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A Consistent Test for Linearity for a Class of General First Order Nonlinear Time Series †

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ABSTRACT

Problem of testing linearity among general class of first order nonlinear time series models is discussed. The null hypotheses of linearity is identified via conditional expectations. A consistent test is then suggested and relevant limiting results are derived. It is worth indicating that any specific alternatives are not specified.

Keywords: Test for linearity; Consistent test; Martingale CLT; First order nonlinear time series

1. INTRODUCTION

Consider the zero mean stationary time series $\{X_t\}$ generated by the equation :

$$X_t = F(X_{t-1}, Z_t\ ;\ \theta) + \epsilon_t \tag{1.1}$$

where θ is a vector of model parameters, $\{(Z_t, \epsilon_t)\}$ are unobservable zero mean i.i.d. random vectors with $\{Z_t\}$ representing the uncertainty about the parameter θ. This class of model in (1.1) embodies standard nonlinear time series models by appropriate choices of the real valued function F. Threshold autoregression, exponential autoregression and random coefficient autoregression are seen to the very special cases of (1.1). See Tong(1990) for these and related models. Random coefficient threshold(exponential) models and generalized random coefficient processes suggested by Hwang and Basawa (1997) are also included in model (1.1).

Notice that (1.1) is a Markov process taking values in R and hence one may impose conditions directly on F for stationarity due to Tweedie (1975) concerning the condition under which there exist a unique distribution for X_0 such that the process $\{X_t,\ t \geq 0\}$ is stationary and ergodic. It will be assumed throughout that

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(C.0) The model (1.1) is a zero mean stationary and ergodic time series.

As with F, F is assumed completely unknown but satisfies the stationarity condition. Consequently, (1.1) can be viewed as a fairly general class of first order nonlinear time series including random coefficient models.

The goal of this paper is to construct a consistent test for linearity. To be more specific, given the data $\{X_0, X_1, \cdots, X_n\}$ from (1.1), we are concerned with testing hypothesis

$$\begin{aligned} H_0 &: F = \rho X_{t-1} \quad (linear) \\ H_1 &: not \quad H_0 \quad (nonlinear) \end{aligned} \tag{1.2}$$

where ρ is the ACF of lag 1 which can be consistently (both under H_0 and under H_1) estimated by sample ACF of lag 1

$$\hat{\rho} = \frac{\sum_{t=1}^{n} X_t X_{t-1}}{\sum_{t=0}^{n} X_t^2} \tag{1.3}$$

In time series literature, many authors discussed testing linearity problem. Ashley et al. (1986), Hinich (1982) and Keenan (1985) among others developed good result for testing (1.2). Their methods, however, assume specific nonlinear time series as alternatives. Diagnostic type linearity testing based on the squared residuals among ARMA models are proposed by Granger and Andersen (1978). Refer to Tong (1990, ch.5) for comprehensive treatments on the linearity testing and references therein. More recently, An and Cheng (1991) developed a Kolmogorov-Smirnov type statistic for testing linearity among fixed coefficient first order nonlinear models. Seemingly related problems may arise : test for conditional heteroscedasticity are considered by Chen and An (1997) for first order ARCH models and Chen (1997) for the p-th order ARCH setting. Testing randomness of the coefficient is on the similar vein(See, Nicholls and Quinn (1982) and Park et al. (1998), for instance).

We here, unlike the works mentioned above, do not assume any specific alternatives in the class of general first order nonlinear time series including both fixed and random coefficient models. Furthermore, the consistency of the test statistic will be focused on. Our results can also be viewed as a generalization of those for randomness testing and for test of conditional heteroscedasticity.

2. MAIN RESULTS

For testing (1.2) based on the data $\{X_0, X_1, \cdots, X_n\}$ from model (1.1), we first identify the hypotheses in terms of conditional expectations.

Define

$$u_t = X_t - \rho X_{t-1} \tag{2.1}$$

and

$$v_t = X_t^2 - \rho^2 X_{t-1}^2 - \sigma_\epsilon^2 \tag{2.2}$$

where $\sigma_\epsilon^2 = Var(\epsilon_t)$.

The conditional expectation of a random quantity T given X_{t-1}, $E(T|X_{t-1})$, is denoted by $E_*(T) \equiv E(T|X_{t-1})$ for simplicity and this notation will be used in the sequel.

