Journal of the Korean Statistical Society (1998), **27**: *3, pp 289–303*

Generalized Measure of Departure From Global Symmetry for Square Contingency Tables with Ordered Categories

Sadao Tomizawa and Kayo Saitoh[1]

ABSTRACT

For square contingency tables with ordered categories, Tomizawa (1995) considered two kinds of measures to represent the degree of departure from global symmetry, which means that the probability that an observation will fall in one of cells in the upper-right triangle of square table is equal to the probability that the observation falls in one of cells in the lower-left triangle of it. This paper proposes a generalization of those measures. The proposed measure is expressed by using Cressie and Read's (1984) power divergence or Patil and Taillie's (1982) diversity index. Special cases of the proposed measure include Tomizawa's measures. The proposed measure would be useful for comparing the degree of departure from global symmetry in several tables.

Keywords: Diversity index; Gini concentration; Kullback-Leibler information; Pearson's chi-squared type discrepancy; Power divergence; Shannon entropy

1. INTRODUCTION

For an $R \times R$ square contingency table with the same row and column classifications, let X and Y denote the row and column variables, respectively, and let $\Pr(X = i, Y = j) = p_{ij}$ $(i = 1, 2, \dots, R; j = 1, 2, \dots, R)$, where $\sum_{i=1}^{R}\sum_{j=1}^{R} p_{ij} = 1$. Read's (1977) global symmetry (GS) model is defined by

$$\delta_U = \delta_L,$$

where $\delta_U = \sum\sum_{i<j} p_{ij}$ $[= \Pr(X < Y)]$ and $\delta_L = \sum\sum_{i>j} p_{ij}$ $[= \Pr(X > Y)]$. Note that the GS model is applied to *ordinal* categorical data, because the GS

[1]Department of Information Sciences, Faculty of Science and Technology, Science University of Tokyo, Noda City, Chiba 278-8510, Japan

model is not invariant under the same arbitrary permutations of row and column categories. Tomizawa (1995) proposed two kinds of measures to represent the degree of departure from GS (see Appendix). One measure, ϕ_{GS}, is expressed by using the Kullback-Leibler information (or the Shannon entropy) and the other measure, ψ_{GS}, is expressed by using the Pearson's chi-squared type discrepancy (or the Gini concentration); see Appendix.

By the way, Cressie and Read (1984), and Read and Cressie (1988) considered the power-divergence statistic for testing goodness of fit, which includes the likelihood ratio and the Pearson's chi-squared statistics in special cases (see Section 5). We are interested in a measure of departure from GS based on the power-divergence discrepancy.

The purpose of this paper is to propose a power-divergence type measure which represents the degree of departure from GS for square contingency tables. The measure would be useful for *comparing* the degree of departure from GS in several tables.

2. MEASURE OF DEPARTURE FROM GLOBAL SYMMETRY

Assume that $\delta_U + \delta_L \neq 0$. Let $\delta_U^* = \delta_U/(\delta_U + \delta_L)$ [$= \Pr(X<Y|X\neq Y)$] and $\delta_L^* = \delta_L/(\delta_U + \delta_L)$ [$= \Pr(X>Y|X\neq Y)$]. Note that (i) δ_U^* (δ_L^*) indicates the probability that an observation falls in one of cells in the upper-right triangle (in one of cells in the lower-left triangle) of square table on condition that the observation will fall in one of the off-diagonal cells of square table, and (ii) $\delta_U^* = \frac{1}{2}$ (or $\delta_L^* = \frac{1}{2}$) if and only if the GS model holds.

