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C^*-ALGEBRAS ASSOCIATED WITH LENS SPACES

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Abstract. We define the rational lens algebra $\mathbb{L}_{\frac{m}{k}}(n)$ as the crossed product by an action of $\mathbb{Z}$ on $C(S^{2n+1})$. Assume the fibres are $M_k(\mathbb{C})$. We prove that $\mathbb{L}_{\frac{m}{k}}(n) \otimes M_p(\mathbb{C})$ is not isomorphic to $C(\text{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_{kp}(\mathbb{C})$ if $k > 1$, and that $\mathbb{L}_{\frac{m}{k}}(n) \otimes M_{p^\infty}$ is isomorphic to $C(\text{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of k is a subset of the set of prime factors of p.

It is moreover shown that if $k > 1$ then $\mathbb{L}_{\frac{m}{k}}(n)$ is not stably isomorphic to $C(\text{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_k(\mathbb{C})$.

1. Introduction

Given a locally compact abelian group G and a multiplier ω on G, one can associate to them the twisted group C^*-algebra $C^*(G, \omega)$. The twisted group C^*-algebra $C^*(\mathbb{Z}^l, \omega)$ is called a *non-commutative torus.* The simplest non-trivial non-commutative tori arise when $G = \mathbb{Z}^2$. In this case we may assume ω is antisymmetric and $\omega((1,0),(0,1)) = e^{\pi i\theta}$. When $\theta = \frac{m}{k}$, one obtains a *rational rotation algebra*, and denoted by $A_{\frac{m}{k}}$.

The rational rotation algebra $A_{\frac{m}{k}}$ can be obtained by the crossed product by an action of $\mathbb{Z}$ on $C(S^1)$ (see [2]). On can canonically replace $C(S^1)$ in the crossed product $C(S^1) \times_\beta \mathbb{Z}$ representing $A_{\frac{m}{k}}$ by $C(S^{2n+1})$.

Definition 1.1. The crossed product by the action α of $\mathbb{Z}$ on the commutative C^*-algebra $C(S^{2n+1})$, which is induced from the homeo-

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morphism

$$(z_0, z_1, \cdots, z_n) \in S^{2n+1} \mapsto (e^{2\pi i \frac{m}{k}} z_0, e^{2\pi i \frac{m}{k}} z_1, \cdots, e^{2\pi i \frac{m}{k}} z_n) \in S^{2n+1}$$

for k and m relatively prime, is said to be a *rational lens algebra*, and denoted by $\mathbb{L}_{\frac{m}{k}}(n)$.

A well-known theorem of Tomiyama-Takesaki [5] asserts that each k-homogeneous C^*-algebra A over a compact Hausdorff space M is isomorphic to the C^*-algebra of sections of a locally trivial C^*-algebra bundle with base space M, fibre $M_k(\mathbb{C})$, and structure group $\mathrm{Aut}(M_k(\mathbb{C})) \cong PU(k)$.

The cyclic group $\mathbb{Z}/k\mathbb{Z}$ acts freely on S^{2n+1} by the homeomorphism given as above and $S^{2n+1}/(\mathbb{Z}/k\mathbb{Z})$ is homeomorphic to the lens space $L^k(n)$. So the cyclic group $\mathbb{Z}/k\mathbb{Z}$ acts on $C(S^{2n+1})$ and the crossed product by the action of $\mathbb{Z}/k\mathbb{Z}$ on $C(S^{2n+1})$ is isomorphic to $C(L^k(n)) \otimes M_k(\mathbb{C})$ by the Mackey machine for a crossed product. The cyclic group $\mathbb{Z}/k\mathbb{Z}$ acts freely on S^1 and $S^1/(\mathbb{Z}/k\mathbb{Z})$ is homeomorphic to S^1. So the cyclic group $\mathbb{Z}/k\mathbb{Z}$ acts on $C(S^1)$ and the crossed product by the action of $\mathbb{Z}/k\mathbb{Z}$ on $C(S^1)$ is isomorphic to $C(S^1) \otimes M_k(\mathbb{C})$. Thus the fibre at each point of $L^k(n) \times S^1$ is $M_k(\mathbb{C})$, and so $\mathbb{L}_{\frac{m}{k}}(n)$ is a k-homogeneous C^*-algebra over $L^k(n) \times S^1$. Hence $\mathbb{L}_{\frac{m}{k}}(n)$ is isomorphic to the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n)) = S^{2n+1}/(\mathbb{Z}/k\mathbb{Z}) \times \widehat{k\mathbb{Z}} = L^k(n) \times S^1$ with fibres $M_k(\mathbb{C})$.

