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ISHIKAWA AND MANN ITERATION METHODS FOR STRONGLY ACCRETIVE OPERATORS

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ABSTRACT. Let E be a smooth Banach space. Suppose $T : E \to E$ is a strongly accretive map. It is proved that each of the two well known fixed point iteration methods (the Mann and Ishikawa iteration methods), under suitable conditions, converges strongly to a solution of the equation $Tx = f$.

1. Introduction

Let E be a real normed linear space. A mapping T with domain $D(T)$ and range $R(T)$ in E is called accretive[2] if for all x, y in $D(T)$ and all $t \geq 0$, we have that

$$\|x - y\| \leq \|x - y + t(Tx - Ty)\|.$$

Interest in the accretive operators (which were introduced by Browder[2] and Kato[11] independently in 1967) stems mainly from their connection with the solvability of evolution equations. An early fundamental result of Browder[2] states that the initial value problem

$$\frac{du}{dt} + Tu = 0, \quad u(0) = u_0$$

is solvable if T is locally Lipschitzian and accretive on E.

Let E be a real Banach space and let E^* be its dual. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. The norm of E is said to be Gateaux differentiable (or E is said to be smooth) if

$$\lim_{t \to \infty} \frac{\|x + ty\| - \|x\|}{t}$$

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exists for each $x, y \in U$. We shall denote by J the normalized duality map from E to 2^{E^*} given by

$$Jx = \{f^* \in E^* : \|f^*\|^2 = \|x\|^2 = \langle x, f^* \rangle\},$$

where $\langle , \rangle$ denotes the generalized duality pairing. It is well known that if E is smooth, then duality mapping J is single-valued.

Let C be a nonempty subset of E. A mapping $T : C \to C$ is called strongly accretive if for each x, y in C there exists $w \in J(x - y)$ such that

$$\langle Tx - Ty, w \rangle \geq k\|x - y\|^2,$$

for some real constant $k > 0$. Without loss of generality we may assume $k \in (0, 1)$. Strongly accretive operators have been studied by various authors (see e.g., [3,9]).

In [13], the following surjectivity result is proved:

THEOREM M. *Let E be a Banach space and $T : E \to E$ be continuous and strongly accretive. Then T is surjective.*

An obvious consequence of Theorem M is that for each $f \in E$, the equation $Tx = f$ has a solution in E. Recently two well known fixed point iteration methods (the Mann and Ishikawa iteration methods) have been successfully employed to approximate a solution of this equation in L_p spaces, $p \geq 2$ (see e.g., [6]). Chidume[6] proved that the Mann iteration process converges strongly to a solution of $Tx = f$ when T is Lipschitzian and strongly accretive. More recently, it has been proved (see e.g., [8]) that the Ishikawa iteration process also converges to the solution of $Tx = f$ in L_p spaces, $p \geq 2$, again where T is Lipschitzian and strongly accretive.

It is our purpose in this paper to prove that if E is any smooth Banach space, and $T : E \to E$ is a strongly accretive map then both the Mann and Ishikawa iteration schemes can be used to approximate the unique solution of the equation $Tx = f$. Also, we prove that if T is a strongly pseudocontractive mapping, then the Ishikawa iteration process, under suitable conditions on the real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$, converges strongly to the unique fixed point of T. Our result extends the result of [14].

2. Preliminaries

In this section we introduce two iterative schemes which will be needed in the sequel.

[A] The Ishikawa iteration (see e.g., [10]) is defined as follows: For C a convex subset of a real Banach space E and T a mapping of C into itself, the sequence $\{x_n\}_{n=0}^{\infty}$ in C is defined by

$$x_0 \in C,$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n,$$
$$y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 0,$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ satisfy $0 \leq \alpha_n \leq \beta_n < 1$ for all n; $\lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$.