Consider a measure defined by

$$\Phi_{\text{GS}}^{(\lambda)} = \frac{\lambda(\lambda+1)}{2^\lambda - 1} I^{(\lambda)} \left(\{\delta_U^*, \delta_L^*\}; \left\{ \frac{1}{2}, \frac{1}{2} \right\} \right) \qquad \text{for } \lambda > -1,$$

where

$$I^{(\lambda)}(\cdot;\cdot) = \frac{1}{\lambda(\lambda+1)} \left[\delta_U^* \left\{ \left(\frac{\delta_U^*}{1/2} \right)^\lambda - 1 \right\} + \delta_L^* \left\{ \left(\frac{\delta_L^*}{1/2} \right)^\lambda - 1 \right\} \right]$$

and the value at $\lambda = 0$ is taken to be continuous limit as $\lambda \to 0$, and where λ is a real-value that is chosen by the user. [Note that $\Phi_{\text{GS}}^{(0)}$ and $\Phi_{\text{GS}}^{(1)}$ are the same to ϕ_{GS} and ψ_{GS}, respectively, in Appendix.] We note that $I^{(\lambda)}$ $(\{\delta_U^*, \delta_L^*\}; \{1/2, 1/2\})$

is the power divergence between two conditional distributions $\{\delta_U^*,\ \delta_L^*\}$ and $\{1/2, 1/2\}$, on condition that an observation falls in one of the off-diagonal cells of square table. [In addition, we note that the power divergence includes the Kullback-Leibler information (when $\lambda = 0$) and the Pearson chi-squared type discrepancy (when $\lambda = 1$) in special cases. For the more details of the power divergence, see Cressie and Read (1984) and Read and Cressie (1988).]

This measure may also be expressed as

$$\Phi_{\mathrm{GS}}^{(\lambda)} = 1 - \frac{\lambda \cdot 2^\lambda}{2^\lambda - 1} H^{(\lambda)}\left(\{\delta_U^*, \delta_L^*\}\right) \qquad \text{for } \lambda > -1,$$

where

$$H^{(\lambda)}(\cdot) = \frac{1}{\lambda}\left[1 - (\delta_U^*)^{\lambda+1} - (\delta_L^*)^{\lambda+1}\right]$$

and the value at $\lambda = 0$ is taken to be continuous limit as $\lambda \to 0$. We note that $H^{(\lambda)}\left(\{\delta_U^*, \delta_L^*\}\right)$ is the Patil and Taillie's (1982) diversity index of degree-λ for the conditional distribution $\{\delta_U^*, \delta_L^*\}$ which includes the Shannon entropy (when $\lambda = 0$) and the Gini concentration (when $\lambda = 1$) in special cases. The measure $\Phi_{\mathrm{GS}}^{(\lambda)}$ is a generalization of Tomizawa's measures ϕ_{GS} and ψ_{GS}.

Noting that $I^{(\lambda)}\left(\{\delta_U^*, \delta_L^*\}; \{1/2, 1/2\}\right) \geq 0$ and $H^{(\lambda)}\left(\{\delta_U^*, \delta_L^*\}\right) \geq 0$, we see that $\Phi_{\mathrm{GS}}^{(\lambda)}$ must lie between 0 and 1. Also, for each $\lambda (> -1)$, (i) there is a structure of GS in the $R \times R$ table if and only if $\Phi_{\mathrm{GS}}^{(\lambda)} = 0$ and (ii) the degree of departure from GS is the largest in the sense that $\delta_U^* = 1$ (then $\delta_L^* = 0$) or $\delta_L^* = 1$ (then $\delta_U^* = 0$); if and only if $\Phi_{\mathrm{GS}}^{(\lambda)} = 1$. According to the power divergence or the Patil and Taillie's diversity index, $\Phi_{\mathrm{GS}}^{(\lambda)}$ represents the degree of departure from GS, and the degree increases as the value of $\Phi_{\mathrm{GS}}^{(\lambda)}$ increases.

3. APPROXIMATE CONFIDENCE INTERVALS FOR MEASURES

Let n_{ij} denote the observed frequency in the ith row and jth column of the square table ($i = 1, 2, \ldots, R; j = 1, 2, \ldots, R$). The sample version of $\Phi_{\mathrm{GS}}^{(\lambda)}$, i.e., $\hat{\Phi}_{\mathrm{GS}}^{(\lambda)}$, is given by $\Phi_{\mathrm{GS}}^{(\lambda)}$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, where $\hat{p}_{ij} = n_{ij}/n$ and $n = \sum\sum n_{ij}$. Assuming that the $\{n_{ij}\}$ result from full multinominal sampling, we shall consider an approximate standard error for $\hat{\Phi}_{\mathrm{GS}}^{(\lambda)}$ and large-sample confidence interval for $\Phi_{\mathrm{GS}}^{(\lambda)}$ using the delta method (see Bishop et al. (1975, Sec.14.6) and Agresti (1990, Sec.12.1)). Using the delta method, $\sqrt{n}(\hat{\Phi}_{\mathrm{GS}}^{(\lambda)} - \Phi_{\mathrm{GS}}^{(\lambda)})$ has asymptotically (as $n \to \infty$) a normal distribution with mean zero and variance,

$$\sigma^2 = \begin{cases} \left\{ \dfrac{2^\lambda(\lambda+1)}{2^\lambda - 1} \right\}^2 \dfrac{\delta_U^* \delta_L^*}{\delta_U + \delta_L} \left\{ (\delta_U^*)^\lambda - (\delta_L^*)^\lambda \right\}^2 & \text{for } \lambda > -1; \lambda \neq 0, \\ \dfrac{\delta_U^* \delta_L^*}{(\log 2)^2 (\delta_U + \delta_L)} \{\log(\delta_U/\delta_L)\}^2 & \text{for } \lambda = 0. \end{cases}$$

Let $\hat{\sigma}^2$ denote σ^2 with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. Then $\hat{\sigma}/\sqrt{n}$ is an estimated approximate standard error for $\hat{\Phi}_{\mathrm{GS}}^{(\lambda)}$, and $\hat{\Phi}_{\mathrm{GS}}^{(\lambda)} \pm z_{p/2}\, \hat{\sigma}/\sqrt{n}$ is an approximate $100(1-p)$ percent confidence interval for $\Phi_{\mathrm{GS}}^{(\lambda)}$, where $z_{p/2}$ is the upper $(p/2)$-th quantile of the standard normal distribution.

4. EXAMPLES

Consider the data in Table 4.1, taken from Tominaga (1979, p53). These data describe the cross-classification of father's and son's occupational status categories in Japan which were examined in 1955, 1965 and 1975.

Since the confidence intervals for $\Phi_{\mathrm{GS}}^{(\lambda)}$ applied to the data in each of Tables 4.1(a), 4.1(b) and 4.1(c) do not include zero for all λ (see Table 4.2), these would indicate that there is not a structure of GS in each table.

Also, we shall investigate the degree of departure from GS in more details. For instance, when $\lambda = 2$, the estimated measure $\hat{\Phi}_{\mathrm{GS}}^{(2)}$ equals to 0.118 for Table 4.1(a), 0.235 for Table 4.1(b), and 0.212 for Table 4.1(c)(see Table 4.2). Thus, (i) for Table 4.1(a), the degree of departure from GS is estimated to be 11.8 percent of the maximum degree of departure from GS, (ii) for Table 4.1(b), it is estimated to be 23.5 percent of the maximum degree of departure from GS, and (iii) for Table 4.1(c), it is estimated to be 21.2 percent of the maximum degree of departure from GS.

Based on the confidence intervals for $\Phi_{\mathrm{GS}}^{(\lambda)}$, we found that the degrees of departure from GS model in Tables 4.1(b), 4.1(c) are greater than those in Table 4.1(a). However, the comparison between them in Tables 4.1(b) and 4.1(c) may be impossible because the values in the confidence interval for Table 4.1(b) are not always greater than the values in the confidence interval for Table 4.1(c).

5. CONCLUDING REMARKS

Let $W^{(\lambda)}$ denote the power-divergence statistic for testing goodness of fit of the GS model, that is

$$W^{(\lambda)} = 2nI^{(\lambda)}\left(\{\hat{p}_{ij}\}\,;\{\hat{p}_{ij}^{ML}\}\right) \qquad \text{for } -\infty < \lambda < \infty,$$

where

$$I^{(\lambda)}(\cdot;\cdot) = \frac{1}{\lambda(\lambda+1)}\sum_{i=1}^{R}\sum_{j=1}^{R}\hat{p}_{ij}\left[\left(\frac{\hat{p}_{ij}}{\hat{p}_{ij}^{ML}}\right)^{\lambda}-1\right],$$