In this paper, using Pimsner-Voiculescu exact sequence for a crossed product, we compute the K-theory of $\mathbb{L}_{\frac{m}{k}}(n)$ and we are going to show that $[1_{\mathbb{L}_{\frac{m}{k}}(n)}] \in K_0(\mathbb{L}_{\frac{m}{k}}(n))$ is primitive. Using the fact that $[1_{\mathbb{L}_{\frac{m}{k}}(n)}] \in K_0(\mathbb{L}_{\frac{m}{k}}(n))$ is primitive, one can show that $\mathbb{L}_{\frac{m}{k}}(n) \otimes M_p(\mathbb{C})$ is not isomorphic to $C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_{kp}(\mathbb{C})$ if $k > 1$, and that the tensor product of $\mathbb{L}_{\frac{m}{k}}(n)$ with a UHF-algebra M_{p^∞} of type p^∞ is isomorphic to $C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of k is a subset of the set of prime factors of p.

By comparison of the K-theory, it is shown that $\mathbb{L}_{\frac{m}{k}}(n)$ is not stably isomorphic to $C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_k(\mathbb{C})$ if $k > 1$.

2. The K-theory of rational lens algebras

We are going to show that $[1_{\mathbb{L}_{\frac{m}{k}}(n)}] \in K_0(\mathbb{L}_{\frac{m}{k}}(n))$ is primitive.

THEOREM 2.1. *$K_0(\mathbb{L}_{\frac{m}{k}}(n)) \cong K_1(\mathbb{L}_{\frac{m}{k}}(n)) \cong \mathbb{Z}^2$, and $[1_{\mathbb{L}_{\frac{m}{k}}(n)}] \in K_0(\mathbb{L}_{\frac{m}{k}}(n))$ is primitive.*

PROOF. $\mathbb{L}_{\frac{m}{k}}(n)$ is given by the crossed product $C(S^{2n+1}) \times_\alpha \mathbb{Z}$, where the action α is given in Definition 1.1.

Note that this action is homotopic to the trivial action, since we can homotope $\frac{m}{k}$ to 0. Hence $\mathbb{Z}$ acts trivially on the K-theory of $C(S^{2n+1})$. The Pimsner-Voiculescu exact sequence for a crossed product gives

$$\cdots \xrightarrow{1-(\alpha)_*} K_0(C(S^{2n+1})) \xrightarrow{\Phi} K_0(\mathbb{L}_{\frac{m}{k}(n)}) \rightarrow K_1(C(S^{2n+1})) \xrightarrow{1-(\alpha)_*} \cdots$$

and similarly for K_1, where the map Φ is induced by inclusion. Since $(\alpha)_* = 1$ and since the K-groups of $C(S^{2n+1})$ are free abelian of rank 1 (see [4, II.1.34]), this reduces a split short exact sequence

$$\{0\} \rightarrow K_0(C(S^{2n+1})) \xrightarrow{\Phi} K_0(\mathbb{L}_{\frac{m}{k}}(n)) \rightarrow K_1(C(S^{2n+1})) \rightarrow \{0\}$$

and similarly for K_1. So $K_j(\mathbb{L}_{\frac{m}{k}}(n))$ are free abelian of rank 2.

Since the inclusion $C(S^{2n+1}) \rightarrow \mathbb{L}_{\frac{m}{k}}(n)$ sends $1_{C(S^{2n+1})}$ to $1_{\mathbb{L}_{\frac{m}{k}}(n)}$, $[1_{\mathbb{L}_{\frac{m}{k}}(n)}]$ is the image of $[1_{C(S^{2n+1})}]$, which is primitive in $K_0(C(S^{2n+1}))$ (see [4, II.1.21]). Hence the image is primitive, since the Pimsner-Voiculescu exact sequence is a split short exact sequence of torsion-free groups.

Therefore, $K_0(\mathbb{L}_{\frac{m}{k}}(n)) \cong K_1(\mathbb{L}_{\frac{m}{k}}(n)) \cong \mathbb{Z}^2$, and the class $[1_{\mathbb{L}_{\frac{m}{k}}(n)}]$ of the unit $1_{\mathbb{L}_{\frac{m}{k}}(n)}$ is primitive. □

COROLLARY 2.2. *Let p be a positive integer. $\mathbb{L}_{\frac{m}{k}}(n) \otimes M_p(\mathbb{C})$ is not isomorphic to $C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_{kp}(\mathbb{C})$ if $k > 1$.*

PROOF. Assume $\mathbb{L}_{\frac{m}{k}}(n) \otimes M_p(\mathbb{C})$ is isomorphic to $C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_{kp}(\mathbb{C})$. Then the unit $1_{\mathbb{L}_{\frac{m}{k}}(n)} \otimes I_p$ maps to the unit $1_{C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n)))} \otimes I_{kp}$, where I_s denotes the $s \times s$ identity matrix. So

$$[1_{\mathbb{L}_{\frac{m}{k}}(n)} \otimes I_p] = [1_{C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n)))} \otimes I_{kp}] = (kp)[1_{C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n)))}].$$

But $[1_{\mathbb{L}_{\frac{m}{k}}(n)} \otimes I_p] = p[1_{\mathbb{L}_{\frac{m}{k}}(n)}]$. Thus there is a projection $e \in \mathbb{L}_{\frac{m}{k}}(n)$ such that $p[1_{\mathbb{L}_{\frac{m}{k}}(n)}] = (kp)[e]$. But $K_0(\mathbb{L}_{\frac{m}{k}}(n)) \cong \mathbb{Z}^2$ is torsion-free, so $[1_{\mathbb{L}_{\frac{m}{k}}(n)}] = k[e]$. This contradicts Theorem 2.1 if $k > 1$. Hence $\mathbb{L}_{\frac{m}{k}}(n) \otimes M_p(\mathbb{C})$ is not isomorphic to $C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_{kp}(\mathbb{C})$ if $k > 1$. □

By a similar fashion, one can easily show that no non-trivial matrix algebra can be factored out of $\mathbb{L}_{\frac{m}{k}}(n)$.

By comparison of the K-theory, one can show that $\mathbb{L}_{\frac{m}{k}}(n) \otimes \mathcal{K}(\mathcal{H})$ has a non-trivial bundle structure.

COROLLARY 2.3. *For a positive integer $k > 1$, $\mathbb{L}_{\frac{m}{k}}(n)$ is not stably isomorphic to $C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_k(\mathbb{C})$.*

PROOF. By Theorem 2.1, $K_0(\mathbb{L}_{\frac{m}{k}}(n)) \cong \mathbb{Z}^2$, which is torsion-free. On the other hand,

$$\begin{aligned} &K_0(C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_k(\mathbb{C})) \\ &\cong K_0(C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n)))) \cong K_0(L^k(n) \times S^1) \\ &\cong K_0(L^k(n)) \otimes K_0(C(S^1)) \oplus K_1(L^k(n)) \otimes K_1(C(S^1)) \end{aligned}$$

by Künneth Theorem (see [1, Theorem 23.1.3]). But

$$K_0(C(L^k(n))) \otimes K_0(C(S^1)) \cong (\mathbb{Z}/k^n\mathbb{Z} \oplus \mathbb{Z}) \otimes \mathbb{Z} \cong \mathbb{Z}/k^n\mathbb{Z} \oplus \mathbb{Z}$$

(see [4, IV.2.11]). So $K_0(C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_k(\mathbb{C}))$ is not torsion-free if $k > 1$. Hence $\mathbb{L}_{\frac{m}{k}}(n)$ is not stably isomorphic to $C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_k(\mathbb{C})$ if $k > 1$. □

We have obtained that $[1_{\mathbb{L}_{\frac{m}{k}}(n)}] \in K_0(\mathbb{L}_{\frac{m}{k}}(n))$ is primitive. This result is very useful to investigate the bundle structure of the tensor products of rational lens algebras with UHF-algebras.

3. The tensor products of rational lens algebras with UHF-algebras

In this section, we investigate the bundle structure of the tensor product of $\mathbb{L}_{\frac{m}{k}}(n)$ with a UHF-algebra M_{p^∞} of type p^∞.

The following is useful.

