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Projective Objects in the Category of Compact Spaces and $\sigma Z^{\#}$-irreducible Maps*

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Abstract

Observing that for any compact space X, the minimal basically disconnected cover $\Lambda_X : \Lambda X \rightarrow X$ is $\sigma Z^{\#}$-irreducible, we will show that the projective objects in the category of compact spaces and $\sigma Z^{\#}$-irreducible maps are precisely basically disconnected spaces.

0. Historical background and introduction

Gleason [4] showed that the projective objects in the category of compact spaces and continuous maps are precisely the extremally disconnected spaces and that each compact space has an essentially unique projective cover, namely its absolute (EX, k_X). Iliads [6] and Banaschewski [2] proved similar results for the category of Hausdorff spaces (regular spaces, resp.) and perfect maps(See Chapters 6 and 9 of [8] for an extensive discussion of this topic). Henriksen, Vermeer and Woods [6] showed that the quasi F-spaces are the projective objects in the category $\underline{\text{Tych}_z}$ of Tychonoff spaces and $Z^{\#}$-irreducible maps and that a Tychonoff space X is a project cover in $\underline{\text{Tych}_z}$ if and only if $QF(\beta X) = \beta(QF(X))$.

The purpose in this paper is to show that the projective objects in the category of compact spaces and $\sigma Z^{\#}$-irreducible maps are precisely basically disconnected spaces. For the terminology, we refer to [1] and [8].

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All spaces in this paper are Tychonoff spaces and for any Tychonoff space X, $\beta_X : X \to \beta X$ denotes the Stone-Čech Compactification of X.

1. $\sigma Z^{\#}$-irreducible maps

For any space X, let $Z(X)$ denote the set of all zero-sets in X, $R(X)$ the set of all regular closed sets in X and $Z(X)^{\#} = \{ \mathrm{cl}_X(\mathrm{int}_X(Z)) : Z \in Z(X)\}$. Then $R(X)$ is a complete Boolean algebra, in which $\vee$, $\wedge$ and $'$ are defined as follows: if $A \in R(X)$ and $\{A_i : i \in I\} \subseteq R(X)$, then $\vee\{A_i : i \in I\} = \mathrm{cl}_X(\cup\{A_i : i \in I\})$, $\wedge\{A_i : i \in I\} = \mathrm{cl}_X(\mathrm{int}_X(\cap\{A_i : i \in I\}))$ and $A' = \mathrm{cl}_X(X - A)$. For any space X, $Z(X)^{\#}$ is a sublattice of $R(X)$.

A Boolean algebra L is called *σ-complete* if L has countable joins and hence countable meets. We note that any intersection of σ-complete Boolean subalgebra of a complete Boolean algebra L is again σ-complete and hence for any sublattice M of a complete Boolean algebra L, there is the smallest σ-complete Boolean subalgebra of L containing M, which will be denoted by σM.

Definition 1.1. (a) A map $f : Y \to X$ is called a *covering map* if it is a perfect irreducible continuous surjection.

(b) A covering map $f : Y \to X$ is called *$Z^{\#}$-irreducible* if $\{f(A) : A \in Z(Y)^{\#}\} = Z(X)^{\#}$.

(c) A covering map $f : Y \to X$ is called *$\sigma Z^{\#}$-irreducible* if $\{f(A) : A \in \sigma Z(Y)^{\#}\} = \sigma Z(X)^{\#}$.

For any covering map $f : Y \to X$, $\mathcal{A} \subseteq P(Y)$ and $\mathcal{B} \subseteq P(X)$, let $f(\mathcal{A})$ denote $\{f(A) : A \in \mathcal{A}\}$ and $f^{-1}(\mathcal{B})$ denote $\{\mathrm{cl}_Y(f^{-1}(\mathrm{int}_X(B))) : B \in \mathcal{B}\}$.

For any covering map $f : Y \to X$, $f^{-1}(Z(X)^{\#}) \subseteq Z(Y)^{\#}$ and hence $\sigma Z(X)^{\#} \subseteq f(\sigma Z(Y)^{\#})$. Thus a covering map $f : Y \to X$ is $\sigma Z^{\#}$-irreducible if and only if $f(\sigma Z(Y)^{\#}) \subseteq \sigma Z(X)^{\#}$ and if f is $Z^{\#}$-irreducible, then it is $\sigma Z^{\#}$-irreducible.