$$\hat{p}_{ij}^{ML} = \begin{cases} \hat{p}_{ij}/(2\hat{\delta}_U^*) & \text{for } i<j, \\ \hat{p}_{ij}/(2\hat{\delta}_L^*) & \text{for } i>j, \\ \hat{p}_{ii} & \text{for } i=j, \end{cases}$$

and $\hat{\delta}_U^*(\hat{\delta}_L^*)$ denotes $\delta_U^*(\delta_L^*)$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$ and where the values at $\lambda = -1$ and $\lambda = 0$ are taken to be continuous limit as $\lambda \to -1$ and $\lambda \to 0$, respectively. [Note that $\{\hat{p}_{ij}^{ML}\}$ are the maximum likelihood estimates of $\{p_{ij}\}$ under the GS model, and also especially $W^{(0)}$ and $W^{(1)}$ are the likelihood ratio and the Pearson's chi-squared statistics, respectively. See Table 5.1 for the values of $W^{(\lambda)}$ applied to the data in Tables 4.1(a), 4.1(b) and 4.1(c).] Then we note that

$$\hat{\Phi}_{\text{GS}}^{(\lambda)} = \frac{\lambda(\lambda+1)}{2(2^{\lambda}-1)\sum\sum_{i\neq j}n_{ij}}W^{(\lambda)} \qquad \text{for } \lambda > -1.$$

Consider the artificial data in Table 5.2. The values of $W^{(\lambda)}$ (with one degree of freedom) applied to these data are given in Table 5.3. ¿From $W^{(\lambda)}$ with $0.8 < \lambda < 2.4$, we see that the GS model fits the data in Table 5.2(a) worse than that in Table 5.2(b). However, from $W^{(\lambda)}$ with $\lambda < 0.6$ or $\lambda > 2.6$, we see that the GS model fits the data in Table 5.2(a) better than that in Table 5.2(b). On the other side, for any fixed $\lambda(> -1)$, the value of $\hat{\Phi}_{\text{GS}}^{(\lambda)}$ for Table 5.2(a) is less than that for Table 5.2(b) (see Table 5.4). Since the value of $\hat{\delta}_U/\hat{\delta}_L$ $(= \hat{\delta}_U^*/\hat{\delta}_L^*)$ is 0.57 for Table 5.2(a) and 0.12 for Table 5.2(b), and also it is equal to 1 when the GS model holds, it seems natural to conclude that the degree of departure from GS for Table 5.2(a) is less than that for Table 5.2(b). Therefore $\hat{\Phi}_{\text{GS}}^{(\lambda)}$ may be preferable to $W^{(\lambda)}$ for *comparing* the degree of departure from GS.

The GS model imposes no restriction on the diagonal cell probabilities $\{p_{ii}\}$. Therefore, the structure of GS based on the probabilities $\{p_{ij}\}$, i.e., $\delta_U = \delta_L$, may also be expressed as $\delta_U^* = \delta_L^*$ using the conditional probabilities $\{p_{ij}^*\}$, $i \neq j$, where $p_{ij}^* = p_{ij}/(\sum\sum_{i\neq j}p_{ij})$. In sample version, $W^{(\lambda)}/n$ (for a given λ) is a measure based on $\{\hat{p}_{ij}\}$, and $\hat{\Phi}_{\text{GS}}^{(\lambda)}$ is essentially the corresponding measure based on $\{\hat{p}^*_{ij}\}$, $i \neq j$, because $\hat{\Phi}^{(\lambda)}_{\mathrm{GS}}$ may also be expressed as

$$\hat{\Phi}^{(\lambda)}_{\mathrm{GS}} = \frac{\lambda(\lambda+1)}{2^\lambda - 1} I^{(\lambda)}\left(\{\hat{p}^*_{ij}\}; \{\hat{p}^{*ML}_{ij}\}\right), \tag{5.1}$$

where $\hat{p}^{*ML}_{ij} = \hat{p}^{ML}_{ij}/(\sum\sum_{i\neq j}\hat{p}^{ML}_{ij})$. It may seem, to many readers, that both are reasonable measures, for representing the degree of departure from GS. However, $\hat{\Phi}^{(\lambda)}_{\mathrm{GS}}$ rather than $W^{(\lambda)}/n$ would be useful for comparing the degree of departure from GS in several tables because the range of $W^{(\lambda)}/n$ (for $\lambda > -1$) depends on the diagonal proportions, i.e.,

$$0 \leq \frac{W^{(\lambda)}}{n} \leq \frac{2(2^\lambda - 1)}{\lambda(\lambda+1)}\left(1 - \sum_{i=1}^{R}\frac{n_{ii}}{n}\right);$$

but $\hat{\Phi}^{(\lambda)}_{\mathrm{GS}}$ always range between 0 and 1 without depending on the diagonal proportions.

The $\hat{\Phi}^{(\lambda)}_{\mathrm{GS}}$ would be useful when one want to measure how far the conditional probability distribution $\{\hat{p}^*_{ij}\}$, $i \neq j$, are distant from those with a structure of GS [though the $W^{(\lambda)}/n$ would be useful when one want to measure how far the unconditional probability distribution $\{\hat{p}_{ij}\}$ are distant from those with a structure of GS].

By the way, since $\hat{p}^{*ML}_{ij}$ in (5.1) is the maximum likelihood estimate of p^*_{ij}, it is easily seen that especially the measure $\hat{\Phi}^{(0)}_{\mathrm{GS}}$ can be expressed as

$$\hat{\Phi}^{(0)}_{\mathrm{GS}} = \frac{1}{\log 2}\min_{\{\hat{p}^{*es}_{ij}\}} I^{(0)}\left(\{\hat{p}^*_{ij}\}; \{\hat{p}^{*es}_{ij}\}\right), \tag{5.2}$$

where $\sum\sum_{i<j}\hat{p}^{*es}_{ij} = \sum\sum_{i>j}\hat{p}^{*es}_{ij}$ and $\sum\sum_{i\neq j}\hat{p}^{*es}_{ij} = 1$ with $\hat{p}^{*es}_{ij} \geq 0$. Therefore we note that $\hat{p}^{*ML}_{ij}$ in (5.1) is the value of $\hat{p}^{*es}_{ij}$ such that the Kullback-Leibler (KL) distance $I^{(0)}(\cdot;\cdot)$ (i.e., the KL distance between the sample conditional distribution $\{\hat{p}^*_{ij}\}$ and the estimated conditional distribution $\{\hat{p}^{*es}_{ij}\}$ with a GS structure) is *minimum*. [Note that the reader may also be interested in (5.2) with $I^{(0)}(\cdot;\cdot)$ replaced by the power-divergence $I^{(\lambda)}(\cdot;\cdot)$; however, it is difficult to obtain the value of $\hat{p}^{*es}_{ij}$ such that the corresponding power-divergence is minimum, and also difficult to obtain the maximum value of such a measure.]

The reader may be interested in which value of λ is preferred for a given table. However, it seems difficult to discuss it. It seems to be important that for given tables, the analyst calculates the values of $\hat{\Phi}^{(\lambda)}_{\mathrm{GS}}$ for various values of λ and discuss the degree of departure from GS in terms of them. [However, the case of $\lambda = 0$, i.e., $\hat{\Phi}^{(0)}_{\mathrm{GS}}$ may be useful in terms of the expression (5.2) when the analyst wants to see with a *minimum* distance measure how far the sample conditional probability distribution is distant from the estimated conditional probability distribution with a GS structure.]

Finally we observe that the measure should be applied to square contingency tables with ordered categories because it is not invariant under the same arbitrary permutations of row and column categories.

ACKNOWLEDGEMENTS

The authors would like to thank Dr. Takashi Seo, Science University of Tokyo, for his helpful comments and suggestions. They are also grateful to the referees and the editor for many helpful comments and suggestions on improving this paper.

