The Korean Communications
in Statistics Vol. 5, No. 3, 1998
pp. 935-944

Testing Goodness-of-Fit for No Effect Models[1)]

Sungho Lee[2)], Jongtae Kim[3)] and GyoungAe Moon[4)]

Abstract

This paper investigates the problem of goodness of fit tests for no effect model. The proposed test statistic Z_{mn} is obtained by multiplying constant on the model free curve estimation techniques. The small and large sample properties of Z_{mn} are investigated and the good results of power studies for the proposed test are illustrated.

1. Introduction

Assume that responses $y_1, y_2, \ldots, y_n$ are obtained from the model

$$y_i = \mu + f(x_i) + \varepsilon_i \quad , \quad i = 1, 2, \ldots, n, \tag{1.1}$$

where the x_i are design points of a predictor value x, μ is an unknown constant, f is some unknown function and the ε_i are independent and identically distributed *(i.i.d.)* normal random errors with zero mean and variance σ^2. Under this model we wish to test the hypothesis that x has no influence on the response, that is, we want to test for $H_0 : f = 0$.

Let $L_2[0,1]$ be the function space consisting of all functions f that satisfy $\|f\|^2 = \int_0^1 f^2(x)\,dx < \infty$ and $\int_0^1 f(x)\,dx = 0$. And let $\{\varphi_j\}_{j=1}^{\infty}$ be a complete orthonormal sequence (CONS) for $L_2[0,1]$ with norm $\|\cdot\|$. Assume that $f \in L_2[0,1]$, and define its Fourier coefficients corresponding to φ_j by $\alpha_j = \int_0^1 f(x)\,\varphi_j(x)\,dx$. Then the unknown function f can be expressed as a Fourier series expansion

$$f(x) = \sum_{j=1}^{\infty} \alpha_j\,\varphi_j(x) \quad , \quad x \in [0, 1].$$

One of strategies for finding $\hat{f}$ relies on a sequence of numbers to optimize the estimator. Specifically, each Fourier cosine series coefficient, $\hat{\alpha}_{jn}$, $j = 1, 2, \ldots$, of the expansion $\hat{f}$

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1) This research was supported by the Taegu University Research Grant, 1998.
2) Associate Professor, Department of Statistics, Taegu University, Taegu, 712-714, Korea
3) Assistant Professor, Department of Statistics, Taegu University, Taegu, 712-714, Korea
4) Assistant Professor, Department of Office Automation, Donghae Junior College, Donghae, Korea.

is multiplied by a real number b_j called multiplier which decreases as j increases so that coefficients of $\hat{f}$ can be gradually tapered to zero instead of being sharply truncated, and given by

$$\hat{f}(x) = \sum_{j=1}^{\infty} b_j \hat{\alpha}_{jn} \varphi_j(x), \quad x \in [0,1].$$

In this paper, we will propose the test based on a functional of each sample Fourier coefficient multiplied by a real number gradually tapered to zero in the expansion of $\hat{f}$.

2. Review of Tests For No Effect

von Neumann(1941) suggested a test statistic based on the sum of squares of first differences of the data to be reciprocal of

$$T_N = \frac{1}{n-1} \frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{\sum_{i=1}^{n-1} (y_{i+1} - y_i)^2}, \tag{2.1}$$

which rejects the null model for large values of (2.1). Recently, Munson and Jernigan(1989) have developed a modified version of the von Neumann(1941) ratio statistic, which is equivalent to the Durbin and Waston(1971) statistic when the regression is constant. Eubank and Hart(1993) showed that it is asymptotically equivalent to the von Neumann(1941) ratio statistic, which is given by

$$T_N = \sum_{j=1}^{n-1} \frac{\hat{\alpha}_{jn}^2}{\hat{\sigma}^2}, \tag{2.2}$$

where the Fourier cosine series coefficient is

$$\hat{\alpha}_{jn} = \frac{\sqrt{2}}{\sqrt{n}} \sum_{i=1}^{n} y_i \cos(j\pi x_i), \tag{2.3}$$

and

$$\hat{\sigma}^2 = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} (y_i - y_{i+1})^2, \tag{2.4}$$

$\hat{\sigma}^2$ is the consistent estimator of σ^2 proposed by Rice(1984) based on successive differences $y_i - y_{i+1}$.

