionic Kählerian manifold is necessarily infinitesimal automorphism.

1. Introduction

A quaternionic Kählerian manifold is defined as a Riemannian manifold whose holonomy group is a subgroup of $Sp(m) \cdot Sp(1)$. Recently several authors ([1],[2],[3],[4]) have studied quaternionic Kählerian manifolds and obtained many interesting results. In particular, Ishihara ([4]) has introduced the notion of infinitesimal Q- transformations preserving the quaternionic structure, and obtained the following result: *In a compact quaternionic Kählerian manifold, a Killing vector field is necessarily an infinitesimal automorphism.*

In this paper, we consider conformal and projective Killing vector fields whose local 1- parameter group preserves the quaternionic structure, and prove the following theorems:

THEOREM A. *An infinitesimal conformal automorphism on a $4m(m \geq 2)$-dimensional compact quaternionic Kählerian manifold is necessarily an infinitesimal automorphism.*

Received October 31, 1997. Revised July 20, 1998.

1991 Mathematics Subject Classification: 53C25.

Key words and phrases: affine, projective and conformal Killing vectors, quaternionic Kählerian manifold, infinitesimal transformation.

This study was partially supported by the Basic Science Research Institute Program, Ministry of Education, 1997, Project No. BSRI-97-1404 and TGRC-KOSEF.

THEOREM B. *An infinitesimal projective automorphism on a $4m(m \geq 2)$-dimensional compact quaternionic Kählerian manifold is necessarily an infinitesimal automorphism.*

Manifolds, mappings, tensor fields and other geometric objects we discuss are assumed to be differentiable and of class C^∞. The indices h, i, j, k, l, r, s, t run over the range $\{1, 2, \cdots, n\}, (n = 4m, m \geq 1)$ and the summation convention will be used with respect to this system of indices.

2. Preliminaries

Let (M, g, V) be an $n(= 4m)$-dimensional quaternionic Kählerian manifold covered by a system of coordinate neighborhoods U and denote its quaternionic Kählerian structure by (g, V) or briefly V. Then there exists a canonical local basis $\{F, G, H\}$ of the 3-dimensional vector bundle V consisting of tensors of type (1,1) over M such that

$$\begin{aligned} &F^2 = -I, && G^2 = -I, && H^2 = -I, \\ &GH = -HG = F, && HF = -FH = G, && FG = -GF = H \end{aligned} \tag{2.1}$$

in each local coordinate neighborhood U, where I denotes the identity tensor field of type (1,1) in M. Moreover, the local tensor fields F, G and H are almost Hermitian with respect to g and

$$\begin{aligned} \nabla_X F &= && r(X)G && - q(X)H, \\ \nabla_X G &= -r(X)F && && + p(X)H, \\ \nabla_X H &= q(X)F && - p(X)G && \end{aligned} \tag{2.2}$$

are satisfied for any vector field X on M, where p, q and r are local 1-forms defined on U and ∇ denotes the Riemannian connection with respect to g (cf. [3],[4]).

We put in U

$$\Lambda = F \otimes F + G \otimes G + H \otimes H, \tag{2.3}$$

then it follows that Λ determines a global tensor field of type (2,2) in M, which will be also denoted by Λ. Then, using (2.2), we easily find

$$\nabla \Lambda = 0. \tag{2.4}$$

It is well known (cf. [1],[3]) that any quaternionic Kählerian manifold is an Einstein space, provided $dimM \geq 8$, i.e., that the Ricci tensor S of M has the form

$$S = \frac{k}{4m} g, \tag{2.5}$$

k being the scalar curvature of M, which is a constant if M is connected. We denote by $K_{kji}{}^h$ components of the curvature tensor of (M, g) and put $K_{kjih} = K_{kji}{}^s g_{hs}$, where g_{ji} are components of g. We denote by $(g^{ji}) = (g_{ji})^{-1}$ and put $F^{ih} = g^{is} F_s{}^h, G^{ih} = g^{is} G_s{}^h, H^{ih} = g^{is} H_s{}^h$, which are all skew- symmetric. Then we can easily see that if $m > 1$,

$$K_{ksih} F_j{}^s F^{ih} = K_{ksih} G_j{}^s G^{ih} = K_{ksih} H_j{}^s H^{ih} = -\frac{k}{2(m+2)} g_{kj}. \tag{2.6}$$

In particular, in the case of $m = 1$, we have

$$K_{ksih}(F_j{}^s F^{ih} + G_j{}^s G^{ih} + H_j{}^s H^{ih}) = -\frac{k}{6} g_{kj} \tag{2.7}$$

(cf. [4]).