Proposition 1.2. For any covering maps $g: Y \to W$, $h: W \to X$, $h \circ g$ is $\sigma Z^{\#}$-irreducible if and only if h and g are $\sigma Z^{\#}$-irreducible.

Proof. Assume that $h \circ g$ is $\sigma Z^{\#}$-irreducible and take any $A \in \sigma Z(W)^{\#}$. Since g is a covering map, $\mathrm{cl}_Y(g^{-1}(\mathrm{int}_W(A))) \in \sigma Z(Y)^{\#}$. Then $h \circ g(\mathrm{cl}_Y g^{-1}(\mathrm{int}_W(A))) = h(A) \in \sigma Z(X)^{\#}$ [6] and hence h is $\sigma Z^{\#}$-irreducible. Let $B \in \sigma Z(Y)^{\#}$. Then $h(g(B)) \in \sigma Z(X)^{\#}$ and hence $\mathrm{cl}_W(h^{-1}(\mathrm{int}_X(h(g(A))))) = g(A) \in \sigma Z(W)^{\#}$. So h is $\sigma Z^{\#}$-irreducible. The converse is trivial.

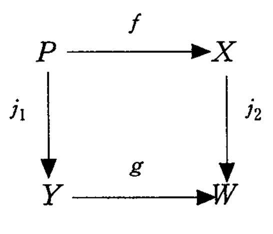
Definition 1.3. Let X be a subspace of a topological space Y. Then X is called *$\sigma Z^{\#}$-embedded* in Y if for any $A \in Z(X)^{\#}$, there is a $B \in \sigma Z(Y)^{\#}$, with $A = B \cap X$.

A subspace X of a space Y is called *Z-embedded*(*$Z^{\#}$-embedded*, resp.) in Y if for any A in $Z(X)$($Z(X)^{\#}$, resp.), there is a B in $Z(Y)$($Z(Y)^{\#}$, resp.) with $A = B \cap X$.

Proposition 1.4. If X is a dense Z-embedded subspace of a space Y, then X is $\sigma Z^{\#}$-embedded in Y.

Proof. Since X is a dense subspace in Y, the map $\phi: R(Y) \to R(X)$, defined by $\phi(A) = A \cap X$, is a Boolean isomorphism [6] and since X is a dense Z-embedded subspace of Y, X is $Z^{\#}$-embedded in Y. So X is $\sigma Z^{\#}$-embedded in Y.

Theorem 1.5. Consider the following commutative diagram:



where j_1, j_2 are dense embeddings and f, g are covering maps. Then g is $\sigma Z^{\#}$-irreducible and P is $\sigma Z^{\#}$-embedded in Y if and only if f is $\sigma Z^{\#}$-irreducible and X is $\sigma Z^{\#}$-embedded in W.

Proof. ($\Rightarrow$) Take any $A \in \sigma Z(P)^{\#}$. Since P is $\sigma Z^{\#}$-embedded in Y, there is a $B \in \sigma Z(Y)^{\#}$ such that $A = B \cap P$. Note that $f(A) = f(B \cap P) = g(B) \cap X$ [8]. Since g is $\sigma Z^{\#}$-irreducible, $f(A) \in \sigma Z(X)^{\#}$. Thus f is $\sigma Z^{\#}$-irreducible. Let $C \in \sigma Z(X)^{\#}$. Then $\mathrm{cl}_P(f^{-1}(\mathrm{int}_X(C))) \in \sigma Z(P)^{\#}$. Since P is $\sigma Z^{\#}$-embedded in Y, there is a $D \in \sigma Z(Y)^{\#}$ such that $D \cap P = \mathrm{cl}_P(f^{-1}(\mathrm{int}_X(C)))$. Then $C = f(D \cap P) = g(D) \cap X$. Since g is $\sigma Z^{\#}$-irreducible, $g(D) \in \sigma Z(W)^{\#}$; therefore X is $\sigma Z^{\#}$-embedded in W.

($\Leftarrow$) Take any $A \in \sigma Z(Y)^{\#}$. Since j_1 is a dense embedding, $A \cap P \in \sigma Z(P)^{\#}$ and $f(A \cap P) = g(A \cap P) = g(A) \cap X$. Since f is $\sigma Z^{\#}$-irreducible, $g(A) \cap X \in \sigma Z(X)^{\#}$. Since X is $\sigma Z^{\#}$-embedded in W, there is $B \in \sigma Z(W)^{\#}$ with $g(A) \cap X = B \cap X$. Since j_2 is a dense embedding and $g(A)$, B are regular closed in W, $g(A) = B$. Thus g is $\sigma Z^{\#}$-irreducible.

Take any $C \in \sigma Z(P)^{\#}$. Since f is a $\sigma Z^{\#}$-irreducible, $f(C) \in \sigma Z(X)^{\#}$. Since X is $\sigma Z^{\#}$-embedded in W, there is $D \in \sigma Z(W)^{\#}$ such that $f(C) = D \cap X$. Since g is a covering map, $\mathrm{cl}_Y(g^{-1}(\mathrm{int}_W(D))) \in \sigma Z(Y)^{\#}$. Then $f(\mathrm{cl}_Y(g^{-1}(\mathrm{int}_W(D))) \cap P) = g(\mathrm{cl}_Y(g^{-1}(\mathrm{int}_W(D)))) \cap X = D \cap X = f(C)$. Hence $\mathrm{cl}_Y(g^{-1}(\mathrm{int}_W(D))) \cap P = C$. Thus P is $\sigma Z^{\#}$-embedded in Y.

2. Project objects in $\underline{\mathrm{Comp}_\sigma}$

A space X is said to be *basically disconnected* if for any zero-set Z in X, $\mathrm{int}_X(Z)$ is closed in X. It is known that X is basically disconnected if and only if βX is basically disconnected [9].

Lemma 2.1. A space X is basically disconnected if and only if $Z(X)^{\#} = \sigma Z(X)^{\#} = B(X)$, where $B(X)$ is the set of all clopen sets in X.

Proof. Suppose that X is a basically disconnected space. Clearly, $Z(X)^{\#} \subseteq \sigma Z(X)^{\#}$ and $B(X) = Z(X)^{\#}$. Enough to show that $Z(X)^{\#}$ is a σ-complete Boolean algebra. Take any sequence $\{A_n : n \in N\}$ in $Z(X)^{\#}$. Then $\vee\{A_n : n \in N\}$

$= \mathrm{cl}_X(\bigcup\{A_n : n \in N\})$. For any $n \in N$, A_n is a cozero-set in X and hence $\bigcup\{A_n : n \in N\}$ is a cozero-set in X. Since X is basically disconnected, $\mathrm{cl}_X(\bigcup\{A_n : n \in N\})$ is clopen in X. Thus $\bigvee\{A_n : n \in N\} \in Z(X)^{\#}$.

Recall that a pair (Y, f) is said to be a *cover* of a space X if $f : Y \to X$ is a covering map.

Definition 2.2. For any X, (a) a pair (Y, f) is called a *basically disconnected cover of* X if (Y, f) is a cover of X and Y is a basically disconnected space.

(b) a basically disconnected cover (Y, f) is called a *minimal basically disconnected cover of* X if for any basically disconnected cover (Z, g) of X, there is a covering $h : Z \to Y$ with $f \circ h = g$.

For any compact space X, let ΛX denote the Stone-space $S(\sigma Z(X)^{\#})$ of $\sigma Z(X)^{\#}$ and $\Lambda_X(\alpha) = \bigcap \alpha$. Then $(\Lambda X, \Lambda_X)$ is the minimal basically disconnected cover of a compact space X [9].

Proposition 2.3. Let X be a compact space. Then for any $A \in \sigma Z(X)^{\#}$, $\mathrm{cl}_{\Lambda X}(\Lambda_X^{-1}(\mathrm{int}_X(A))) = A^*$, where $A^* = \{\alpha : \alpha \in A\}$.

Proof. Let $A \in \sigma Z(X)^{\#}$. Clearly, for any $\alpha \in A^*$, $\Lambda_X(\alpha) \in A$. Let $x \in \mathrm{int}_X(A)$. Since Λ_X is onto, there is an $\alpha \in \Lambda X$ with $\Lambda_X(\alpha) = x$. For any $C \in \alpha$, $x \in \mathrm{cl}_X(\mathrm{int}_X(C))$ and hence $\mathrm{int}_X(C) \cap \mathrm{int}_X(A) \neq \phi$. Thus for any $C \in \alpha$, $C \wedge A \neq \phi$. Since α is a $\sigma Z(X)^{\#}$-ultrafilter, $A \in \alpha$. Then $x = \Lambda_X(\alpha) \in \Lambda_X(A^*)$ and so $\mathrm{int}(A) \subseteq \Lambda_X(A^*)$. Since Λ_X is closed, $\mathrm{cl}_X(\mathrm{int}_X(A)) \subseteq \Lambda_X(A^*)$. Thus $\Lambda_X(A^*) = A$. Since $\Lambda_X(\mathrm{cl}_{\Lambda X}(\Lambda_X^{-1}(\mathrm{int}_X(A)))) = A$, A^* is clopen in ΛX and Λ_X is a covering map, $A^* = \mathrm{cl}_{\Lambda X}(\Lambda_X^{-1}(\mathrm{int}_X(A)))$.

