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Bayesian Analysis in Generalized Log-Gamma Censored Regression Model

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Abstract

For industrial and medical lifetime data, the generalized log-gamma regression model is considered. Then the Bayesian analysis for the generalized log-gamma regression with censored data are explained and following the data augmentation (Tanner and Wang; 1987), the censored data is replaced by simulated data. To overcome the complicated Bayesian computation, Makov Chain Monte Carlo (MCMC) method is employed. Then some modified algorithms are proposed to implement MCMC. Finally, one example is presented.

1. Introduction

In this paper, the generalized log-gamma regression model is considered. The reason for examming this model is as follows : This model includes the log-exponential, log-gamma, and lognormal as special cases. As such, it is often suggested as a model for industrial and medical lifetime data (Farewell and Prentice 1977 ; Lawless 1982) useful for discriminating among these common distributions, as well as for providing a flexible parametric family of distributions for modelling the data. Formally, the generalized log-gamma model(Lawless, 1982) is given by

$$Y_i = T_i^T \beta + \sigma \varepsilon_i , i = 1, \cdots , n$$

where Y_i is the log lifetime, $\beta = (\beta_0, \cdots, \beta_{p-1})$ is a $p*1$ vector and \$ ε\$ has the generalized log gamma distribution with density function

$$f(\varepsilon \mid k) = \begin{cases} \frac{k^{k-\frac{1}{2}}}{\Gamma(k)} \exp[k^{\frac{1}{2}} \varepsilon - k e^{\frac{\varepsilon}{\sqrt{k}}}], & 0 < k < \infty \\ \frac{1}{\sqrt{2\pi}} \exp(-\frac{\varepsilon^2}{2}), & k = \infty \end{cases} \tag{1.1}$$

With the distribution of ε written in the form (1.1), the extreme value(log Weibull) and

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normal(log log normal) distributions are given by the case $k=1$ and $k=\infty$, respectively. Since the model depends on the value of k, Farewell and Prentice(1977) considered the maximum likelihood estimator(MLE) of k in life testing in order to study distributional shape and to discriminate between special cases.

We assume that $n-m\ (m \leq n)$ observations are censored since the life time data is sometimes censored. For frequentist analysis with incomplete data, Dempster, Laird and Rubin(1977), Ibrahim(1990) investegated the maximum likelihood estimate of parameters using EM algorithm. Gelfand and Smith(1990), Chib(1992) and Chung(1997) have studied Bayesian analysis with incompete data. Chib(1992) studied Bayes inference in the Tobit censored regression model using Gibbs sampler. We study the Bayesian analysis of the generalized log-gamma regression model asssuming that k's are different fixed values.

The purpose of this paper is to investigate the approaches to solve the computational difficulty in Bayesian Inference.

The paper is organized as follows: In section 2, the generalized log-gamma distribution and the Bayesian analysis for generalized log-gamma regression with censored data are explained. Then the sampling methods are proposed to apply Gibbs Sampler. In section 3, one example (Schmee and Hahn, 1979) is presented.

2. Bayesian Inference for Censored Regression Model

In this section, we consider the generalized log-gamma regression model with censored data.

2.1 Generalized log gamma distribution

Weibull and lognormal models are commonly used to represent failure times. Specifically, if failure time is denoted by t and $y=\log t$, then the generalized gamma models give, for $a \in R,\ b,\ k>0,\ y=a+bv$ where the probability density function of v is $\Gamma(k)^{-1}\exp(kv-e^{v})$ and Γ denotes the gamma integral. The generalized gamma distributio n(g.g.d.) is a three-parameter distribution with probability density function of the form

$$f(t)=\frac{\beta}{\Gamma(k)}\frac{t^{\beta k-1}}{\alpha^{\beta k}}\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right],\quad t>0. \tag{2.1}$$

We will reparameterize the model(2.1); suppose that T has p.d.f.(2.1), and let $Y=\log T$. Then it easily follows from (2.1) that Y has p.d.f.