3. Infinitesimal transformations

Let ∇' and ∇ be affine connections on Riemannian manifolds M' and M, respectively.
A map $f : M' \to M$ is called *affine* if $f_* \nabla'_X Y = \nabla_{f_* X} f_* Y$ for vector fields X, Y in M' (cf. [5]). An *affine transformation* of (M, ∇) is an affine diffeomorphism of M. A vector field X on M is called *affine* if the local 1- parameter group of X consists of local affine transformations.

A map $f : (M', \nabla') \to (M, \nabla)$ of manifolds is called *projective* if for each geodesic γ of ∇', $f \circ \gamma$ is a reparametrization of a geodesic of ∇ (cf. [5]). A *projective transformation* of M is a diffeomorphism which is projective. A vector field X on M is called *projective* if the local 1-parameter group of X consists of local projective transformations.

A map $f : (M', g') \to (M, g)$ of Riemannian manifolds is called *conformal* if $f^* g$ is conformally equivalent to g' (cf. [5]). A conformal diffeomorphism of a Riemannian manifold M is called a *conformal transformation* of M. A vector field X on M is called *conformal* if the local 1- parameter group of X consists of local conformal transformations.

Let (M, g, V) be a quaternionic Kählerian manifold. If a transformation $f : M \to M$ leaves the bundle V invariant, then f is called a *Q-transformation* of (M, g, V) ([4]). Let $\{F, G, H\}$ be a canonical local basis of V in a coordinate neighborhood U of M. Then a transformation $f : M \to M$ is a Q-transformation of (M, g, V) if and only if $\{f^*F, f^*G, f^*H\}$ is a canonical local basis of V in $f(U)$, where f^*F denotes the tensor field induced by f from F and so on. Thus, a Q-transformation f preserves the tensor field Λ defined by (2.3) globally in M. Conversely, using (2.1), we can easily prove that a transformation $f : M \to M$ is a Q-transformation of (M, g, V) if f preserves the tensor field Λ invariant.

A vector field X in (M, g, V) is called an *infinitesimal Q-transformation* of (M, g, V), if the local 1- parameter group of X is a Q-transformation of (M, g, V).

If a transformation $f : M \to M$ is a Q-transformation of (M, g, V) and at the same time an isometry (resp. a conformal, a projective, an affine) of M, then f is called an *automorphism* (*resp. a conformal automorphism, a projective automorphism, an affine automorphism*) of (M, g, V).

If, for a vector field X in M, the local 1- parameter group of X is an automorphism (resp. a conformal automorphism, a projective automorphism, an affine automorphism) of (M, g, V), then X is called an *infinitesimal automorphism* (*resp. infinitesimal conformal automorphism, infinitesimal projective automorphism, infinitesimal affine automorphism*) of (M, g, V).

From now on we let (M, g, V) be a quaternionic Kählerian manifold. Consider a vector field X in (M, g, V) and denote by X^h components of X and put $X_i = g_{is}X^s$ in a coordinate neighborhood U of M. By means of $\nabla\Lambda = 0$, which is a direct consequence of (2.2), we can obtain in U

$$(3.1)\quad \mathcal{L}_X \Lambda_k{}^j{}_i{}^h = \nabla_k X^s \Lambda_s{}^j{}_i{}^h - \nabla_t X^j \Lambda_k{}^t{}_i{}^h + \nabla_i X^s \Lambda_k{}^j{}_s{}^h - \nabla_t X^h \Lambda_k{}^j{}_i{}^t,$$

where

$$(3.2)\qquad \Lambda_k{}^j{}_i{}^h = F_k{}^j F_i{}^h + G_k{}^j G_i^h + H_k{}^j H_i^h$$

are components of Λ in U and $\mathcal{L}_X$ denotes the Lie derivation with respect to X (cf. [4]).

Finally we prepare the following lemmas for later use.

LEMMA 3.1 ([4]). *Let (M, g, V) be a compact quaternionic Kählerian manifold. Then a necessary and sufficient condition for a vector field X in (M, g, V) to be an infinitesimal Q-transformation is that*

$$(3.3) \qquad 3m\left[\nabla^l\nabla_l X^j + \frac{k}{4(m+2)}X^j\right] - \Lambda^{kjih}\nabla_k\nabla_i X_h = 0,$$

where $\Lambda^{kjih} = g^{kt}g^{is}\Lambda_t{}^j{}_s{}^h$, $\nabla^l = g^{ls}\nabla_s$.