[B] The Mann iteration process (see e.g., [12]) is similar to the Ishikawa iteration process but with $\beta_n = 0$ and different conditions placed on α_n. More precisely, with E, C, and x_0 as in part [A], the Mann iteration process is defined by

$$x_0 \in C,$$
$$x_{n+1} = (1-c_n)x_n + c_n T x_n, \quad n \geq 0,$$

where $\{c_n\}_{n=0}^{\infty}$ is a real sequence satisfying $c_0 = 1$, $0 \leq c_n < 1$ for all $n \geq 1$, and $\sum_{n=0}^{\infty} c_n = \infty$.

The iteration processes [A] and [B] have been successfully employed by various authors to approximate solutions of nonlinear operator equations in Banach spaces (see e.g., [4,5]).

For our next theorems we shall need the following lemmas [14].

LEMMA 2.1. *Let $\{\beta_n\}$ be a nonnegative real sequence and suppose $\{\beta_n\}$ satisfies the following inequality*

$$\beta_{n+1} \leq (1-\alpha_n)\beta_n + \varepsilon\alpha_n,$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\alpha_n \in (0,1)$, and $\varepsilon > 0$. Then $0 \leq \limsup_{n\to\infty} \beta_n \leq \varepsilon$.

LEMMA 2.2. *Let E be a smooth Banach space. Suppose one of the followings holds.*

(i) J is uniformly continuous on any bounded subset of E.

(ii) $\langle J(x) - J(y), x - y\rangle \leq \|x - y\|^2$ for all x, y in E.

(iii) For any bounded subset D of E there is a c such that

$$\langle J(x) - J(y), x - y\rangle \leq c(\|x - y\|)$$

for all x, y in D where c satisfies $\lim_{t\to 0^+} \frac{c(t)}{t} = 0$.

Then for any $\varepsilon > 0$ and any bounded subset C there is $\delta > 0$ such that

$$\|tx + (1-t)y\|^2 \leq 2\langle x, J(y)\rangle t + 2\varepsilon t + (1-2t)\|y\|^2 \tag{1}$$

for any $x, y \in C$ and $t \in [0, \delta)$.

For the remainder of this paper, the Lipschitz constant of T will be denoted by $L(\geq 1)$ and the constant appearing in the definition of a strongly accretive map will be denoted by $k \in (0, 1)$.

3. Strong convergence theorems for the solution of $Tx = f$ when T is strongly accretive

In this section we prove that the Mann and Ishikawa iteration processes converge strongly to a solution of the equation $Tx = f$ when T is strongly accretive.

THEOREM 3.1. *Let E be a smooth Banach space under one of the assumptions in* Lemma 2.2 *and C a closed bounded convex subset of E. Suppose $T : E \to E$ is a continuous strongly accretive map. For a given $f \in E$, define $S : E \to E$ by $Sx = f - Tx + x$ for each $x \in E$. Define the sequence $\{x_n\}_{n=0}^{\infty}$ in C iteratively by $x_0 \in E$,*

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Sx_n \tag{2}$$

for $n \geq 0$, where $\{\lambda_n\}_{n=0}^{\infty}$ is a real sequence satisfying

(i) $0 < \lambda_n < 1$ for all $n \geq 0$;

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(iii) $\lim_{n\to\infty} \lambda_n = 0$.

Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the solution of $Tx = f$.

PROOF. The existence of a solution to $Tx = f$ follows from Morales[13]. Let x^* denote a solution. Clearly, x^* is a fixed point of S and for arbitrary $x, y \in E$ we have

$$\langle Sx - Sy, J(x-y)\rangle \le (1-k)\|x-y\|^2. \tag{3}$$

For given $\varepsilon > 0$, we have $\delta > 0$ in Lemma 2.2. Since $\lim_{n\to\infty} \lambda_n = 0$, there exists N such that for all $n \ge N$, $\lambda_n < \delta$. For such an n, using (1), (2), and (3), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\lambda_n(Sx_n - Sx^*) + (1-\lambda_n)(x_n - x^*)\|^2 \\ &\le 2\langle J(x_n - x^*), Sx_n - Sx^*\rangle\lambda_n + 2\varepsilon\lambda_n \\ &\quad + (1-2\lambda_n)\|x_n - x^*\|^2 \\ &\le 2(1-k)\lambda_n\|x_n - x^*\|^2 + 2\varepsilon\lambda_n \\ &\quad + (1-2\lambda_n)\|x_n - x^*\|^2 \\ &= (1-2k\lambda_n)\|x_n - x^*\|^2 + 2\varepsilon\lambda_n.\end{aligned}$$