APPENDIX

For an $R \times R$ table, two kinds of measures of departure from GS, considered by Tomizawa (1995), are given as follows:

$$\begin{aligned} \phi_{GS} &= \frac{1}{\log 2} I\left(\{\delta_U^*, \delta_L^*\}; \left\{\frac{1}{2}, \frac{1}{2}\right\}\right) \\ &= 1 - \frac{1}{\log 2} H\left(\{\delta_U^*, \delta_L^*\}\right), \end{aligned}$$

where

$$\begin{aligned} I(\cdot;\cdot) &= \delta_U^* \log \frac{\delta_U^*}{1/2} + \delta_L^* \log \frac{\delta_L^*}{1/2}, \\ H(\cdot) &= -\delta_U^* \log \delta_U^* - \delta_L^* \log \delta_L^*, \end{aligned}$$

with $\{\delta_U^*\}$ and $\{\delta_L^*\}$ defined in Section 2, and

$$\begin{aligned} \psi_{GS} &= D\left(\{\delta_U^*, \delta_L^*\}; \left\{\frac{1}{2}, \frac{1}{2}\right\}\right) \\ &= 1 - 2 \cdot C\left(\{\delta_U^*, \delta_L^*\}\right), \end{aligned}$$

where

$$\begin{aligned} D(\cdot;\cdot) &= \frac{\left(\delta_U^* - \frac{1}{2}\right)^2}{1/2} + \frac{\left(\delta_L^* - \frac{1}{2}\right)^2}{1/2}, \\ C(\cdot) &= 1 - \left(\delta_U^{*2} + \delta_L^{*2}\right). \end{aligned}$$

Table 4.1: Occupational status for Japanese father-son pairs; from Tominaga (1979, p53).

(a) Examined in 1955

				Son's	status				
Father's status	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	Total
(1)	36	4	14	7	8	2	3	8	82
(2)	20	20	27	24	11	11	2	11	126
(3)	9	6	23	12	9	5	3	16	83
(4)	15	14	39	81	17	16	11	15	208
(5)	6	7	22	13	72	20	6	13	159
(6)	3	2	5	12	18	19	9	7	75
(7)	5	3	10	11	21	15	38	25	128
(8)	39	30	76	80	69	52	45	614	1005
Total	133	86	216	240	225	140	117	709	1866

(b) Examined in 1965

				Son's	status				
Father's status	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	Total
(1)	27	10	16	3	6	6	1	2	71
(2)	15	38	30	20	8	4	3	7	125
(3)	13	17	32	17	7	16	6	5	113
(4)	12	36	40	132	22	30	13	6	291
(5)	8	22	38	41	91	42	22	9	273
(6)	2	2	7	12	13	16	3	2	57
(7)	3	2	11	11	13	26	30	6	102
(8)	38	44	95	101	132	114	60	309	893
Total	118	171	269	337	292	254	138	346	1925

(c) Examined in 1975

				Son's	status				
Father's status	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	Total
(1)	44	18	28	8	6	8	1	5	118
(2)	15	50	45	20	18	17	4	7	176
(3)	18	25	47	30	24	18	5	7	174
(4)	16	27	53	77	40	29	9	6	257
(5)	18	25	42	31	122	43	17	13	311
(6)	12	15	21	15	36	33	3	8	143
(7)	3	5	8	7	26	21	9	3	82
(8)	44	65	114	92	184	195	58	325	1077
Total	170	230	358	280	456	364	106	374	2338

Note: Status(1) is Professional; (2), Managers; (3), Clerical; (4), Sales; (5), Skilled Manual; (6), Semiskilled manual; (7), Unskilled manual; and (8), Farmers.

Table 4.2: Estimate of $\Phi_{\rm GS}^{(\lambda)}$, estimated approximate standard error for $\hat{\Phi}_{\rm GS}^{(\lambda)}$ and approximate 95% confidence interval for $\Phi_{\rm GS}^{(\lambda)}$, applied to Tables 4.1(a), 4.1(b) and 4.1(c).