Lemma 2.1. *Suppose that $\{\epsilon_t\}$ and $\{z_t\}$ are independent and (C.0) holds. Then H_0 in (1.2) is true if and only if*

$$E_* u_t = E_* v_t = 0. \tag{2.3}$$

Proof: Necessity can be shown easily and we prove sufficient part only. Consider

$$E[\{F(X_{t-1}, z_t; \theta) - \rho X_{t-1}\}^2]$$

$$= EE_*[\{F(X_{t-1}, z_t; \theta) - \rho X_{t-1}\}^2]. \tag{2.4}$$

Using $E_* u_t = 0$, i.e., , $E_* X_t = \rho X_{t-1}$ and after some tedious algebra, the conditional expectation in (2.4) reduces to

$$E_*[X_t^2 - \rho^2 X_{t-1}^2 - 2X_t\epsilon_t + \sigma_\epsilon^2]$$

$$= E_*[v_t - 2\epsilon_t F(X_{t-1}, z_t; \theta)]$$

Consequently, using $E_* v_t = 0$, it follows that

$$E[\{F(X_{t-1}, z_t; \theta) - \rho X_{t-1}\}^2]$$

$$= -2EE_*\{\epsilon_t F(X_{t-1}, z_t; \theta)\} \tag{2.5}$$

which will vanish by the independence of $\{\epsilon_t\}$ and $\{z_t\}$. This essentially concludes the proof. □

It may be noticed that the term in (2.5) remains zero for the class of generalized random coefficient processes(cf. Hwang and Basawa(1998)) where $\{\epsilon_t\}$ and $\{z_t\}$ are permitted correlated and $F(X_{t-1}, z_t; \theta) = (\theta + z_t)X_{t-1}$,which enables one to incorporate these models into (1.1) enlarging the class of models under consideration.

Write that

$$U_n = n^{-1/2} \sum_{t=1}^{n} u_t$$

$$V_n = n^{-1/2} \sum_{t=1}^{n} v_t \tag{2.6}$$

and

$$T_n = (U_n, V_n)'. \tag{2.7}$$

It then follows from Lemma 2.1 that under H_0 , U_n and V_n are sum of zero mean martingale differences and hence the martingale CLT due to Billingsley (1961) can be employed for the limiting behavior of T_n under H_0. This is summarized below without a detailed proof.

(C.1) The fourth order moment of the stationary distribution exists.

Theorem 2.1. *Under (C.0) and (C.1), T_n is asymptotically normal under H_0. To be more precise,*

$$T_n \xrightarrow{d} N(o, \Sigma), \quad \textit{under } H_0 \tag{2.8}$$

where

$$\Sigma = \begin{pmatrix} Eu_t^2 & Eu_t v_t \\ Eu_t v_t & Ev_t^2 \end{pmatrix} \tag{2.9}$$

with u_t and v_t being defined as in (2.1) and (2.2).

For the practical point of view , ρ and σ_ϵ^2 must be consistently estimated (under H_0) by $\hat{\rho}$ and $\hat{\sigma}_\epsilon^2$ of the form

$$\hat{\rho} = \frac{\sum_{t=1}^{n} X_t X_{t-1}}{\sum_{t=0}^{n} X_t^2}$$

and

$$\hat{\sigma}_\epsilon^2 = n^{-1} \sum_{t=1}^{n} (X_t - \hat{\rho} X_{t-1})^2. \tag{2.10}$$

Omitting details, it can be verified that results in Theorem 2.1 remain valid replacing ρ and σ_ϵ^2 by $\hat{\rho}$ and $\hat{\sigma}_\epsilon^2$ respectively. Consequently , it follows that $\hat{T}_n$ obtained from T_n after plugging (in place of ρ and σ_ϵ^2) $\hat{\rho}$ and $\hat{\sigma}_\epsilon^2$, is asymptotically normal, viz.,

$$\hat{T}_n \xrightarrow{d} N(o, \Sigma), \quad under\ H_0 \tag{2.11}$$

Equivalently,

$$\hat{Q}_n = \hat{T}_n' \hat{\Sigma}^{-1} \hat{T}_n \tag{2.12}$$

converges in law to the chi square distribution with two degrees of freedom where $\hat{\Sigma}$ is a consistent estimator of Σ which can be obtained by replacing the expectations in (2.9) by sample averages. The consistency of $\hat{\Sigma}$ follows readily from the ergodic theorem.

To ensure the consistency of the test statistic $\hat{Q}_n$, we must show $\hat{Q}_n \longrightarrow \infty$, in probability, under H_1. Unfortunately, nothing is known on the limiting behavior of $\hat{Q}_n$ under H_1 except $\hat{Q}_n \geq 0$. This, together with employing the similar ideas of Chen and An (1997) and Chen (1997), leads us to construct another statistic $\hat{\delta}_n$ satisfying

$$\begin{aligned} \hat{\delta}_n &\xrightarrow{p} 0, \quad under\ H_0 \\ \hat{\delta}_n &\xrightarrow{p} \infty, \quad under\ H_1 \end{aligned} \tag{2.13}$$

Such a statistic $\hat{\delta}_n$ can be used to derive a consistent test W_n for linearity,

$$W_n = \hat{Q}_n + \hat{\delta}_n. \tag{2.14}$$

Summarizing the lines above, one can obtain

Theorem 2.2. *Under (C.0) and (C.1), the test statistic W_n for linearity defined in (2.14) is consistent for testing H_0 .vs. H_1.*

Proof: Using (2.12) and (2.13), one can easily see, under H_0, W_n in (2.14) converges to the chi square distribution with two degrees of freedom. Also, under H_1 , $\hat{Q}_n \geq 0$ and $\hat{\delta}_n \longrightarrow \infty$ in probability, which in turn implies the consistency of W_n . The rejection region is clearly $W_n \geq \chi_2^2(\alpha)$. □

Let us now proceed to derive $\hat{\delta}_n$ enjoying (2.13). Recall that H_0 holds if and only if $E_* u_t = E_* v_t = 0$. By the definition of the conditional expectation (cf. Chow and Teicher(1978)), $E_* u_t = 0$ $[E_* v_t = 0]$ is equivalent to

$$E u_t I_{[X_{t-1} \leq x]} = 0 \ \ [E v_t I_{[X_{t-1} \leq x]} = 0] \quad for\ all\ x \in R$$

Denote, for each fixed $x \in R$,

$$\alpha_n(x) = n^{-1} \sum_{t=1}^{n} u_t I_{[X_{t-1} \leq x]} \tag{2.15}$$

and

$$\beta_n(x) = n^{-1} \sum_{t=1}^{n} v_t I_{[X_{t-1} \leq x]} \tag{2.16}$$

Stationarity of $\{X_t\}$ entails that, under H_0, $\alpha_n(x) \longrightarrow 0$ and $\beta_n(x) \longrightarrow 0$ for each x . Arguments as in An and Cheng (1991), Chen(1997), and Chen and An (1997) lead us to consider

$$\delta_n(\rho, \sigma_\epsilon^2) = \gamma_n \int_{-\infty}^{\infty} [\alpha_n^2(x) + \beta_n^2(x)] dM(x) \tag{2.17}$$

where $\{\gamma_n\}$ is an arbitrary sequence of real numbers with $\gamma_n \longrightarrow \infty$ and

$$n^{-1}\gamma_n \longrightarrow 0, \quad as \quad n \longrightarrow \infty \tag{2.18}$$

and $M(x)$ is a distribution function with support $(-\infty, \infty)$. One may choose, for instance, $\gamma_n = logn$ and the distribution function of $N(0,1)$ for $M(x)$ in (2.17).

The following lemma is concerned with the limiting results of $\delta_n(\rho, \sigma_\epsilon^2)$.

Lemma 2.2. *Assume (C.0) and (C.1). we then have , in probability,*

$$\delta_n(\rho, \sigma_\epsilon^2) \longrightarrow 0, \quad under \ H_0 \tag{2.19}$$

and

$$\delta_n(\rho, \sigma_\epsilon^2) \longrightarrow \infty, \quad under \ H_1 \tag{2.20}$$

Proof: Notice first that, under H_0 , $n^{1/2}\alpha_n(x)$ and $n^{1/2}\beta_n(x)$ converge in law to normal distribution with zero mean and appropriate variances of which the existence can be secured by the moment condition (C.1). It then follows that for each $x \in (-\infty, \infty)$, $n^{1/2}\alpha_n(x)$ and $n^{1/2}\beta_n(x)$ are bounded in probability under H_0 and this argument holds uniformly in x. Consequently,

$$\int_{-\infty}^{\infty} n[\alpha_n^2(x) + \beta_n^2(x)] dM(x)$$

is $O_p(1)$, which leads to (2.19) . For (2.20), at least one of $E_* u_t$ and $E_* v_t$ do not vanish, under H_1. Equivalently, for some $x_0 \in R$,

$$E u_t I_{[X_{t-1} \leq x_0]} \neq 0 \quad or \quad E v_t I_{[X_{t-1} \leq x_0]} \neq 0$$

Following similar lines as in Chen(1997), suppose, for example, $Eu_tI_{[X_{t-1}\leq x_0]} \neq 0$ which implies that there exist a constant $\Delta > 0$ and an open neighborhood $N(x_0)$ of x_0 such that

$$|Eu_tI_{[X_{t-1}\leq x]}| > \Delta \quad for\ all \quad x \in N(x_0)$$

Also, ergodic theorem tells us that $\alpha_n^2(x) \longrightarrow \{Eu_tI_{[X_{t-1}\leq x]}\}^2$ which exceeds Δ^2 for all $x \in N(x_0)$. Consequently, under H_1, $\delta_n(\rho, \sigma_\epsilon^2) \longrightarrow \infty$ due to $\gamma_n \longrightarrow \infty$, as $n \longrightarrow \infty$. This concludes the proof. □

We are now in a position to construct $\hat{\delta}_n := \delta_n(\hat{\rho}, \hat{\sigma}_\epsilon^2)$ replacing the unknown parameters ρ and σ_ϵ^2 in $\delta_n(\rho, \sigma_\epsilon^2)$ by their consistent estimators $\hat{\rho}$ and $\hat{\sigma}_\epsilon^2$ defined in (2.10). It can also be verified that $\hat{\delta}_n$ is equivalent to $\delta_n(\rho, \sigma_\epsilon^2)$ up to terms negligible in probability both under H_0 and under H_1. Consequently the results in Lemma 2.2 continue to hold for $\hat{\delta}_n$. Details are omitted for brevity.

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