$$f(y)=\frac{1}{\Gamma(k)b}\exp\left[k\left(\frac{y-u}{b}\right)-\exp\left(\frac{y-u}{b}\right)\right],\qquad -\infty<y<\infty$$

where $u=\log\alpha$ and $b=\beta^{-1}$. Here, $\frac{Y-u}{b}$ has a log gamma distribution. As $k\rightarrow\infty$

the mean and variance of $W_1 = \frac{Y-u}{b}$ become infinite: thus we make a further transformation and consider the variate. Let $W = \sqrt{k}\,(W_1 - \log k) = \frac{Y-u}{\sigma}$ where $\sigma = \frac{b}{\sqrt{k}}$ and $\mu = u + b\log k$. Then the probability density function of W, for $0 < k < \infty$, is readily found to be

$$f(w) = \frac{k^{k-\frac{1}{2}}}{\Gamma(k)} \exp[\sqrt{k}w - ke^{\frac{w}{\sqrt{k}}}],\ -\infty < w < \infty. \tag{2.2}$$

The distribution of W does not change greatly as k gets large. It is easily shown that as $k \to \infty$, (2.2) approaches the standard normal probability density function $\frac{1}{\sqrt{2\pi}} e^{-\frac{W^2}{2}}$ (see Abramowitz and Stegun, 1965, Chapter 6). The log exponential distribution corresponds to the special case $k=1$ and $\sigma = 1$ in the parameterization, whereas the (log) two-parameter gamma distribution, which given by $\beta = 1$ in the old parameterization, is now given by $\sigma\sqrt{k} = 1$.

2.2. Linear regression model with censored data

Densities are denoted generically by brackets, so joint, conditional, and marginal forms, for example, appear as $[X, Y]$, $[X|Y]$ and $[X]$, respectively. We consider the linear regression model with censored data as follows :

$$t_i = \beta_0 + \sum_{j=1}^{p-1} \beta_j \nu_{ij} + \sigma\varepsilon_i \quad i = 1, \cdots, n \tag{2.3}$$

where ε_i is distributed to generalized log gamma distribution (2.2).

Since we have little information about (β, σ^2), it is reasonable assumption that the prior distribution $[\beta_0, \beta_1, \cdots, \beta_{p-1}, \sigma^2] \propto \frac{1}{\sigma^2}$. Then the joint posterior density can be written as

$$\begin{aligned}
&[\beta_0, \beta_1, \cdots, \beta_{p-1}, \sigma^2 | T] \\
&\propto \prod_{i=1}^{n} [t_i | \beta_0, \beta_1, \cdots, \beta_{p-1}, \sigma^2][\beta_0, \cdots, \beta_{p-1}, \sigma^2] \\
&\propto \frac{1}{\sigma^{n+2}} \exp\left\{ \frac{\sqrt{k}}{\sigma} \sum_{i=1}^{n} (t_i - \beta_0 - \sum_{j=1}^{p-1} \beta_j \nu_{ij}) - k \sum_{i=1}^{n} \exp \frac{(t_i - \beta_0 - \sum_{j=1}^{p-1} \beta_j \nu_{ij})}{\sqrt{k}\sigma} \right\} \\
&= g(\beta_0, \beta_1, \cdots, \beta_{p-1}, \sigma^2).
\end{aligned} \tag{2.4}$$

Since the life time data is sometimes censored, we assume that $n-m$ observations are censored. Without the loss of generality, the right-censored observations are assumed. Then we will reorder the data so that the first m observations are uncensored and remaining $n-m$ are right-censored(c_i denotes a censored event time). The essential idea(following data augmentation) is quite simple. Suppose that along with the censored observation y_i, it is available the corresponding latent data, say z_i Define Z_j as the (unobserved) variate for the case of T_j such that $Z_j > c_j$. That is, let $T=(y_1,\cdots,y_m, z_{m+1},\cdots,z_n)$ To apply the Gibbs sampler, we need the full conditional distributions(FCD) as follows:

$$[\beta_0 \mid \beta_1,\cdots,\beta_{p-1},\sigma^2,T] \propto \exp\Big\{-\frac{\sqrt{k}}{\sigma}n\beta_0 - k\Big\{\sum_{i=1}^{m}\exp[\frac{y_i-\beta_0-\sum_{j=1}^{p-1}\beta_j v_{ij}}{\sqrt{k}\sigma}]+\sum_{i=m+1}^{n}\exp[\frac{z_i-\beta_0-\sum_{j=1}^{p-1}\beta_j v_{ij}}{\sqrt{k}\sigma}]\Big\}\Big\} \tag{2.5}$$