By the above proposition, we have the following :

Corollary 2.4. For any compact space X, $\Lambda_X : \Lambda X \to X$ is $\sigma Z^{\#}$-irreducible.

In [7], it is shown that for any space X, $\Lambda^{-1}(X)$ is C^*-embedded in $\Lambda\beta X$ if and only if $\Lambda^{-1}(X)$ is $Z^\#$-embedded (or Z-embedded) in $\Lambda\beta X$, where $(\Lambda\beta X, \Lambda)$ is the minimal basically disconnected cover of βX.

Theorem 2.5. For any space X, $\Lambda_X : \Lambda X \to X$ is $\sigma Z^\#$-irreducible if and only if $\Lambda^{-1}(X)$ is C^*-embedded in $\Lambda\beta X$.

Proof. Suppose that Λ_X is $\sigma Z^\#$-irreducible. Let Λ_r be the restriction and corestriction of Λ to $\Lambda^{-1}(X)$ and X, respectively. Then there is a covering map $g: \beta\Lambda X \to \beta X$ such that $\beta_X \circ \Lambda_X = g \circ \beta_{\Lambda X}$. Since $\beta\Lambda X$ is a basically disconnected space, there is a covering map $h: \beta\Lambda X \to \Lambda\beta X$ with $\Lambda \circ h = g$. Hence there is a covering map $k: \Lambda X \to \Lambda^{-1}(X)$ with $\Lambda_r \circ k = \Lambda_X$ and $j \circ k = h \circ \beta_{\Lambda X}$, where j: $\Lambda^{-1}(X) \to \Lambda\beta X$ is the inclusion map.

Since Λ_X is $\sigma Z^\#$-irreducible, by Proposition 2.2., k and h are $\sigma Z^\#$-irreducible maps. By Theorem 1.5., $\Lambda^{-1}(X)$ is $\sigma Z^\#$-embedded in $\Lambda\beta X$. Take any $A \in Z(\Lambda^{-1}(X))^\#$. Then $A \in \sigma Z(\Lambda^{-1}(X))^\#$ and $\Lambda^{-1}(X)$ is $\sigma Z^\#$-embedded in $\Lambda\beta X$ and $\Lambda\beta X$ is basically disconnected, there is $B \in Z(\Lambda\beta X)^\#$ such that $A = B \cap \Lambda^{-1}(X)$. Thus $\Lambda^{-1}(X)$ is $Z^\#$-embedded in $\Lambda\beta X$ and so $\Lambda^{-1}(X)$ is C^*-embedded in $\Lambda\beta X$. The converse is trivial.

Lemma 2.6. A compact space X is basically disconnected if and only if for any zero-set Z in X, $cl_X(int_X(Z)) \cup (X-Z)$ is C^*-embedded in X.

Proof. Suppose that X is basically disconnected. Let Z be a zero set in X. Then $cl_X(int_X(Z)) \cup (X-Z)$ is dense Lindelöf subspace of X and hence $cl_X(int_X(Z)) \cup (X-Z)$ is Z-embedded in X [3]. Since X is a basically disconnected space, $cl_X(int_X(Z)) \cup (X-Z)$ is C^*-embedded in X.

Suppose that for any zero-set Z in X, $cl_X(int_X(Z)) \cup (X-Z)$ is C^*-embedded in X. Let Z be a zero-set in X and $T = cl_X(int_X(Z)) \cup (X-Z)$. Then $cl_X(int_X(Z))$ and $(X-Z)$ are disjoint clopen sets in T and hence zero-sets in T. Since T is C^*-embedded in X, $cl_X(int_X(Z)) \cap cl_X(X-Z) = (X - int_X(Z)) \cap$

$cl_X(int_X(Z)) = \phi$. So $cl_X(int_X(Z)) \subseteq int_X(Z)$. Thus X is a basically disconnected space.

Proposition 2.7. Let (Y,f) be a cover of a compact space X. If f is $\sigma Z^\#$-irreducible, then there is a $\sigma Z^\#$-irreducible map $k: \Lambda X \to Y$ with $f \circ k = \Lambda_X$.

Proof. Since X is a compact space, Y is a compact space and hence Λ_Y is $\sigma Z^\#$-irreducible. Since $f \circ \Lambda_Y$ is a covering map, there is a covering map $g: \Lambda Y \to \Lambda X$ with $f \circ \Lambda_Y = \Lambda_X \circ g$. By Proposition 1.2. and Corollary 2.4., g is $\sigma Z^\#$-irreducible. Hence g is a homeomorphism, because ΛX and ΛY are basically disconnected spaces. Let $k = \Lambda_Y \circ g^{-1}$. Then k is $\sigma Z^\#$-irreducible map and $f \circ k = \Lambda_X$.

Definition 2.8. Let $\underline{C}$ be a subcategory of the category $\underline{\text{Top}}$ of topological spaces and continuous maps and $X \in \underline{C}$. Then X is called *projective* in $\underline{C}$ if for any $Y, Z \in \underline{C}$, any morphism $f: X \to Y$ in $\underline{C}$ and onto morphism $g: Z \to Y$ in $\underline{C}$, there is a morphism $h: X \to Z$ in $\underline{C}$ with $g \circ h = f$.

Let $\underline{\text{Comp}_\sigma}$ be the category of compact spaces and $\sigma Z^\#$-irreducible maps.

Theorem 2.9. In $\underline{\text{Comp}_\sigma}$, the projective objects are precisely the basically disconnected spaces.

Proof. Let X be a basically disconnected space, $f: X \to Y$ and $g: Z \to Y$ $\sigma Z^\#$-irreducible maps. By Proposition 2.7., there is a $\sigma Z^\#$-irreducible map $k: \Lambda Y \to Z$ with $g \circ k = \Lambda_Y$. Since (X,f) is a basically disconnected cover of Y, there is a covering map $h: X \to \Lambda Y$ with $f = \Lambda_Y \circ h$. By Proposition 1.2., h is a $\sigma Z^\#$-irreducible map. Thus X is a projective object in $\underline{\text{Comp}_\sigma}$.

Suppose that X is a projective object in $\underline{\text{Comp}_\sigma}$ and X is not basically disconnected. By Lemma 2.6., there is a zero-set Z in X such that $(X-Z) \cup cl_X(int_X(Z))$ is not C^*-embedded in X. Let $T = (X-Z) \cup cl_X(int_X(Z))$. Then $\beta T \neq X$ and there is a continuous map $g: \beta T \to X$ with $g \circ \beta_T = j_T$, where $j_T: T \to X$ is the inclusion map. Since X is a compact space, T is a Lindelöf space and hence T is Z-embedded in X. Since T is dense in X, T is $Z^{\#}$-embedded in X. Take any zero-set A in βT. Then there is a zero-set B in X such that $\mathrm{cl}_{\beta T}(\mathrm{int}_{\beta T}(A)) \cap T = \mathrm{cl}_X(\mathrm{int}_X(B)) \cap T$. Since g is a covering map, $g(\mathrm{cl}_{\beta T}((\mathrm{int}_{\beta T}(A)) \cap T) = g(\mathrm{cl}_{\beta T}(\mathrm{int}_{\beta T}(A))) \cap T = \mathrm{cl}_X(\mathrm{int}_X(B)) \cap T$.

Since $g(\mathrm{cl}_{\beta T}((\mathrm{int}_{\beta T}(A))) \cap T = \mathrm{cl}_X(\mathrm{int}_X(B)) \cap T$ and T is dense in X, $g(\mathrm{cl}_{\beta T}(\mathrm{int}_{\beta T}(A))) = \mathrm{cl}_X(\mathrm{int}_X(B))$. So g is $Z^{\#}$-irreducible and hence $\sigma Z^{\#}$-irreducible. Since the identity map $1_X : X \to X$ belongs to $\underline{\mathrm{Comp}}_{\sigma}$, there is a $\sigma Z^{\#}$-irreducible map $h : X \to \beta T$ with $g \circ h = 1_X$. Hence h is a homeomorphism, that is, $\beta T = X$. This is a contradiction.

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