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학술논문

A New Metric for A Class of 2-D Parametric Curves

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ABSTRACT

We propose the area between a pair of non-self-intersecting 2-D parametric curves with same endpoints as an alternative distance metric between the curves. This metric is used when a curve is approximated with another in a simpler form to evaluate how good the approximation is. The traditional set-theoretic Hausdorff distance can be defined for any pair of curves but requires expensive calculations. Our proposed metric is not only intuitively appealing but also very easy to numerically compute. We present the numerical schemes and test it on some examples to show that our proposed metric converges in a few steps within a high accuracy.

Key words : CAD, Curve, Hausdorff distance

1. Introduction

In Computer Aided Geometric Design and its various engineering applications, it is common to approximate the shape of a curve with another curve in a simpler form. Naturally we need a metric to quantify how good the approximation is. This is a motivation for defining a "distance" metric between two curves. The traditional set-theoretic distance applied to the class of parametric curves is the so-called Hausdorff distance. Hausdorff distance is defined as for parametric curves $\gamma_1(t)$, $\gamma_1(s)$, where $d(x, y)$ is any metric on 2-dimensional plane.

$$dist(\gamma_1, \gamma_2) = \max\{(\max_t \min_s d(\gamma_1(t), \gamma_2(s)) \\ (\max_s \min_t d(\gamma_1(t), \gamma_2(s))\}$$

Although Hausdorff distance has been the most widely known, it has been pointed out[4] to be very expensive to compute, requiring to compute the distance between every point on one curve and every point on the other. There has been a few attempts to overcome this computational difficulty, but none of them is satisfactory. Emery[2] developed an algorithm to compute the distance between piecewise linear curves and showed that a curve with a bounded curvature can be approximated by a piecewise linear curve to arbitrary accuracy. The implementation of their algorithm requires a 2-step procedure and the overall efficiency is not known. Fritsch and Nielson[4] proposed another approach which takes a finite number of pairs of sampling points equally spaced in arclength along each curve, but there is no theoretical grounds to support the adequacy of their arbitrary sampling points as well as their convergence to the correct distance.

In this paper, we propose the area between two curves as a new alternative distance metric. For several years, one of the authors has developed computerized font design and generation system for Korean and Chinese characters. In this font system, each font designed by a font designer is represented as a mathematical entity: the path of the tip of the brush is the medial axis of the curves representing the boundary shape and the size or the thickness of the brush is the radius of the medial axis transform. To determine how good our mathematical approximation is for the original font, we tried to use Hausdorff distance but it was hard to compute. As an alternative, it was very natural to consider the area between two

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curves as a measure of the goodness of fit and we found this alternative computationally satisfactory. While the Hausdorff distance can be defined between any curves, it is very hard to compute except for a very restricted class of curves, let's say, a class of piecewise linear curves. On the other hand, our proposed metric is defined for a pair of non-self-intersecting curves with same endpoints but it is much easier to compute and intuitively appealing. The definition of our metric might appear to be quite restrictive. However, we think that our assumptions of non self-intersecting curves sharing same end points poses no real loss in most practical applications. Thus our proposed metric provide an alternative for 2D curves when Hausdorff distance doesn't perform well. The possibility of using the area between a pair of parametric curves as a distance is briefly mentioned in [4] as a future research, but to the best of our knowledge, no attempt has been made to further investigate this issue.

In the rest of our paper, first we give a formal definition of our proposed metric and show how to compute it using the well-known Green's theorem. Then we test our proposed metric on several examples. We use the same examples given in [4] to show the computational superiority of our proposed metric over the Hausdorff distance; the exact value of our proposed metric can be computed in only a few steps using a simple numerical integration scheme such as Gauss quadratures method.

2. Proposed Metric

Let $\gamma_1(t)$ $(t_1 \le t \le t_2)$ and $\gamma_1(s)$ $(s_1 \le s \le s_2)$ be two parametric curves without self-intersections and assume that $\gamma_1(t_1)=g_1(s_1)$ and $\gamma_1(t_2)=\gamma_2(s_2)$.

2.1 Definition 1

Let the domain closed and bounded by $\gamma_1(t)$ and $\gamma_2(s)$ be $\boldsymbol{R}$ and $d(\gamma_1, \gamma_2)$ be the distance between γ_1 and γ_2. Define $d(\gamma_1, \gamma_2)$=Area of $\boldsymbol{R}$.