for $1 \le a \le p-1$,

$$[\beta_a \mid \beta_0,\cdots,\beta_{a-1},\beta_{a+1},\cdots,\beta_{p-1},\sigma^2,T] \propto \exp\Big\{-\frac{\sqrt{k}}{\sigma}\sum_{i=1}^{n}v_{ia}\beta_a - k\Big\{\sum_{i=1}^{m}\exp[\frac{y_i-\beta_0-\sum_{j=1}^{p-1}\beta_j v_{ij}}{\sqrt{k}\sigma}]+\sum_{i=m+1}^{n}\exp[\frac{z_i-\beta_0-\sum_{j=1}^{p-1}\beta_j v_{ij}}{\sqrt{k}\sigma}]\Big\}\Big\}, \tag{2.6}$$

$$[\sigma^2 \mid \beta_0,\beta_1,\cdots,\beta_{p-1},T] \propto \sigma^{-(n+2)}\times \exp\Big\{\sum_{i=1}^{m}\frac{\sqrt{k}}{\sigma}(y_i-\beta_0-\sum_{j=1}^{p-1}\beta_j v_{ij})+\sum_{i=m+1}^{n}\frac{\sqrt{k}}{\sigma}(z_i-\beta_0-\sum_{j=1}^{p-1}\beta_j v_{ij})\ n - k\ [\sum_{i=1}^{m}\exp(\frac{y_i-\beta_0-\sum_{j=1}^{p-1}\beta_j v_{ij}}{\sqrt{k}\sigma})+\sum_{i=m+1}^{n}\exp(\frac{z_i-\beta_0-\sum_{j=1}^{p-1}\beta_j v_{ij}}{\sqrt{k}\sigma})]\Big\} \tag{2.7}$$

and

$$[z_i \mid \beta_0,\beta_1,\cdots,\beta_{p-1},\sigma^2,Y]=\frac{g(z_i \mid \beta_0,\cdots,\beta_{p-1},\sigma^2)}{1-G(c_i)}, \quad z_i > c_i, \tag{2.8}$$

where g is the density funstion of Z_i and $G(c_i)$ denotes the low tail probability upto c_i under the density g.

To simulate random variables from FCD, the methods used in the computations are described as follows:

The following lemma is useful to sample β_0 from $[\beta_0 \mid \beta_1,\cdots,\beta_{p-1},\sigma^2,T]$ in (2.5).

Lemma 2.1. Suppose that γ is generated from gamma distribution with kn and 1. That is, γ is distributed to $f(\gamma) = \frac{1}{\Gamma(kn)} \gamma^{kn-1} e^{-\gamma}$. Let

$$a = k\left\{ \sum_{i=1}^{m} \exp\left[\frac{(y_i - \sum_{j=1}^{p-1} \beta_j v_{ij})}{\sqrt{k}\sigma}\right] + \sum_{i=m+1}^{n} \exp\left[\frac{(z_i - \sum_{j=1}^{p-1} \beta_j v_{ij})}{\sqrt{k}\sigma}\right] \right\}.$$

Then $\beta_0 = \sqrt{k}\,\sigma \log\left(\frac{a}{\gamma}\right)$ is distributed to $[\beta_0 | \beta_1, \cdots, \beta_{p-1}, \sigma^2, T]$.

