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Testing the Randomness of the Coefficients In First Order Autoregressive Processes †

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ABSTRACT

In this paper, we are concerned with the problem of testing the randomness of the coefficients in a first order autoregressive model. A consistent test based on prediction error is suggested. It is shown that under the null hypothesis, the test statistic is asymptotically normal.

Keywords: CLT for Martingales; Randomness Test; Consistent Test

1. INTRODUCTION

Consider the time series model

$$Y_t = (\phi + b_t)Y_{t-1} + \epsilon_t, \tag{1.1}$$

where (b_t, ϵ_t) are iid random vectors independent of $Y_s, s \leq t-1$ with $E(\epsilon_t) = E(b_t) = 0, Var(b_t) = {\sigma_b}^2$, $Var(\epsilon_t) = {\sigma_\epsilon}^2$, $Cov(b_t, \epsilon_t) = \sigma_{b\epsilon}$, and $E({\epsilon_t}^4 + {b_t}^4) < \infty$.

The model (1.1) is referred to as a generalized first order random coefficient model (cf. Hwang and Basawa, 1996). Particularly , when b_t is identically 0, the model (1.1) is the usual first order autoregressive model (cf. Brockwell and Davis, 1992). In order to provide the stationarity for the model (1.1), assume that

$$\phi^2 + {\sigma_b}^2 < 1 \tag{1.2}$$

It is well-known that under (1.2), Y_t can be rewritten as

$$Y_t = \epsilon_t + \sum_{j=1}^{\infty}(\prod_{i=1}^{j-1}(\phi + b_i))\epsilon_{t-i} \quad a.s. \tag{1.3}$$

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(cf. Hwang and Basawa, 1996), and Y_t in (1.3) is the unique stationary solution of the difference equation (1.1).

The objective of this paper is to test whether b_t is identically 0 or not. Nicholls and Quinn (1982) gives a good introduction to random coefficient autoregressive models and inference problems. In the text, there is a concern for testing randomness in the model (1.1). The proposed test statistic is as follows:

$$NQ = \frac{(1/2)\sum_{t=1}^{n} Y_{t-1}^2\{(\hat{\epsilon}_t^2/\hat{\sigma}_n^2)-1\}}{\{(1/2)[\sum_{t=1}^{n} Y_{t-1}^4 - (1/n)(\sum_{t=1}^{n} Y_{t-1}^2)^2]\}^{1/2}},$$

where $\hat{\epsilon}_t = Y_t - \hat{\phi}_n Y_{t-1}$, $\hat{\phi}_n$ is the least-squares estimator of ϕ, and $\hat{\sigma_n}^2 = n^{-1}\sum_{t=1}^{n}\hat{\epsilon_t}^2$.

The statistic NQ performs very well in broad situations. However, the test based on NQ may not be consistent when ϵ_t and b_t are correlated. Ramanathan and Rajarshi (1994) provided a test based on the ranks of residuals, say, $\hat{\epsilon}_t$. As with Nicholls and Quinn, they also imposed the independence assumption on ϵ_t and b_t. (See also Ramanathan and Rajarshi (1992), which deals with linear regression models). In this paper, unlike the papers mentioned above, we relax the independence assumption on ϵ_t and b_t. The detailed testing procedure is addressed next.

For testing $\mathrm{H}_0 : {\sigma_b}^2 = 0$ vs. $\mathrm{H}_1 : {\sigma_b}^2 > 0$, based on $Y_1, \cdots, Y_n$, we introduce the least squares estimator of ϕ

$$\hat{\phi}_n = \frac{\sum_{t=1}^{n} Y_{t-1}Y_t}{\sum_{t=1}^{n} {Y_{t-1}}^2}, \qquad Y_0 \equiv 0.$$

Notice that

$$\hat{\phi}_n - \phi = \frac{\sum_{t=1}^{n} Y_{t-1}b_t}{\sum_{t=1}^{n} {Y_{t-1}}^2} + \frac{\sum_{t=1}^{n} Y_{t-1}\epsilon_t}{\sum_{t=1}^{n} {Y_{t-1}}^2}\ , \tag{1.4}$$

and by our assumption, $n^{1/2}(\hat{\phi}_n - \phi) = O_p(1)$ under both H_0 and H_1.