The area of $\boldsymbol{R}$ can be computed by applying the well-known Green's theorem in the plane. We state the following theorem derived from Green's theorem without any proof.

2.2 Theorem 2.1

Let $\boldsymbol{R}$ be a closed bounded region in the xy-plane whose boundary $\boldsymbol{C}$ consists of finitely many smooth curves. Then

Area of $\boldsymbol{R}$

$$= \iint_R dxdy = 1/2 \int_C (xdy - ydx)$$

, where the integration of $\int_C$ is taken along the entire boundary $\boldsymbol{C}$ of $\boldsymbol{R}$ such that $\boldsymbol{R}$ is on the left as one advances in the direction of integration.

Theorem 2.1 expresses the area of a domain in terms of a line integral over the boundary. We apply Theorem 2.1 to our problem to get a working formula for area between a pair of parametric curves in the following Lemma.

2.3 Lemma 1

Let $\gamma_1(t)=(x_1(t), y_1(t))$ $(t_1 \le t \le t_2)$, $\gamma_2(s)=(x_2(s), y_2(s))$ $(s_1 \le s \le s_2)$ and $\boldsymbol{R}$ be defined as in Definition 1 as well as in Fig. 1. Then

$$\text{Area of } \boldsymbol{R} = 1/2 \int_C (xdy - ydx) = \int_C xdy$$
$$= \int_{t_1}^{t_2} x_1(t)\, dy_1(t) - \int_{s_1}^{s_2} x_2(s)\, dy_2(s).$$

Lemma 1 is obvious from the fact that if one advances on from t_1 to t_2 and then continues on from s_2 to s_1, then $\boldsymbol{R}$ is always on the left.

Remark: In case that γ_1 and γ_2 intersect at $\gamma_1(t_3)$=$\gamma_2(s_3)$ for t_1 t_3 t_2 and s_1 s_3 s_2 as in Fig. 2, $d(\gamma_1, \gamma_2)$= Area of $\boldsymbol{R}_1$+Area of $\boldsymbol{R}_2$, where the each area of $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ can be computed by Lemma 1.

As for the computational schemes of the proposed metric, there are many efficient numerical integration

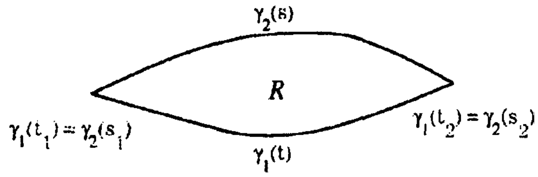

Fig. 1. Area bounded by two non-intersecting curves.

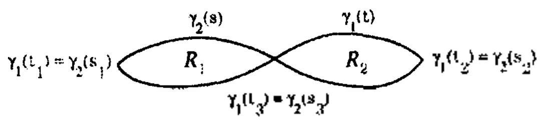

Fig. 2. Area bounded by two intersecting curves.

methods one can choose from depending on the situations. Here we use the method of Gauss quadratures. Gauss quadratures of function $f(x)$ over the interval $[a, b]$ using N points is given by

$$\int_a^b f(x)dx \cong \frac{b-a}{2}\sum_{k=1}^{N} w_k f\left(\frac{(b-a)x_k+a+b}{2}\right)$$

, where x_k are the roots of Legendre polynomial of order N and w_k are the corresponding weights. See [8]. Although Gauss quadratures do not use the equispaced intervals, it is known to have higher accuracy than other integration method and converges very fast. It is also known that in general, using more than 3 points for Gauss quadratures does not make a significant improvement in the accuracy. Thus in the following numerical examples, for the Gauss quadratures of function $f(x)$ over the given interval $[a, b]$, we fix the number of Gauss points between 3 and 5. When we want to improve the accuracy of the integration of function $f(x)$ over the given interval $[a, b]$, we divide the given interval $[a, b]$ into two subintervals $[a, \frac{a+b}{2}]$ $[\frac{a+b}{2}, b]$ and compute the Gauss quadratures of function $f(x)$ over each subinterval using the fixed number of Gauss points.

3. Numerical Examples

In this section, we study two examples given in [4]. We show that the numerical computation of our proposed metric converges very fast to the correct value even in the cases where the approaches of Fritsch and Nielson[4] fail.

3.1 First example

The first example considers two curves which are the same but with different parametrization. In this case, both the Hausdorff distance and our proposed distance should be zero. The curve, $P(t)$, is given as follows:

$$P(t)=\begin{cases}(0, 1/3)+f(1-t)(-1, 2/3), & 0 \; t \; 1; \\ (0, 1/3)+f(t-1)(1, 2/3), & 1 \; t \; 2\end{cases} \quad (1)$$

Here f is any increasing function in $C^2[0, 1]$ with $f(0)=0$, $f(1)=1$. Obviously this curve consists of two *straight line segments as shown Fig. 3.*

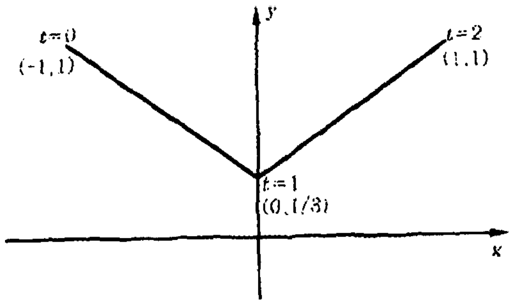

Fig. 3. Curve representing Eq. 1.

Let $f(u)=u^{k+1}$, $k>0$. Now we consider two different parameterization of $P(t)$ by choosing two different values of k. Table 1~3 summarize the results of our proposed metric in (a) and the results of Hausdorff distance obtained by Fritsch and Nielson[4] in (b). N in (a) is the total number of Gauss points taken on each curve, while N in (b) is the number of sampling points taken on each curve, More specifically, in the first iteration of Table 1(a), the original parameter interval [0, 2] of each curve is divided into two subintervals [0, 1] [1, 2] and 3 Gauss points are used on each subinterval to make the total of 6 Gauss

Table 1. Example using Eq. (1) (k=2 versus k=0)

(a)		(b)	
N	Proposed metric	N	F&N's metric
6	8.92577e-11	20	2. 06542e-2
12	2.56616e-11	40	2.12025e-3
4	6.62461e-12	80	2.42353e-4
48	1.66917e-12	160	2.90172e-5
96	4.18177e-13	320	3.55117e-6
192	1.04636e-13	640	4.39261e-7
384	2.60866e-14	1280	5.46213e-8

Table 2. Example using Eq. (1) (k=3 versus k=0)

(a)		(b)	
N	Proposed metric	N	F&N's metric
10	1.14959e-10	20	6.29210e-2
20	3.59474e-11	40	5.95409e-3
40	9.62305e-12	80	6.57588e-4
80	2.44849e-12	160	7.74469e-5
160	6.14801e-13	320	9.40142e-6
320	1.53823e-13	640	1.15822e-6
640	3.87219e-14	1280	1.43733e-7

Table 3. Example using Eq. (1) (k=2 versus k=3)

(a)		(b)	
N	Proposed metric	N	F&N's metric
10	4.42754e-11	20	4.22668e-2
20	1.58829e-11	40	3.83383e-3
40	4.45736e-12	80	4.15235e-4
80	1.14767e-12	160	4.96186e-5
160	2.89076e-13	320	6.19683e-6
320	7.23738e-14	640	7.73892e-7
640	1.80862e-14	1280	9.66810e-8

points. In each of the following iterations, each subinterval is divided into two subintervals with equal length and again 3 Gauss points are taken on each subinterval to double the total number of Gauss points. In Table 2(a) and 3(a), 5 Gauss points are used on each subinterval. Although N has a different meaning in each of (a) and (b), the complexity of algorithm is $O(N)$ for both cases. Thus we can directly compare the Tables (a) to (b) without further consideration.

One can see that there exists a consistent pattern for each case regardless of k's. The numerical scheme for our proposed metric produces a result accurate within 10^{-10} for $N \leq 10$ while that by F&N for Hausdorff distance is accurate only within 10^{-2} in for N=20 and converges slowly achieving the accuracy within 10^{-8} for N=1280. Thus our proposed metric can be computed in only a few steps. In fact each case of k=0, 2, 3 involves the integration of polynomial functions of order 1, 5, 7, respectively. It is a fact that Gauss quadratures using number of points is exact when any polynomial of order 2N-1 or less is integrated. Therefore the errors in Table 1(a), 2(a), and 3(a) are entirely due to the computational round-off errors.***

3.2 Second example

The second example is the case of two different planar curves, for which both our proposed distance and the Hausdorff distance are not zero but known.