REMARK. We can easily verify that (3.3) is established even if $m = 1$ or $m > 1$ (cf. [4]).

LEMMA 3.2 ([4]). *In a compact quaternionic Kählerian manifold, a Killing vector field is necessarily an infinitesimal automorphism.*

4. Proofs of Theorem A and B

PROOF OF THEOREM A. Let X be an infinitesimal conformal automorphism in a compact quaternionic Kählerian manifold (M, g, V). Then, as is well known (cf. [5]), we have

$$(4.1) \qquad \nabla_k\nabla_l X^i + K_{jkl}{}^i X^j = \Phi_k\delta_l{}^i + \Phi_l\delta_k{}^i - \Phi^i g_{kl},$$

where Φ is a positive function that characterizes a conformal change of the metric, $\Phi_i = \nabla_i\Phi$ and $\Phi^h = g^{hi}\Phi_i$. Transvecting (4.1) with g^{kl} and using (2.5), we obtain

$$(4.2) \qquad \nabla^l\nabla_l X^j + \frac{k}{4m}X^j = (2-4m)\Phi^j.$$

On the other hand, (4.1) can be rewritten as the form

$$\nabla_k\nabla_i X_h + K_{lkih}X^l = \Phi_k g_{ih} + \Phi_i g_{kh} - \Phi_h g_{ki},$$

from which, multiplying Λ^{kjih} and using (2.1) and (2.6), we have

$$(4.3) \qquad \Lambda^{kjih}\nabla_k\nabla_i X_h + \frac{3k}{2(m+2)}X^j = -6\Phi^j.$$

Substituting (4.2) and (4.3) into (3.3), we can find $\Phi^j = 0$ because m is a positive integer, which means that Φ is a constant. Since a homothetic transformation of a compact orientable Riemannian manifold is an isometry (cf. [5]), we complete the proof Theorem A. □

PROOF OF THEOREM B. Let X be an infinitesimal projective automorphism in a compact quaternionic Kählerian manifold. Then we have

$$\nabla_k \nabla_l X^i + K_{jkl}{}^i X^j = \delta_k{}^i p_l + \delta_l{}^i p_k, \tag{4.4}$$

where p^i is a certain gradient vector ([5]). Transvecting (4.4) with g^{kl} and using (2.5), we have

$$\nabla^l \nabla_l X^j + \frac{k}{4m} X^j = 2p^j. \tag{4.5}$$

On the other side, (4.4) can be rewritten as the form

$$\nabla_k \nabla_i X_h + K_{lkih} X^l = g_{hk} p_i + g_{hi} p_k,$$

from which, multiplying Λ^{kjih} and using (2.1) and (2.6), we can obtain

$$\Lambda^{kjih} \nabla_k \nabla_i X_h + \frac{3k}{2(m+2)} X^j = -3p^j. \tag{4.6}$$

Substituting (4.5) and (4.6) into (3.3), it follows that $p^j = 0$ because m is a positive integer. Hence X is an infinitesimal affine automorphism. By the way, an affine vector field is a Killing vector field in a compact orientable Riemannian manifold (cf. [5]). Thus Lemma 3.2 yields Theorem B. □

References

[1] D. V. Alekseevskii, *Compact quaternion spaces*, Funktsional'nyi Analiz i Ego Prilozhenyia **2** (1968), 11-20.

[2] A. Gray, *A note on manifolds whose holonomy group is a subgroup of* $Sp(m) \cdot Sp(1)$, Michigan Math. J. **16** (1968), 125- 128.

[3] S. Ishihara, *Quaternion Kählerian manifolds*, J. Diff. Geo. **9** (1974), 483-500.
[4] ______, *Integral formula and their applications in quaternionic Kählerian manifolds*, Kodai Math. Sem. Rep. **28** (1976), 63-71.
[5] K. Yano, *Some integral formulas and their applications*, Michigan Math. J. **5** (1958), 63-73.
[6] ______, *Differential geometry on complex and almost complex space*, Pergamon Press, 1965.
[7] K. Yano and S. Bochner, *Curvature and Betti numbers*, Princeton Univ. Press, 1953.

Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea