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FROBENIUS MAP ON THE EXTENSIONS OF T-MODULES

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ABSTRACT. On the group of all extensions of elliptic modules by the Carlitz module we define Frobenius map and by using a concrete description of the extension group we give an explicit description of the Frobenius map.

1. Introduction

In order to get good duality theory on elliptic modules we have looked at extensions of elliptic modules by the Carlitz module [3]. In case of rank 2, we could find a certain subclass of extensions which fit into the theory of duality [4]. However if the rank gets bigger than the group of extensions of an elliptic module E by the Carlitz module C, written $Ext(E,C)$, becomes bigger so that we cannot use the same trick. Hence we have to try another way – we want to furnish the structure of an elliptic module to the group $Ext(E,C)$. In this paper we obtain a well defined Frobenius map on the group $Ext(E,C)$ (Theorem 3) which will be essential to define an elliptic module structure on $Ext(E,C)$. And we give an explicit expression for the Frobenius map (Proposition 2) by using concrete description of the group $Ext(E,C)$.

2. Elliptic modules and t-modules

Throughout this paper we fix the following notations: p is a fixed prime, A is the polynomial ring $\mathbb{F}_p[t]$ where $\mathbb{F}_p$ is the field of p elements, K is a perfect field containing A and T is the image of t in K. It is well

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known that the ring of endomorphisms $End_K(\mathbb{G}_a)$ is a noncommutative ring $K[\tau]$ with a commutation relation,

$$\tau x = x^p \tau \text{ for } x \in K.$$

DEFINITION 1. *An elliptic module or a Drinfeld module E of rank r is the additive group scheme $\mathbb{G}_a$ together with an A-action*

$$\psi : A \longrightarrow End_K(\mathbb{G}_a) = K[\tau]$$

such that

(1) *degree of ψ_a in τ is the same as $deg(a)r$, where we denote the image of $a \in A$ by ψ_a,*
(2) *the constant term of ψ_a is the same as a.*

If (E_1, ψ_1) and (E_2, ψ_2) are elliptic modules then a morphism from E_1 to E_2 is defined to be an endomorphism u of $\mathbb{G}_a$ such that

$$u \circ \psi_1 = \psi_2 \circ u.$$

See [2] for more detail.

Anderson [1] gave a definition of higher dimensional analogue of elliptic modules.

DEFINITION 2. *An abelian t-module over K is the A-module valued functor E such that*

(1) *as a group valued functor E is isomorphic to $\mathbb{G}_a^n$ for some n,*
(2) *$(t-T)^n$ $Lie(E) = 0$ for some positive integer n,*
(3) *there is a finite dimensional subspace V of the group $Hom(E, \mathbb{G}_a)$ of the morphisms of K-algebraic groups such that*

$$Hom(E, \mathbb{G}_a) = \sum_{j=0}^{\infty} V \circ t^j.$$

A *morphism* between t-modules is simply a natural transformation of the functors.

Let $K[t, \tau]$ be the noncommutative ring generated by t and τ with the relations generated by

$$t\tau = \tau t, xt = tx, \tau x = x^p \tau \text{ for } x \in K.$$

We will often write R for the ring $K[t, \tau]$.

Definition 3. *A t-motive M is a left $K[t,\tau]$-module with the following properties,*

(1) *M is free of finite rank over $K\ [t\ ]$,*
(2) *$(t-T)^N(M/\tau M)=0$ for some positive integer N,*
(3) *M is finitely generated over $K[\tau]$.*

A *morphism* between t-motives is a $K[t,\tau]$-linear map.

Anderson [1] showed that the category of t-modules is anti-equivalent to the category of t-motives. To state his theorem let E be a t-module and let $M(E)$ be the set of all morphisms from E to $\mathbb{G}_a$ of K-algebraic groups equipped with $K[t,\tau]$-module structure with

$$\begin{cases} (xm)(e) = x(m(e)), \\ \tau(m)(e) = m(e)^p, \ \text{for } e \in E \\ tm(e) = m(t(e)). \end{cases}$$

Theorem 1 (Anderson). *The functor sending E to M(E) is an anti-equivalence of the categories between t-modules and t-motives.*

In [3], we have shown that an extension of a t-module by another t-module is again a t-module:

Proposition 1. *Let M_1 and M_2 be t-motives. If*

$$0 \to M_1 \to M \to M_2 \to 0$$

is an exact sequence of $K[t,\ \tau]$-modules then M is again a t-motive. In particular, if E_1 and E_2 are t-modules then, we have an isomorphism

$$Ext_{t-mod}(E_1,E_2) \cong Ext_{K[t,\tau]}(M(E_1),M(E_2)).$$

3. Frobenius map

Let C be the Carlitz module. We define $C^{(p)}$ so that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\ \tau\ } & C^{(p)} \\ {\scriptstyle \psi_t^C}\downarrow & & \downarrow{\scriptstyle \psi_t^{C^{(p)}}} \\ C & \xrightarrow{\ \tau\ } & C^{(p)} \end{array}$$

commutes. Hence $C^{(p)}$ is the additive group scheme $\mathbb{G}_a$ together with the t-action given by

$$\psi_t^{C^{(p)}} = \tau + T^p.$$

Note that $C^{(p)}$ is not an elliptic module since the constant term of $\psi_t^{C^{(p)}}$ is not the same as T. Using the A-module scheme we want to define $Ext^{(p)}(E, C)$. It should consist of all push outs of extensions of C by E. Namely $Ext^{(p)}(E, C)$ consists of push outs of the extensions of the form

$$\mathcal{E} : 0 \to C \to \mathcal{E} \to E \to 0$$

under the map $\tau : C \longrightarrow C^{(p)}$. That is $Ext^{(p)}(E, C)$ consists of the corresponding extension $\mathcal{E}^{(p)}$:

$$\begin{array}{ccccccccc} \mathcal{E} : 0 & \longrightarrow & C & \longrightarrow & \mathcal{E} & \longrightarrow & E & \longrightarrow & 0 \\ & & \tau \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{E}^{(p)} : 0 & \longrightarrow & C^{(p)} & \longrightarrow & \mathcal{E} & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

With this motivation, we define $Ext^{(p)}(E, C)$ to be the set of all extensions of E by $C^{(p)}$. That is, we define

$$Ext^{(p)}(E, C) = Ext(E, C^{(p)}).$$

We want to compute the group $Ext^{(p)}(E, C)$ explicitly. We let

$$M(C^{(p)}) = Hom_{alg.group}(C^{(p)}, \mathbb{G}_a).$$

Furnish $M(C^{(p)})$ with $R(= K[t, \tau])$ module structure as we did for $M(E)$ (see [3]).

We have a free resolution of $M(C^{(p)})$:

$$0 \to R \xrightarrow{d_1 = t - \psi_t^{C^{(p)}}} R \xrightarrow{\pi} M(C^{(p)}) \to 0.$$

Here $\pi(\sum a_{ij} t^i \tau^j) = \sum a_{ij} (\psi_t^{C^{(p)}})^i \tau^j$ and $d_1 = t - \psi_t^{C^{(p)}}$. To compute $Ext^{(p)}(E, C)$ we apply the functor $Hom_R(-, M(E))$ to get

$$Hom(M(C^{(p)}), M(E)) \xrightarrow{\pi^*} Hom(R, M(E)) \xrightarrow{d_1^*} Hom(R, M(E)) \xrightarrow{d_2^*} 0.$$

THEOREM 2. *Let E be an elliptic module of rank r and C be the Carlitz module. Then $Ext^1(E, C^{(p)})$ is isomorphic to $K[\tau]/\mathcal{B}$ where $\mathcal{B} = \{\alpha\psi_t^E - \psi_t^{C^{(p)}}\alpha \mid \alpha \in K[\tau]\}$. Further this group is isomorphic to K^r as an (additive) abelian group.*

PROOF. The proof of this is almost the same as the proof of Theorem 3 of [3]. For the first assertion we note that $Hom(R, M(E)) = K[\tau]$ and the image of d_1^* is precisely given by $\mathcal{B}$.

To prove the last assertion, we claim that for a given f there is a unique α such that the degree of $(f - (\alpha\psi_t^E - \psi_t^{C^{(p)}}\alpha))$ is less than r which is the rank of E. To prove this we use induction on the degree of f. If the degree of f is less than r, then we can choose α to be 0. Now suppose that $deg(f) = n+1$. Since we can write $f = b_{n+1}\tau^{n+1} + f_n$ where f_n is a polynomial in τ of degree less than or equal to n and since we are assuming our assertion for f_n, we only need to prove our assertion for $b_{n+1}\tau^{n+1}$. First assume $(n+1) < 2r$. Then by Euclidean algorithm (see [1]) in $K[\tau]$ we see that there are unique α and γ' in $K[\tau]$ such that

$$b_{n+1}\tau^{n+1} = \alpha\psi_t^E + \gamma' \text{ and } deg(\gamma') < r,$$

where $deg(\gamma') < r$ and $deg(\alpha) < r$ since $(n+1) < 2r$. Therefore

$$b_{n+1}\tau^{n+1} = \alpha\psi_t^E - \psi_t^{C^{(p)}}\alpha + \gamma,$$

where $\gamma = \gamma' + \psi_t^{C^{(p)}}\alpha$. Now proceed in the same way to get rid of our extra assumption that $(n+1) < 2r$.

For the last assertion, we simply notice that the map

$$Ext^1(E, C) = K[\tau]/\mathcal{B} \longrightarrow K^r$$

sending f to the coefficients of $(f - (\alpha\psi_t^E - \psi_t^{C^{(p)}}\alpha))$ is obviously an isomorphism of abelian groups. □

Next we want to describe the Frobenius map on $Ext(E, C) \to Ext^{(p)}$

(E, C). For this consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow[d_1 = t - \psi_t^{C(p)}]{} & > R & \xrightarrow{\pi^{(p)}} & M(C^{(p)}) & \longrightarrow & 0 \\
 & & \Big\downarrow \rho_\tau & & \Big\downarrow \rho_\tau & & \Big\downarrow \rho_\tau & & \\
0 & \longrightarrow & R & \xrightarrow[d_1 = t - \psi_t^{C}]{} & > R & \xrightarrow{\pi} & M(C) & \longrightarrow & 0
\end{array}$$

Here ρ_τ is the right multiplication by τ which is R-linear. Applying $Hom_R(-, M(E))$, we have a commutative diagram:

$$\begin{array}{ccccccc}
Hom(M(C), M(E)) & \xrightarrow{\pi^*} & Hom(R, M(E)) & \xrightarrow{d_1^*} & Hom(R, M(E)) & \xrightarrow{d_2^*} & 0 \\
\Big\downarrow \rho_\tau^* & & \Big\downarrow \rho_\tau^* & & \Big\downarrow \rho_\tau^* & & \\
Hom(M(C^{(p)}), M(E)) & \xrightarrow[\pi^{(p)*}]{} & Hom(R, M(E)) & \xrightarrow[d_1^*]{} & vHom(R, M(E)) & \xrightarrow[d_2^*]{} & 0
\end{array}$$

Commutativity of the diagram implies that ρ_τ^* induces a map which we will denote by $\tilde{\tau}$. We summarize these facts in:

THEOREM 3. *Right multiplication by τ induces a map from $Ext(E, C)$ to $Ext^{(p)}(E, C)$. That is we have an additive group homomorphism $\tilde{\tau}$ which is induced by ρ_τ, right multiplication by τ,*

$$\tilde{\tau} : Ext(E, C) \to Ext^{(p)}(E, C).$$

Using the identification

$$Ext(E, C) = K[\tau]/\{\alpha\psi_t^E - \psi_t^C \alpha \mid \alpha \in K[\tau]\} \text{ and}$$
$$Ext^{(p)}(E, C) = K[\tau]/\{\alpha\psi_t^E - \psi_t^{C^{(p)}} \alpha \mid \alpha \in K[\tau]\},$$

we want to describe the map $\tilde{\tau}$ explicitly.

PROPOSITION 2. *Let $\psi_t^E = T + a_1\tau + a_2\tau^2 + \cdots + a_r\tau^r$ and $b_0 + b_1\tau + b_2\tau^2 + \cdots + b_{r-1}\tau^{r-1}$ be an element of $Ext(E, C)$. Let $\alpha = b_{r-1}^p / a_r$. Then we have*

$$\begin{aligned}
&\tilde{\tau}(b_0 + b_1\tau + b_2\tau^2 + \cdots + b_{r-1}\tau^{r-1}) \\
=&(T^p\alpha - T\alpha) + (\alpha^p + b_0^p - b_1\alpha)\tau + (b_1^p - b_2\alpha)\tau^2 \\
&+ (b_2^p - b_3\alpha)\tau^3 + \cdots + (b_{r-2}^p - b_{r-1}\alpha)\tau^r.
\end{aligned}$$

PROOF. To prove this formula, we chase the maps carefully. First we claim $\rho_\tau^*(\alpha) = \tau \cdot \alpha$. (Despite the fact that ρ_τ^* is induced by the right multiplication it turns out to be the left multiplication.) In fact, ρ_τ^* is the composition,

$$R \xrightarrow{\rho_\tau} R \xrightarrow{\alpha} M(E).$$

The first map maps r to $r\tau$ and the second map maps r to $r\alpha(1)$. (Here we identify $\alpha \in Hom(R, M(E))$ with $\alpha(1)$.) Hence the composition maps 1 to $\tau \cdot \alpha(1)$ as desired.

Now we compute

$$\begin{aligned}\tilde{\tau}(b_{r-1}\tau^r) &= b_{r-1}^p\tau^r - (\alpha\psi_t^E - \psi_t^{C^{(p)}}\alpha) \\ &= b_{r-1}^p\tau^r - \{(\alpha T + \alpha b_1\tau + \cdots + b_{r-1}^p\tau^r) - (T^p + \tau)\alpha\} \\ &= (T^p\alpha - T\alpha) + (\alpha^p - \alpha b_1)\tau - \alpha b_2\tau^2 - \cdots - \alpha b_{r-1}\tau^{r-1}.\end{aligned}$$

Now the result follows immediately. □

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