Buckley(1991) sought to detect any smooth variation in function f in viewpoint of Bayesian, not to specify a parametric alternatives, assuming that $f = h(n)g$, where $h(n)$ allows an arbitrary departure from the null.

Given a particular prior distribution $(g(x_1), g(x_2), \ldots, g(x_n))'$ a statistic is given proportional to

$$T_B = \frac{1}{n^2} \sum_{j=1}^{n} \frac{\left[\sum_{i=1}^{j} (y_i - \bar{y})^2 \right]^2}{\hat{\sigma}^2} \tag{2.5}$$

with $\hat{\sigma}^2$ used for (2.5), where $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$ is given for testing H_0. These kinds of statistics have been considered by Nair(1986) and Hirotsu(1986) against ordered alternatives. Eubank and Hart(1993) showed that T_B can be represented as follows :

$$T_B = \left(\sum_{j=1}^{n-1} \frac{\hat{\alpha}^2_{jn}}{\gamma_j} \right) / \hat{\sigma}^2 \tag{2.6}$$

with $\gamma_j = \left\{ 2n \sin(\frac{j\pi}{2n}) \right\}^2$, by using the fact by Nair(1986) and the Fourier cosine series coefficient $\hat{\alpha}_{jn}$ used in (2.3).

As noted by Eubank and Hart(1993), T_B and T_N have some problems detecting alternatives with lower or higher frequency. In order to overcome difficulties arising when T_B and T_N are used as test statistics, Eubank and Hart(1993) have proposed some tests. First, they considered a test derived by using a standard nonparametric estimator of f, $\hat{f}_m(t) = \frac{\sqrt{2}}{\sqrt{n}} \sum_{j=1}^{m} \hat{\alpha}_{jn} \cos(j\pi x)$, where $1 \le m \le n$ is an integer. Then, the test statistic is

$$T_m = \sum_{j=1}^{m} \frac{\hat{\alpha}^2_{jn}}{\hat{\sigma}^2}, \tag{2.7}$$

and the null hypothesis is rejected for large value of (2.7).

Although, $T_m = T_N$ for $m = n - 1$, m is often expected to be much smaller than n. The statistic T_m assigns weights 0 or 1 to the $\hat{\alpha}^2_{jn}$ according to whether or not m is larger than j, while T_B down-weights $\hat{\alpha}^2_{jn}$ by $\frac{1}{\gamma_j}$.

3. The Proposed Test

We now propose two new tests for testing $H_0 : f = 0$ under the model

$$y_i = \mu + h(n) g\left(\frac{2i-1}{2n}\right) + \varepsilon_i, \quad i = 1, 2, \ldots, n, \tag{3.1}$$

where μ is an unknown parameter, g is some unknown function satisfying $\int_0^1 g(x)dx = 0$ and $h(n)$ is some function of the sample size that satisfies $h(n) \to 0$ as $n \to \infty$ and the ε_i are independent and identically distributed normal random errors with mean zero and variance σ^2. If the null hypothesis is true, the residuals $e_i = y_i - \hat{y}$ from fitting the null model (3.1) with $g = 0$ should have no pattern as a function of x. Thus, given some nonparametric regression fit $\hat{g}$ to the residuals, we can base a test on quadratic functional form of g, for example, $\sum_{i=1}^{n} \hat{g}^2(x_i)$. The test statistics derived arise from the above perspective.

For some integer $1 \le m \le n$, the Fourier cosine series estimator of g with m terms to the residuals e_i can be considered as follows :

$$\hat{g}(x) = \sqrt{2} \sum_{j=1}^{m} b_j \hat{a}_{jn} \cos(j\pi x), \tag{3.2}$$

with the multiplier sequence

$$b_j = 1 - \frac{j}{m+1} \tag{3.3}$$

that makes coefficients of $g(x)$, to be gradually tapered to zero and was noticed by Tarter and Lock (1993), and

$$\hat{a}_{jn} = \frac{\sqrt{2}}{\sqrt{n}} \sum_{i=1}^{n} y_i \cos(j\pi x_i) , \tag{3.4}$$

as an estimator of each Fourier coefficient $a_j = \sqrt{2} \int_0^1 g(x) \cos(j\pi x) dx$ in the expansion of $g(x)$.

we propose the new test statistic based on $\sum_{i=1}^{n} \hat{g}^2(x_i)$ as follows :

$$Z_m = \frac{n \sum_{j=1}^{m} b_j^2 \widehat{a^2}_{jn} - \hat{\sigma} \sum_{j=1}^{m} b_j^2}{\hat{\sigma}^2 \left(2 \sum_{i=1}^{m} b_j^4\right)^{1/2}} , \tag{3.5}$$

where $\hat{\sigma}^2$ is any consistent estimator of σ^2.

Theorem 3.1. (Kim and Moon(1994)) Under the assumptions of model (3.1), assume that $m \to \infty$, $n \to \infty$ in such a way that $\sup_{1 \le j \le m} m^{1/2} \gamma(m, n) \to 0$, where $\gamma(m, n) = | a_{jn} - a_{jm} |$. Then if $h(n) = m^{1/4}/\sqrt{n}$, $Z_m \xrightarrow{d} Z$ where Z is a normal random variable with unit variance and mean $\frac{\sqrt{5} \| g \|^2}{\sqrt{2}\sigma^2}$.

One of our interests for test is to see whether the test is consistent. The next theorem provides the consistency of the test.

Theorem 3.2 Define $\alpha_{jn} = \frac{\sqrt{2}}{\sqrt{n}} \sum_{i=1}^{n} f(x_i) \cos(j\pi x_i)$ and $\alpha_j = \sqrt{2} \int_0^1 f(x) \cos(j\pi x)\, dx$. For any integer m assume that as $n, m \to \infty$,

$$m^{1/2} \sup_{1 \le j \le m} |\alpha_{jn} - \alpha_j| \to 0 .$$

Proof. We will also assume that σ^2 is known as in Theorem 3.1. First note that $P(Z_m \ge z_m \alpha) = P(A_n + B_n + C_n \ge z_m \alpha)$ with $\sigma_m = \sigma^2 \left(2 \sum_{j=1}^{m} b_j^4 \right)^{1/2}$,

$$A_n = \left(n \sum_{j=1}^{m} b_j^2 (\hat{a}_{jn} - \alpha_{jn})^2 - \sigma^2 \sum_{j=1}^{m} b_j^2 \right) / \sigma_m ,$$

$$B_n = 2n \sum_{j=1}^{m} b_j^2 (\hat{a}_{jn} - \alpha_{jn}) \alpha_{jn} / \sigma_m \text{ and}$$

$$C_n = n \sum_{j=1}^{m} b_j^2 \alpha_{jn}^2 / \sigma_m .$$

A_n converges in distribution to a normal random variable with mean zero and unit variance by the Lindeberg-Feller theorem. Now, $E(B_n) = 0$ and by Lemma 3.1,

$$Var\{B_n\} \le O\left(\frac{n}{m}\right) (m \gamma^2(m,n) + 2 \|f\| m^{1/2} \gamma(m,n) + \|f\|^2) .$$

Thus, Chebyshev's inequality shows B_n to $O_p\left(\left(\frac{n}{m}\right)^{1/2}\right)$. Since

$$C_n \le \frac{1}{\sqrt{2}\sigma^2} O\left(\frac{n}{m}\right) (\gamma^2(m,n) m + 2\|f\| m^{1/2} \gamma(m,n) + \|f\|^2)$$

by Cauchy-Schwartz inequality, It follows that $\frac{\sqrt{m}}{n} Z_m \xrightarrow{d} \frac{\|f\|^2}{\sqrt{2}\sigma^2}$ and theorem has been proved.

By considering alternatives corresponding to certain subsets of the collection of all square summable sequences, we can get an analogue result for comparing Z_m with T_B, T_m and which is a stronger result than Theorem 3.3.

Theorem 3.3. For any integer m and constants $0 < \gamma_1 < \gamma_2 < \infty$, and define

$$\varrho_m(\gamma_1, \gamma_2) = \left\{ \xi = (\ldots, \xi_{-1}, \xi_0, \xi_1, \ldots) \in l_2 : \sum_{|j| \le m} \xi_j^2 / j^4 \ge \gamma_1 \text{ and } \sum_{j=-\infty}^{\infty} \xi_j^2 \le \gamma_2 \right\} .$$

Assuming that $n, m \to \infty$ and $\varphi_j(x) = \sqrt{2} \cos(j\pi x)$, then for any $\alpha \in (\alpha, 1)$ there exists γ_1 and γ_2 such that

$$\lim_{n \to \infty} \inf_{\xi \in \varrho_m(\gamma_1, \gamma_2)} P\left(Z_m \ge z_m \alpha \mid \frac{m^{1/4}}{\sqrt{n}} \sum_{j=1}^{m} b_j^{-1} \xi_j \varphi_j / \sqrt{n} \right) \ge \beta .$$

Proof. We will assume in our proof that σ^2 is known since the extension to the case where σ^2 is replaced by a consistent estimator is handled as in Theorem 3.1. First note that

$$P\left(Z_m \geq z_m \alpha \mid \frac{m^{1/4}}{\sqrt{n}} \sum_{j=1}^{m} b_j^{-1} \xi_j \varphi_j \right)$$

$$= P\left(A_n + B_n + C_n \geq z_m \alpha \mid \frac{m^{1/4}}{\sqrt{n}} \sum_{j=1}^{m} b_j^{-1} \xi_j \varphi_j \right)$$

with

$$A_n = \frac{n \sum_{j=1}^{m} b_j^2 \left(\hat{a}_{jn} - \frac{m^{1/4}}{\sqrt{n}} \sum_{j=1}^{m} b_j^{-1} \xi_j \right)^2 - \sigma^2 \sum_{j=1}^{m} b_j^2}{\sigma^2 \left(2 \sum_{j=1}^{m} b_j^4 \right)^{1/2}},$$

$$B_n = \frac{2\sqrt{n}\, m^{1/4} \sum_{j=1}^{m} b_j \left(\hat{a}_{jn} - \frac{m^{1/4}}{\sqrt{n}} b_j^{-1} \xi_j \right) \xi_j}{\sigma^2 \left(2 \sum_{j=1}^{m} b_j^4 \right)^{1/2}}$$

and

$$C_n = m^{1/2} \sum_{j=1}^{m} \xi_j^2 \Big/ \left(\sigma^2 (2 \sum_{j=1}^{m} b_j^4)^{1/2} \right),$$

since $C_n = m^{1/2} \sum_{j=1}^{m} \xi_j^2 \Big/ \left(\sigma^2 \left(2 \sum_{j=1}^{m} b_j^4 \right)^{1/2} \right) \geq \gamma_1 / \sqrt{2}\, \sigma^2$, the least equality is

$$P\left(A_n + B_n + C_n \geq z_m \alpha - \frac{\gamma_1}{\sqrt{2}\sigma^2} \mid \frac{m^{1/4}}{\sqrt{n}} \sum_{j=1}^{m} b_j^{-1} \xi_j \varphi_j \right)$$

$$\geq P\left(A_n + B_n \geq z_m \alpha - \frac{\gamma_1}{\sqrt{2}\sigma^2} \mid \frac{m^{1/4}}{\sqrt{n}} \sum_{j=1}^{m} b_j^{-1} \xi_j \varphi_j \right).$$

Now, $E\{B_n\} = 0$ because $E\left\{ \hat{a}_{jn} - \frac{m^{1/4}}{\sqrt{n}} b_j^{-1} \xi_j \right\} = 0$. And by (2) of Lemma 3.1.

$$Var\{B_n\} \leq \frac{2\, m^{1/2} \sum_{j=1}^{m} \xi_j^2}{\sigma^2 \sum_{j=1}^{m} b_j^4} = O\left(\frac{1}{\sqrt{m}} \right)$$

for any alternative corresponding to $\mathcal{C}_m(\gamma_1, \gamma_2)$. Now,

$$P\left(A_n + B_n \geq z_m \alpha - \frac{\gamma_1}{\sqrt{2}\sigma^2} \mid \frac{m^{1/4}}{\sqrt{n}} \sum_{j=1}^{m} b_j^{-1} \xi_j \varphi_j \right)$$

$$= P\left(A_n + B_n \geq z_m \alpha - \frac{\gamma_1}{\sqrt{2}\sigma^2},\ |B_n| > \varepsilon \mid \frac{m^{1/4}}{\sqrt{n}} \sum_{j=1}^{m} b_j^{-1} \xi_j \varphi_j \right)$$

$$+ P\left(A_n + B_n \geq z_m \alpha - \frac{\gamma_1}{\sqrt{2}\sigma^2},\ |B_n| \leq \varepsilon \mid \frac{m^{1/4}}{\sqrt{n}} \sum_{j=1}^{m} b_j^{-1} \xi_j \varphi_j \right).$$

The first term of right-hand side is

$$P\left(A_n + B_n \geq z_m \alpha - \frac{\gamma_1}{\sqrt{2}\sigma^2} \; , \; |B_n| > \varepsilon \;\middle|\; \frac{m^{1/4}}{\sqrt{n}} \sum_{j=1}^{m} b_j^{-1} \xi_j \varphi_j \right)$$
$$\leq P(\, |B_n| > \varepsilon \,) \; \leq \; \frac{o(1)}{\varepsilon^2}$$

with $o(1) \to 0$ as $n \to \infty$ uniformly over $\xi \in \varrho (\gamma_1, \gamma_2)$ by Chebyshev's inequality and the fact that $Var(B_n) = O(1/\sqrt{m})$ Now, it remains to show that

$$\lim_{n \to \infty} P\left(A_n + B_n \geq z_\alpha - \frac{\gamma_1}{\sqrt{2}\sigma^2} \;\middle|\; |B_n| \leq \varepsilon \right) \geq \alpha \, .$$

Since $-\varepsilon \leq B_n \leq \varepsilon$, for any $\varepsilon > 0$ there exists an n_0 such that for all $n > n_0$ (3.6) is bounded below by

$$P\left(A_n \geq z_\alpha - \frac{\gamma_1}{\sqrt{2}\sigma^2} + \varepsilon \right) .$$

Now A_n converges in distribution to a normal random variable with mean zero and unit variance by Lindebuger-Fellow theorem. Thus, letting $n \to \infty$ and $\varepsilon \to 0$, the desired result is obtained.

Theorem 3.3 states that Z_m has also comparable power against alternatives of the form $\frac{m^{1/4}}{\sqrt{n}} \sum_{j=1}^{m} b_j^{-1} \xi_j \varphi_j$ while T_B has comparable power against alternatives of the form $\sum_{j=1}^{m} j \xi_j \varphi_j / \sqrt{n}$. But we can see that the higher frequency components corresponding to $\varphi_j(x_i)$ with $j > m^{1/4} b_j^{-1}$ will be farther away from the null in the alternatives of T_B than in those of Z_m for not too small value of m. This fact gives some theoretical justification for the experimental conclusion that Z_m has better power than T_B type statistics for many alternatives if we allow the number of temrs not to be much small. But compared with T_m , the higher frequency components corresponding to $\varphi_j(x_i)$ probably will be closer to the null in alternatives of T_m than in those of Z_m since except for smaller j and $m^{1/4} < m^{1/4} b_j^{-1}$ when m is more or less small. Therefore, we may expect that under alternatives with lower frequency, Z_m will have about the same power as T_B to say nothing of T_m . In contrast, it is predicted that Z_m is perhaps less effective detecting alternatives with higher frequency for a smaller m . But Z_m may have comparable powers with T_m for a little larger m relative to its frequency against both lower and higher alternatives.

4. A Monte Carlo Simulation

We examine the small sample properties of our test for fixed alternatives in order to investigate whether our asymptotic results can be extanded in finite samples through the Monte Carlo simulation study in this section. The Monte Calro simulation performed by samples of size equal to 20 and 40 which were generated from model (1.1) using the ε_i uncorrelated normal random errors. The variance of error, σ^2 was assumed to be known and without loss of generality, to be equal to unit. And x_i were taken to be equally spaced. For the function f in (1.1), we used

$$f_1(x) = b\left(e^{4x} - \frac{(e^4-1)}{4}\right)\left(\frac{(e^8-1)}{8} - \frac{(e^4-1)^2}{4}\right)^{-1/2},$$

where $b = 0.25, 0.5, 0.75, 0.1$, and

$$f_2(x) = \gamma \cos(j\pi x),$$

where $\gamma = 1$, and $j = 1, 3, 5, 7$ is to be manipulated to obtain higher and lower frequency alternatives.

4.1 Powers Against The Alternative f_1 when $\alpha = 0.05$

Sample Size	Type	b 0.25	0.50	0.75	1.00
20	T_N	0.11000	0.27400	0.55100	0.80700
	T_B	0.16200	0.44500	0.77800	0.94900
	T_m	0.08300	0.17400	0.36800	0.65000
	Z_m	0.16600	0.44200	0.77800	0.94200
40	T_N	0.10700	0.39000	0.77300	0.97200
	T_B	0.26200	0.78000	0.98700	1.00000
	T_m	0.07400	0.25800	0.66800	0.95100
	Z_m	0.24100	0.75100	0.98200	1.00000

For each above combination, first uniform pseudo-random numbers are generated by GGUBS in IMSL package. Using Box-Muller transformation method, we generated the normal random errors with mean 0 and variance 1 for samples of each size. For samples of each size, the proportions of times T_B, T_N, T_m, and Z_m exceeded their approximate upper α - level critical values, 0.05, were recorded. In this case, the approximate upper α - level Table critical values were empirically found by simulation from the null distribution of these statistics since, in general, it is well known that the normal approximation does not always work well. In doing this, we used 10,000 trials to get more satisfactory critical values for these statistics in each case.

Table 4.2 Powers Against The Alternative f_2 when $\alpha=0.05$.

Sample Size	Type	j = 1	3	5	7
20	T_N	0.50900	0.48900	0.45500	0.40300
	T_B	0.81200	0.31400	0.11500	0.06700
	T_m	0.33000	0.33500	0.32700	0.33700
	Z_m	0.81500	0.36800	0.18800	0.09300
40	T_N	0.72200	0.73100	0.72600	0.69400
	T_B	0.98700	0.72300	0.25600	0.10900
	T_m	0.58200	0.63400	0.64700	0.65400
	Z_m	0.98400	0.71100	0.33600	0.20900

In summary, simulation results are likely to support our asymptotic analysis for the most part. As we have seen in the examination of Table 4.1, the Berckley T_B test and Z_m test have excellent power against alternatives f_1 , followed by T_N and then T_m .

In the case of alternatives f_2 in Table 4.2, when $j = 1$, Both Z_m and T_B have more excellent power than T_N , T_m . But, when f_2 has higher frequency, the Berckley T_B test has the poor performance. This mean is that T_B and Z_m test may have some difficulties in detecting higher frequency alternatives. So, as see the result in Table 4.2, when the cases of $j = 5$ and 7, Both T_N and T_m tests have more excellent power than Z_m , T_B tests.

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