(a) For Table 4.1(a):

Values of λ	Estimated measure	Standard error	Confidence interval
-0.8	0.023	0.004	(0.015, 0.032)
-0.6	0.043	0.008	(0.028, 0.059)
-0.4	0.061	0.011	(0.039, 0.082)
-0.2	0.075	0.014	(0.048, 0.102)
0	0.087	0.016	(0.056, 0.118)
0.2	0.097	0.017	(0.063, 0.131)
0.4	0.105	0.019	(0.068, 0.141)
0.6	0.111	0.020	(0.072, 0.149)
0.8	0.115	0.020	(0.075, 0.155)
1.0	0.118	0.021	(0.077, 0.159)
1.2	0.120	0.021	(0.079, 0.161)
1.4	0.121	0.021	(0.079, 0.162)
1.6	0.121	0.021	(0.079, 0.162)
1.8	0.120	0.021	(0.078, 0.161)
2.0	0.118	0.021	(0.077, 0.159)
2.2	0.116	0.020	(0.076, 0.156)
2.4	0.113	0.020	(0.074, 0.153)
2.6	0.110	0.020	(0.072, 0.149)
2.8	0.107	0.019	(0.069, 0.144)
3.0	0.103	0.019	(0.067, 0.140)

Table 4.2: (continued)

(b) For Table 4.1(b):

Values of λ	Estimated measure	Standard error	Confidence interval
-0.8	0.049	0.006	(0.038, 0.060)
-0.6	0.091	0.010	(0.071, 0.111)
-0.4	0.125	0.014	(0.098, 0.152)
-0.2	0.154	0.017	(0.121, 0.186)
0	0.177	0.019	(0.140, 0.214)
0.2	0.200	0.021	(0.155, 0.236)
0.4	0.210	0.022	(0.167, 0.253)
0.6	0.221	0.023	(0.177, 0.266)
0.8	0.229	0.024	(0.183, 0.276)
1.0	0.235	0.024	(0.188, 0.282)
1.2	0.238	0.024	(0.191, 0.286)
1.4	0.240	0.024	(0.192, 0.287)
1.6	0.239	0.024	(0.192, 0.287)
1.8	0.238	0.024	(0.190, 0.285)
2.0	0.235	0.024	(0.188, 0.282)
2.2	0.231	0.024	(0.185, 0.278)
2.4	0.227	0.023	(0.181, 0.272)
2.6	0.221	0.023	(0.176, 0.266)
2.8	0.216	0.023	(0.171, 0.260)
3.0	0.209	0.022	(0.166, 0.253)

Table 4.2: (continued)

(c) For Table 4.1(c):

Values of λ	Estimated measure	Standard error	Confidence interval
-0.8	0.044	0.005	(0.035, 0.053)
-0.6	0.081	0.008	(0.065, 0.098)
-0.4	0.112	0.011	(0.090, 0.135)
-0.2	0.138	0.014	(0.111, 0.165)
0	0.159	0.016	(0.128, 0.190)
0.2	0.176	0.017	(0.142, 0.210)
0.4	0.190	0.018	(0.153, 0.225)
0.6	0.200	0.019	(0.162, 0.237)
0.8	0.207	0.020	(0.168, 0.246)
1.0	0.212	0.020	(0.172, 0.252)
1.2	0.215	0.020	(0.175, 0.255)
1.4	0.216	0.021	(0.176, 0.257)
1.6	0.216	0.021	(0.176, 0.256)
1.8	0.215	0.020	(0.174, 0.255)
2.0	0.212	0.020	(0.172, 0.252)
2.2	0.208	0.020	(0.169, 0.248)
2.4	0.204	0.020	(0.166, 0.243)
2.6	0.199	0.019	(0.161, 0.237)
2.8	0.194	0.019	(0.157, 0.231)
3.0	0.188	0.019	(0.152, 0.225)

Table 5.1: The values of power-divergence statistic $W^{(\lambda)}$ (with one degree of freedom) for testing goodness of fit of the global symmetry model, applied to Tables 4.1(a), 4.1(b) and 4.1(c).

Values of λ	$W^{(\lambda)}$ applied to Table 1a	$W^{(\lambda)}$ applied Table 1b	$W^{(\lambda)}$ applied Table 1c
-0.8	119.86	327.77	381.27
-0.6	118.76	321.41	374.68
-0.4	117.77	315.78	368.83
-0.2	116.90	310.84	363.68
0	116.12	306.54	359.19
0.2	115.46	302.86	355.34
0.4	114.89	299.77	352.10
0.6	114.42	297.24	349.44
0.8	114.05	295.25	347.35
1.0	113.77	293.79	345.80
1.2	113.59	292.83	344.79
1.4	113.50	292.36	344.29
1.6	113.50	292.37	344.30
1.8	113.59	292.85	344.80
2.0	113.77	293.79	345.80
2.2	114.04	295.19	347.28
2.4	114.40	297.04	349.25
2.6	114.85	299.34	351.69
2.8	115.38	302.09	354.61
3.0	116.01	305.30	358.02