In order to apply Lemma 2.1 we let $\beta_n = \|x_n - x^*\|^2$. Then

$$0 \le \limsup_{n\to\infty} \beta_n \le \varepsilon.$$

Since ε is arbitrary, $\limsup_{n\to\infty} \beta_n = 0$. So $\lim_{n\to\infty} \beta_n = 0$. The proof is complete. □

THEOREM 3.2. *Let E be a smooth Banach space under one of the assumptions in* Lemma 2.2 *and C a closed bounded convex subset of E. Suppose $T : E \to E$ is a Lipschitzian strongly accretive map. For a given $f \in E$, define $S : E \to E$ by $Sx = f - Tx + x$ for each $x \in E$. Let $\{\lambda_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ be real sequences satisfying*

(i) $0 < \lambda_n, \beta_n < 1$ for all $n \ge 0$;

(ii) $\sum_{n=0}^\infty \lambda_n = \infty$;

(iii) $\lim_{n\to\infty} \beta_n = 0$.

For arbitrary $x_0 \in E$ define the sequence $\{x_n\}_{n=0}^\infty$ in C by

$$x_0 \in E,$$

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n S y_n, \tag{4}$$

$$y_n = (1-\beta_n)x_n + \beta_n S x_n, \quad n \ge 0. \tag{5}$$

Then $\{x_n\}_{n=0}^\infty$ converges strongly to the unique solution of $Tx = f$.

PROOF. The existence of a solution follows from Morales[13]. Let x^* be a solution. Clearly x^* is a fixed point of S and S is Lipschitzian with Lipschitz constant $L^* = 1 + L$. Moreover, for each x, y in E we have

$$\langle Sx - Sy, J(x-y)\rangle \le (1-k)\|x-y\|^2. \tag{6}$$

Using (4) we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1-\lambda_n)(x_n - x^*) + \lambda_n(Sy_n - Sx^*)\|^2\\ &\le 2\langle Sy_n - Sx^*, J(x_n - x^*)\rangle\lambda_n + 2\varepsilon\lambda_n\\ &\quad + (1-2\lambda_n)\|x_n - x^*\|^2. \end{aligned} \tag{7}$$

Using inequality (5), (6) and condition (iii) we have for sufficiently large n, $L^*(2+L)\beta_n < \frac{k}{2}$ so that

$$\begin{aligned}&\langle Sy_n - Sx^*, J(x_n - x^*)\rangle\\ &\quad= \langle Sy_n - Sx_n, J(x_n - x^*)\rangle\\ &\qquad+ \langle Sx_n - Sx^*, J(x_n - x^*)\rangle\\ &\quad\le \|Sy_n - Sx_n\|\|x_n - x^*\| + (1-k)\|x_n - x^*\|^2\\ &\quad\le L^*\|y_n - x_n\|\|x_n - x^*\| + (1-k)\|x_n - x^*\|^2\\ &\quad\le L^*\beta_n(\|Sx_n - x^*\| + \|x^* - x_n\|)\|x_n - x^*\|\\ &\qquad+ (1-k)\|x_n - x^*\|^2\\ &\quad\le (\frac{k}{2} + (1-k))\|x_n - x^*\|^2\\ &\quad= (1-\frac{k}{2})\|x_n - x^*\|^2.\end{aligned}$$

Substituting this inequality in (7) yields

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le 2(1-\frac{k}{2})\lambda_n\|x_n - x^*\|^2 + 2\varepsilon\lambda_n\\ &\quad+ (1-2\lambda_n)\|x_n - x^*\|^2\\ &= (1-k\lambda_n)\|x_n - x^*\|^2 + 2\varepsilon\lambda_n.\end{aligned}$$