Now, define $\hat{\epsilon}_t = Y_t - \hat{\phi}_n Y_{t-1}$ and $\hat{\sigma_n}^2 = n^{-1}\sum_{t=1}^{n}\hat{\epsilon_t}^2$. Our analysis (cf. Theorem 2.1) shows that $n^{1/2}(\hat{\sigma}_n^2 - {\sigma_\epsilon}^2)$ is asymptotically normal under H_0, and diverges to ∞ under H_1. According to Theorem 2.1, for constructing a consistent test statistic it suffices to find another estimator $\tilde{\sigma_n}^2$ of ${\sigma_\epsilon}^2$, such that $n^{1/2}(\tilde{\sigma}_n^2 - {\sigma_\epsilon}^2)$ is $O_p(1)$ under both H_0 and H_1 (cf. Lemma 2.1), and $n^{1/2}(\hat{\sigma}_n^2 - \tilde{\sigma}_n^2)$ is asymptotically normal under H_0.

The idea for finding out such $\tilde{\sigma}_n^2$ is to compute the conditional prediction errors : Observe that

$$E[(Y_t - \phi Y_{t-1})^2 | \mathcal{F}_{t-1}] = {\sigma_\epsilon}^2 + 2\sigma_{b\epsilon} Y_{t-1} + {\sigma_b}^2 {Y_{t-1}}^2,$$

where $\mathcal{F}_t$ is the σ-field generated by $Y_s, s \le t$.

This enables us to formulate the following regression model :

$$(Y_t - \hat{\phi}_n Y_{t-1})^2 = {\sigma_\epsilon}^2 + 2\sigma_{b\epsilon} Y_{t-1} + {\sigma_b}^2 {Y_{t-1}}^2 + \xi_t,$$

where ξ_t are error terms. Putting

$$X = \begin{pmatrix} 1 & 2Y_0 & {Y_0}^2 \\ \vdots & \vdots & \vdots \\ 1 & 2Y_{n-1} & {Y_{n-1}}^2 \end{pmatrix} \quad \underline{\beta} = \begin{pmatrix} {\sigma_\epsilon}^2 \\ \sigma_{b\epsilon} \\ {\sigma_b}^2 \end{pmatrix}$$

$$\underline{y} = \begin{pmatrix} (Y_1 - \hat{\phi}_n Y_0)^2 \\ \vdots \\ (Y_n - \hat{\phi}_n Y_{n-1})^2 \end{pmatrix} \quad \underline{\xi} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix},$$

we obtain the equation :

$$\underline{y} = X\underline{\beta} + \underline{\xi}.$$

Here, based on X and $\underline{y}$ the least squares estimator $\hat{\underline{\beta}}_n = (\tilde{\sigma}_n^2, \tilde{\sigma}_{nb\epsilon}, \tilde{\sigma}_{nb}^2)'$ of $\underline{\beta}$ is computed. The $\tilde{\sigma}_n^2$ turns out to be a desired estimator of ${\sigma_\epsilon}^2$ (cf. Theorem 2.2).

2. MAIN RESULT

As mentioned in section 1, our testing procedure is based on prediction errors, and analyzing $\hat{\sigma_n}^2$ is an important task. The result in the following theorem is essential to construct a consistent test.

Theorem 2.1. *Under* H_0, $n^{1/2}(\hat{\sigma}_n^2 - {\sigma_\epsilon}^2)$ *is asymptotically normal. Meanwhile, it diverges to* ∞ *in probability under* H_1.

Proof: Write that

$$\begin{aligned} \hat{\epsilon_t} &= Y_t - \phi Y_{t-1} - (\hat{\phi_n} - \phi) Y_{t-1} \\ &= b_t Y_{t-1} + \epsilon_t - (\hat{\phi_n} - \phi) Y_{t-1} \end{aligned}$$

Note that

$$\begin{aligned} n^{1/2}(\hat{\sigma}_n^2 - \sigma_\epsilon{}^2) &= n^{-1/2}\sum_{t=1}^{n}(\epsilon_t{}^2 - \sigma_\epsilon{}^2) + n^{-1/2}\sum_{t=1}^{n} b_t{}^2 Y_{t-1}{}^2 + 2n^{-1/2}\sum_{t=1}^{n} b_t Y_{t-1}\epsilon_t \\ &\quad -2n^{1/2}(\hat{\phi}_n - \phi)\frac{1}{n}\sum_{t=1}^{n} b_t Y_{t-1}{}^2 - 2n^{1/2}(\hat{\phi}_n - \phi)\frac{1}{n}\sum_{t=1}^{n}\epsilon_t Y_{t-1} \\ &\quad +\frac{1}{\sqrt{n}}[n^{1/2}(\hat{\phi}_n - \phi)]^2\frac{1}{n}\sum_{t=1}^{n} Y_{t-1}{}^2 \end{aligned} \tag{2.1}$$