Table 5.2: Artificial data

(a) $n = 1117$ (sample size)

202	92	7
123	53	140
125	170	205

Note: $\{\hat{\delta}_U, \hat{\delta}_L\} = \{0.21, 0.37\}$, $\{\hat{\delta}_U^*, \hat{\delta}_L^*\} = \{0.36, 0.64\}$

(b) $n = 115$

7	5	0
19	27	3
27	22	5

Note: $\{\hat{\delta}_U, \hat{\delta}_L\} = \{0.07, 0.59\}$, $\{\hat{\delta}_U^*, \hat{\delta}_L^*\} = \{0.11, 0.89\}$

Table 5.3: The values of power-divergence statistic $W^{(\lambda)}$ (with one degree of freedom) for testing goodness of fit of the global symmetry model, applied to Tables 5.2(a) and 5.2(b).

Values of λ	$W^{(\lambda)}$ applied to Table 4a	$W^{(\lambda)}$ applied to Table 4b
-0.8	50.36	68.55
-0.6	50.08	63.88
-0.4	49.82	60.01
-0.2	49.59	56.82
0	49.39	54.21
0.2	49.21	52.09
0.4	49.07	50.40
0.6	48.94	49.07
0.8	48.84	48.08
1.0	48.77	47.37
1.2	48.72	46.92
1.4	48.70	46.71
1.6	48.70	46.73
1.8	48.72	46.95
2.0	48.77	47.37
2.2	48.84	47.98
2.4	48.94	48.78
2.6	49.06	49.76
2.8	49.20	50.93
3.0	49.37	52.29

Table 5.4: Values of $\hat{\Phi}_{GS}^{(\lambda)}$ applied to Tables 5.2(a) and 5.2(b).

Values of λ	$\hat{\Phi}_{GS}^{(\lambda)}$ applied to Table 4a	$\hat{\Phi}_{GS}^{(\lambda)}$ applied to Table 4b
-0.8	0.014	0.170
-0.6	0.027	0.296
-0.4	0.038	0.391
-0.2	0.047	0.462
0	0.054	0.515
0.2	0.060	0.553
0.4	0.065	0.581
0.6	0.069	0.601
0.8	0.072	0.615
1.0	0.074	0.623
1.2	0.075	0.628
1.4	0.076	0.630
1.6	0.076	0.630
1.8	0.075	0.627
2.0	0.074	0.623
2.2	0.073	0.618
2.4	0.071	0.612
2.6	0.069	0.605
2.8	0.067	0.598
3.0	0.064	0.590

References

Agresti, A (1990). *Categorical Data Analysis.* John Wiley, New York.

Bishop, Y.M.M., Fienberg, S.E., Holland, P.W. (1975). *Discrete Multivariate Analysis: Theory and Practice.* Cambridge, Massachusetts: The MIT Press.

Cressie, N. and Read, T.R.C. (1984). Multinominal goodness-of-fit tests. *Journal of the Royal Statistical Society, Series B* **46**, 440-464.

Patil, G.P. and Taillie, C. (1982). "Diversity as a concept and its measurement," *Journal of the American Statistical Association* **77**, 548-561.

Read, C.B. (1977). "Partitioning chi-square in contingency tables : A teaching approach," *Communications in Statistics-Theory and Methods*, **6**, 553-562.

Read, T.R.C., and Cressie, N. (1988). *Goodness-of-Fit Statistics for Discrete Multivariate Data.* Springer-Verlag, New York.

Tominaga, K. (1979). *Nippon no Kaisou Kouzou* (*Japanese Hierarchical Structure*). University of Tokyo Press, Tokyo (in Japanese).

Tomizawa, S (1995). "Measures of departure from global symmetry for square contingency tables with ordered categories," *Behaviormetrika*, **22**, 91-98.