Consider $P_1(t)=(R+a\sin Kt)\ (\cos\frac{p}{2},\ \sin\frac{p}{2})$. As t varies from 0 to 1, $P_1(t)$ draws out a perturbed quarter-circle of radius R. The perturbation has amplitude a and has K extrema over the interval. Now consider an unperturbed quarter circle

***This is pointed out by an anonymous referee.

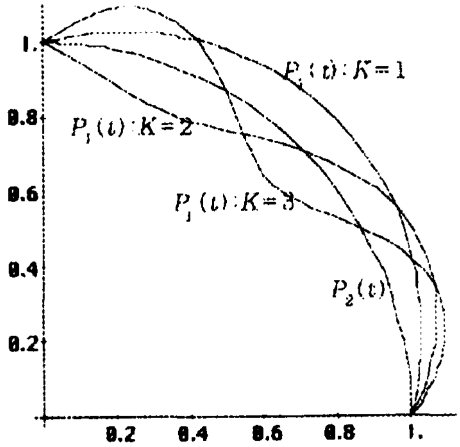

Fig. 4. $P_1(t)$ for K=1, 2, 3 and $P_2(t)$.

$$P_2(t)=R\left(\cos\frac{\pi}{2}t^2, \sin\frac{\pi}{2}t^2\right) \tag{2}$$

Fig. 4 shows the above curves when K varies from 1 to 3 with R=1 and a=0.123.

We summarize the results of the numerical solution in Table 4~6. For this example, both our proposed metric and Hausdorff distance can be analytically computed and they are given under the headings of the second columns. Table 4(a)~6(a) used four Gauss points.

Table 4. Perturbed quarter-circle example, R=1, a=0.123, K=1

(a)			(b)		
N	Proposed metric (0.128941)	Error	N	F&N's metric (0.123)	Error
4	0.128753	- 2.0e-4	20	0.123003	- 2.82e-6
8	0.128941	3.0e-9	40	0.123000	- 1.78e-7
16	0.128941	5.0e-9	80	0.123000	- 1.13e-8
32	0.128941	5.0e-9	160	0.123000	- 7.28e-10
64	0.128941	5.0e-9	320	0.123000	- 4.52e-11

Table 5. Perturbed quarter-circle example, R=1, a=0.123, K=2

(a)			(b)		
N	Proposed metric (0.123)	Error	N	F&N's metric (0.123)	Error
4	0.123330	3.3e-4	20	0.124578	- 1.58e-3
8	0.123056	5.6e-5	40	0.125826	- 2.83e-3
16	0.123000	5.7e-9	80	0.126557	- 3.56e-3
32	0.123000	5.7e-9	160	0.126575	- 3.58e-3

Table 6. Perturbed quarter-circle example, R=1, a=0.123, K=3

(a)			(b)		
N	Proposed metric (0.124980)	Error	N	F&N's metric (0.123)	Error
12	0.124977	−3.0e-6	20	0.123002	−2.20e-6
24	0.124980	4.44e-8	40	0.123002	−1.39e-7
48	0.124980	4.72e-8	80	0.123534	−5.34e-4
96	0.124980	4.72e-8	160	0.123537	−5.37e-4

*Values inside the parenthesis represent exact values.

Our proposed metric converges nicely for all K. Furthermore, for $K>1$, the Fritsch and Nielson's approach converges to some value which is larger than the correct answer. This is due to the fact that the Fritsch and Nielson's approach lacks the theoretical grounds to justify the adequacy of their arbitrary sampling points as well as their convergence to the correct distance.

4. Conclusion

In Computational Geometry, the traditional set-theoretic Hausdorff distance applied to the class of parametric curves is commonly used. However, Hausdorff distance is very expensive to compute. In this paper, we propose the area between two curves as a new alternative distance metric. While the Hausdorff distance can be defined between any curves, our proposed metric is defined for a pair of non-self-intersecting curves with same endpoints, which aren't very restrictive assumptions in the most of the practical applications. The biggest advantage of our proposed metric over Hausdorff distance is that it is much easier to compute and intuitively appealing. As shown in Section 3, a simple numerical scheme converges fast with high accuracy.

As a future work, our idea could be extended to computing the distance between a pair of 3-D surfaces. In this case the volume between the surfaces would be an alternative distance.

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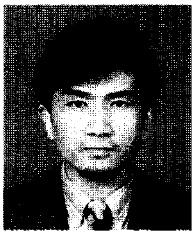

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