The remainder of the argument now follows exactly as in the proof of Theorem 3.1 to give that $x_n \to x^*$ as $n \to \infty$, completing the proof of Theorem 3.2. □

4. Strong convergence theorem for the fixed point of Lipschitz strongly pseudocontractive mapping

We conclude this paper by proving a result which is closely related to Theorem 3.2. We need the following definition: Let C be a nonempty subset of a Banach space E. A mapping $T : C \to C$ is called to be strongly pseudocontractive if there exists $t > 1$ such that the inequality

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|$$

holds for all $x, y \in C$ and $r > 0$. If $t = 1$, then T is called pseudocontractive. The relation between pseudocontractive and accretive maps is that T is pseudocontractive if and only if $(I - T)$ is accretive.

Pseudocontractive maps have been studied by various authors. Borgin[1] studied the connection between strongly pseudocontractive mappings and strongly accretive mappings. He proved that T is a strongly pseudocontractive mapping if and only if $(I - T)$ is a strongly accretive mapping.

THEOREM 4.1. *Let E be a smooth Banach space under one of the assumptions in* Lemma 2.2 *and C a nonempty bounded closed convex subset of E. Let $T : C \to C$ be a Lipschitz strongly pseudocontractive mapping. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences in* $(0,1)$ *satisfying the following conditions:*
(*i*) $\lim_{n\to\infty} \beta_n = 0$;
(*ii*) $\sum_{n=0}^{\infty} \alpha_n = \infty$.
Then for each $x_0 \in C$, the Ishikawa sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n,$$

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad n \geq 0,$$

converges strongly to the unique fixed point of T.

PROOF. The existence of a fixed point x^* follows from Deimling[7]. The uniqueness follows from the strong pseudocontractivity. Since $(I - T)$ is strongly accretive, so there exists $k \in (0,1)$ such that for all $x, y \in C$,

$$\langle (I - T)x - (I - T)y, J(x - y)\rangle \geq k\|x - y\|^2.$$

Using (1) we obtain

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Ty_n - Tx^*)\|^2 \\
&\leq 2\langle Ty_n - Tx^*, J(x_n - x^*)\rangle\alpha_n + 2\varepsilon\alpha_n \\
&\quad + (1-2\alpha_n)\|x_n - x^*\|^2 \qquad (8)
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle Ty_n - Tx^*, J(x_n - x^*)\rangle \\
&\quad = \langle Ty_n - Tx_n, J(x_n - x^*)\rangle \\
&\qquad - \langle (I-T)x_n - (I-T)x^*, J(x_n - x^*)\rangle \\
&\qquad + \langle x_n - x^*, J(x_n - x^*)\rangle \\
&\quad \leq \|Ty_n - Tx_n\|\|x_n - x^*\| - k\|x_n - x^*\|^2 + \|x_n - x^*\|^2 \\
&\quad \leq \beta_n L(1+L)\|x_n - x^*\|^2 - k\|x_n - x^*\|^2 + \|x_n - x^*\|^2 \\
&\quad = \{1 - k + \beta_n L(1+L)\}\|x_n - x^*\|^2.
\end{aligned}
$$

Condition (i) implies that for sufficiently large n, $\beta_n L(1+L) \leq k(1-k)$ so that

$$
\begin{aligned}
\langle Ty_n - Tx^*, J(x_n - x^*)\rangle &\leq \{1 - k + k(1-k)\}\|x_n - x^*\|^2 \\
&= (1-k^2)\|x_n - x^*\|^2.
\end{aligned}
$$

Using this inequality in (8) we obtain

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq 2\alpha_n(1-k^2)\|x_n - x^*\|^2 + 2\varepsilon\alpha_n \\
&\quad + (1-2\alpha_n)\|x_n - x^*\|^2 \\
&= (1 - 2k^2\alpha_n)\|x_n - x^*\|^2 + 2\varepsilon\alpha_n.
\end{aligned}
$$

The result now follows exactly as in the proof of Theorem 3.1. This completes the proof. □

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