By our assumptions and (1.4) all terms except first two terms of right hand side of the above equality are $o_p(1)$ under both H_0 and H_1. Under H_0, the second term is identically 0. Consequently, we have that

$$n^{1/2}(\hat{\sigma}_n^2 - \sigma_\epsilon{}^2) = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}(\epsilon_t{}^2 - \sigma_\epsilon{}^2) + o_p(1). \tag{2.2}$$

On the other hand, under H_1 the second term goes to ∞ in probability. Therefore, the theorem follows. □

Lemma 2.1. *Put*

$$\Gamma = \begin{pmatrix} 1 & 0 & EY_1{}^2 \\ 0 & 4EY_1{}^2 & 2EY_1{}^3 \\ EY_1{}^2 & 2EY_1{}^3 & EY_1{}^4 \end{pmatrix}.$$

Then, under both H_0 *and* H_1,

$$n^{1/2}(\underline{\hat{\beta}}_n - \underline{\beta}) = \Gamma^{-1}(\zeta_{n1}, \zeta_{n2}, \zeta_{n3})' + o_p(1), \tag{2.3}$$

where

$$\zeta_{ni} = n^{-1/2}\sum_{t=1}^{n}\{(\epsilon_t{}^2 - \sigma_\epsilon{}^2)Y_{t-1}{}^{i-1} + 2(b_t\epsilon_t - \sigma_{b\epsilon})Y_{t-1}{}^{i} + (b_t{}^2 - \sigma_b{}^2)Y_{t-1}{}^{i+1}\},\ \ i = 1,2,3$$

Particularly, under H_0 *and* H_1,

$$n^{1/2}(\tilde{\sigma}_n^2 - \sigma_\epsilon{}^2) = O_p(1). \tag{2.4}$$

Proof: By the assumption on the model (1.1) and the fact (1.4), one can easily prove (2.3) and thus (2.4). The details are omitted for brevity. □

In fact Lemma 2.1 implies more than (2.3) : $n^{1/2}(\underline{\hat\beta}_n - \underline{\beta})$ is asymptotically normal under both H_0 and H_1 . Observe that

$$n^{1/2}(\tilde\sigma_n^2 - {\sigma_\epsilon}^2) = (1,0,0)\Gamma^{-1}n^{-1/2}\begin{pmatrix} \sum_{t=1}^n({\epsilon_t}^2 - {\sigma_\epsilon}^2) \\ \sum_{t=1}^n({\epsilon_t}^2 - {\sigma_\epsilon}^2)Y_{t-1} \\ \sum_{t=1}^n({\epsilon_t}^2 - {\sigma_\epsilon}^2){Y_{t-1}}^2 \end{pmatrix} + o_p(1).$$

Since

$$\begin{aligned} {\Gamma_{11}}^{-1} &= \{4E{Y_1}^2E{Y_1}^4 - 4(E{Y_1}^3)^2\}/det(\Gamma) \\ {\Gamma_{12}}^{-1} &= 2E{Y_1}^3E{Y_1}^2/det(\Gamma) \\ {\Gamma_{13}}^{-1} &= 4E{Y_1}^2/det(\Gamma) \\ \text{where } {\Gamma_{ij}}^{-1} &\text{ is the } (i,j)\text{th entry of } \Gamma^{-1} \\ \text{and } det(\Gamma) &= 4E{Y_1}^2E{Y_1}^4 - 4(E{Y_1}^3)^2 - 4(E{Y_1}^2)^3, \end{aligned}$$

one can write that

$$n^{1/2}(\tilde\sigma_n^2 - {\sigma_\epsilon}^2) = n^{-1/2}\sum_{t=1}^n({\epsilon_t}^2 - {\sigma_\epsilon}^2)({\Gamma_{11}}^{-1} + {\Gamma_{12}}^{-1}Y_{t-1} + {\Gamma_{13}}^{-1}{Y_{t-1}}^2) + o_p(1), \tag{2.5}$$

which is asymptotically normal due to central limit theorem for martingales.