THEOREM 3.1 [3, Theorem 7.1]. *Suppose there exists an intertwining of the sequence of C^*-algebra homomorphisms $A_1 \to A_2 \to \cdots$ and $B_1 \to B_2 \to \cdots$. Then the inductive limits $\varinjlim A_i$ and $\varinjlim B_i$ are isomorphic.*

THEOREM 3.2. *Let M_{p^∞} be a UHF-algebra of type p^∞. Then $\mathbb{L}_{\frac{m}{k}}(n) \otimes M_{p^\infty}$ is isomorphic to $C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of k is a subset of the set of prime factors of p.*

PROOF. Assume the set of prime factors of k is a subset of the set of prime factors of p. To show that $\mathbb{L}_{\frac{m}{k}}(n) \otimes M_{p^\infty}$ is isomorphic to $C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$, it is enough to show that $\mathbb{L}_{\frac{m}{k}}(n) \otimes M_{k^\infty}$ is isomorphic to $C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_{k^\infty}$. But there exist the canonical C^*-algebra homomorphisms:

$$\mathbb{L}_{\frac{m}{k}}(n) \hookrightarrow C \otimes M_k(\mathbb{C}) \hookrightarrow \mathbb{L}_{\frac{m}{k}}(n) \otimes M_k(\mathbb{C}) \hookrightarrow C \otimes M_{k^2}(\mathbb{C}) \hookrightarrow \cdots,$$

where $C := C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n)))$. The inductive limit of the odd terms

$$\cdots \to \mathbb{L}_{\frac{m}{k}}(n) \otimes M_{k^d}(\mathbb{C}) \to \mathbb{L}_{\frac{m}{k}}(n) \otimes M_{k^{d+1}}(\mathbb{C}) \to \cdots$$

is $\mathbb{L}_{\frac{m}{k}}(n) \otimes M_{k^\infty}$, and the inductive limit of the even terms

$$\cdots \to C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_{k^d}(\mathbb{C}) \to C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_{k^{d+1}}(\mathbb{C}) \to \cdots$$

is $C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_{k^\infty}$. Thus by Theorem 3.1, $\mathbb{L}_{\frac{m}{k}}(n) \otimes M_{k^\infty}$ is isomorphic to $C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_{k^\infty}$.

Assume $\mathbb{L}_{\frac{m}{k}}(n) \otimes M_{p^\infty}$ is isomorphic to $C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$. Then the unit $1_{\mathbb{L}_{\frac{m}{k}}(n)} \otimes 1_{M_{p^\infty}}$ maps to the unit $1_{C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n)))} \otimes 1_{M_{p^\infty}} \otimes I_k$. So

$$[1_{\mathbb{L}_{\frac{m}{k}}(n)} \otimes 1_{M_{p^\infty}}] = [1_{C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n)))} \otimes 1_{M_{p^\infty}} \otimes I_k].$$

And $[1_{\mathbb{L}_{\frac{m}{k}}(n)} \otimes 1_{M_{p^\infty}}] = [1_{\mathbb{L}_{\frac{m}{k}}(n)}] \otimes [1_{M_{p^\infty}}]$ and $[1_{C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n)))} \otimes 1_{M_{p^\infty}} \otimes I_k] = k([1_{C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n)))}] \otimes [1_{M_{p^\infty}}])$. But $K_0(\mathbb{L}_{\frac{m}{k}}(n) \otimes M_{p^\infty}) \cong [\frac{1}{p}]$ $(K_0(\mathbb{L}_{\frac{m}{k}}(n)))$ and $K_0(C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_{p^\infty} \otimes M_k(\mathbb{C})) \cong k[\frac{1}{p}]$ $(K_0(C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n)))))$. If there is a prime factor q of k such that $q \nmid p$, then $[1_{M_{p^\infty}}] \neq q[e_\infty]$ for e_∞ a projection in M_{p^∞} under the assumption that the unit $1_{\mathbb{L}_{\frac{m}{k}}(n)} \otimes 1_{M_{p^\infty}}$ maps to the unit $1_{C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n)))} \otimes 1_{M_{p^\infty}} \otimes I_k$. So there is a projection $e \in \mathbb{L}_{\frac{m}{k}}(n)$ such that $[1_{\mathbb{L}_{\frac{m}{k}}(n)}] = q[e]$. This contradicts Theorem 2.1. Thus the set of prime factors of k is a subset of the set of prime factors of p.

Therefore, $\mathbb{L}_{\frac{m}{k}}(n) \otimes M_{p^\infty}$ is isomorphic to $C(\mathrm{Prim}(\mathbb{L}_{\frac{m}{k}}(n))) \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of k is a subset of the set of prime factors of p. □

We have obtained that $\mathbb{L}_{\frac{m}{k}}(n) \otimes M_{p^\infty}$ has the trivial bundle structure if and only if the set of prime factors of k is a subset of the set of prime factors of p.

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