Since $n^{1/2}(\hat\sigma_n^2 - \tilde\sigma_n^2) = n^{1/2}[(\hat\sigma_n^2 - {\sigma_\epsilon}^2) - (\tilde\sigma_n^2 - {\sigma_\epsilon}^2)]$, under H_1 $n^{1/2}(\hat\sigma_n^2 - \tilde\sigma_n^2)$ goes to ∞ in probability by Theorem 2.1 and Lemma 2.1. Meanwhile, under H_0, it follows from (2.2) and (2.5) that

$$\begin{aligned} n^{1/2}(\hat\sigma_n^2 - \tilde\sigma_n^2) &= n^{1/2}[(\hat\sigma_n^2 - {\sigma_\epsilon}^2) - (\tilde\sigma_n^2 - {\sigma_\epsilon}^2)] \\ &= n^{-1/2}\sum_{t=1}^n({\epsilon_t}^2 - {\sigma_\epsilon}^2)(1 - {\Gamma_{11}}^{-1} - {\Gamma_{12}}^{-1}Y_{t-1} - {\Gamma_{13}}^{-1}{Y_{t-1}}^2) \\ &\quad + o_p(1). \end{aligned} \tag{2.6}$$

The main results of this section are summarized in the following theorem :

Theorem 2.2. *Put* $T_n = n^{1/2}(\hat\sigma_n^2 - \tilde\sigma_n^2)$, *then* T_n *is a consistent test for* H_0 : ${\sigma_b}^2 = 0$ *vs.* H_1 : ${\sigma_b}^2 > 0$, *and is asymptotically normal under* H_0. *More precisely, under* H_0,

$$T_n \xrightarrow{d} N(0, A^2),$$

where

$$A^2 = E({\epsilon_1}^2 - {\sigma_\epsilon}^2)^2 E(1 - {\Gamma_{11}}^{-1} - {\Gamma_{12}}^{-1}Y_1 - {\Gamma_{13}}^{-1}{Y_1}^2)^2.$$

Proof: The asymptotic normality is due to (2.6) and central limit theorem for martingales. □

For a practical use of Theorem 2.2, we should estimate A^2. Note that A^2 is equal to $(E{\epsilon_1}^4 - {\sigma_\epsilon}^4)h(E{Y_1}^2, E{Y_1}^3, E{Y_1}^4)$, where h is some real valued function of R^3. It is easy to see that if we put

$$\hat{A}^2 = \{n^{-1}\sum_{t=1}^{n}\hat{\epsilon}_t^4 - (n^{-1}\sum_{t=1}^{n}\hat{\epsilon}_t^2)^2\} \cdot h(n^{-1}\sum_{t=1}^{n}{Y_t}^2, n^{-1}\sum_{t=1}^{n}{Y_t}^3, n^{-1}\sum_{t=1}^{n}{Y_t}^4),$$

$\hat{A}^2 \longrightarrow A^2$ in probability under H_0 (cf. (2.1)). Hence, under H_0, $\hat{T}_n \equiv n^{1/2}(\hat{\sigma}_n^2 - \tilde{\sigma}_n^2)/\hat{A}$ is asymptotically normal with mean 0 and variance 1.

As with T_n, we require that $\hat{T}_n$ should be consistent. Now, notice that under H_1, $n^{-1}\sum_{t=1}^{n}\hat{\epsilon}_t^2 \longrightarrow {\sigma_\epsilon}^2 + E{b_2}^2{Y_1}^2$ in probability, which can be checked easily by seeing the argument (2.1). Similarly, one can verify that under H_1, $n^{-1}\sum_{t=1}^{n}\hat{\epsilon}_t^4 \xrightarrow{p}$ constant . Since $h(n^{-1}\sum_{t=1}^{n}{Y_t}^2, n^{-1}\sum_{t=1}^{n}{Y_t}^3, n^{-1}\sum_{t=1}^{n}{Y_t}^4)$ converges to a real number in probability, $\hat{A}^2$ converges to a real number B^2 in probability under H_1 as well as under H_0 . Here B^2 may be different from A^2, but that does not affect the consistency of $\hat{T}_n$ owing to Theorem 2.2. Hence, $\hat{T}_n$ or a transform of $\hat{T}_n$ can be used as a suitable esimator for testing H_0 vs. H_1. For example, one may employ $\hat{T}_n^2$. In this case, given significant level α, one rejects H_0 if $\hat{T}_n^2 > {\chi_1}^2(\alpha)$, where ${\chi_1}^2(\alpha)$ is the α upper quantile point of chi-square distribution of one degree of freedom.

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