Proof. Let $\gamma = a \exp[-\frac{\beta_0}{\sqrt{k}\sigma}]$. Then $\beta_0 = \sqrt{k}\,\sigma \log\left(\frac{a}{\gamma}\right)$ and Jacobian is $|J| = a \exp[-\frac{\beta_0}{\sqrt{k}\sigma}] \frac{1}{\sqrt{k}\sigma}$. So, the probability density function of β_0 is

$$\begin{aligned} f(\beta_0) &= \frac{1}{\Gamma(kn)} a^{kn-1} [\exp(-\frac{\beta_0}{\sqrt{k}\sigma})]^{kn-1} \exp[-a\exp(-\frac{\beta_0}{\sqrt{k}\sigma})] |J| \\ &\propto \exp(-\frac{n\sqrt{k}}{\sigma}\beta_0) \exp[-a\exp(-\frac{\beta_0}{\sqrt{k}\sigma})] \\ &\propto [\beta_0 | \beta_1, \cdots, \beta_{p-1}, \sigma^2, T]. \end{aligned}$$

Thus, the random variate β_0 is distributed to $[\beta_0 | \beta_1, \cdots, \beta_{p-1}, \sigma^2, T]$.

When the random variates are generated from $[\beta_a | \beta_{(-a)}, \sigma^2, T]$ for $a = 1, \cdots, p-1$ in (2.6) where $\beta_{(-a)} = (\beta_0, \cdots, \beta_{a-1}, \beta_{a+1}, \cdots, \beta_{p-1})$. Metropolis algorithm (Metropolis et al. 1953) is employed. However the Metropolis algorithm can not be applied to $[\sigma^2 | \beta_0, \cdots, \beta_{p-1}, T]$ in (2.7) since the support set of σ^2 is positive real line. So the following algorithm is used for sampling from $[\sigma^2 | \beta_0, \cdots, \beta_{p-1}, T]$ in (2.7).

Modified Algorithm 2.1. Since σ^2 is a variable with range in positive real line, we can use a transformation, such as $\sigma^{2'} = \log \sigma^2$, to map$(0, \infty)$ into $(-\infty, \infty)$, then use the transition kernel and applying of the Metropolis algorithm to the density of $\sigma^{2'}$. After one transition of the Metropolis algorithm is done, then we transform $\sigma^{2'}$ back to the original scale means of $\sigma^2 = \exp \sigma^{2'}$.

The following modified algorithm 2.2 is useful to sample from $[z_i | \beta_0, \cdots, \beta_{p-1}, \sigma^2, T]$ for $1 \le i \le n-m$ in (2.8).

Modified Algorithm 2.2. Assume that $k > 1$.

1. Generate Y_i from exponential distribution and set $X_i = Y_i + c_i'$.

   For notational conveneince, let $c_i' = \exp[\frac{1}{\sqrt{k}} \frac{c_i - \beta_0 - \sum_{j=1}^{p-1} \beta_j v_{ij}}{\sigma}]$.

2. Generate U from uniform distribution on [0,1].

   If $U \le g(x_i) = \frac{x_i^{k-1} e^{-(k-1)x_i}}{e^{-(k-1)}}$ then accept x_i.

   Otherwise, go to until accept x_i .

3. Set $\varepsilon_i = \sqrt{k} \log X_i$.

   Then $z_i = \beta_0 + \sum_{j=1}^{p-1} \beta_j v_{ij} + \sigma \varepsilon_i$ is a sample from $[z_i | \beta, \sigma^2, Y]$.

The implementation of Gibbs sampler is briefully described in the following.

Step 1. Starting with the initial guesses at $\beta_1^{(0)}, \ldots, \beta_{p-1}^{(0)}, \sigma^{2(0)}$ and $z_{m+1}^{(0)}, \ldots, z_n^{(0)}$.
Step 2. The usual Gibbs iteration is as follows:

$\beta_0 \sim [\beta_0 | \beta_1^{(0)}, \cdots, \beta_{p-1}^{(0)}, \sigma^{2(0)}, z_{m+1}^{(0)}, \cdots, z_n^{(0)}]$ in (2.5) using lemma 2.1. For $1 \le a \le p-1$,

$\beta_a \sim [\beta_a | \beta_0^{(1)}, \cdots, \beta_{a-1}^{(1)}, \beta_{a+1}^{(0)}, \cdots, \beta_{p-1}^{(0)}, \sigma^{2(0)}, z_{m+1}^{(0)}, \cdots, z_n^{(0)}]$ in (2.6) using Metroplis algorithm.

$\sigma^2 \sim [\sigma^2 | \beta_0^{(1)}, \beta_1^{(1)}, \cdots, \beta_{p-1}^{(1)}, z_{m+1}^{(0)}, \cdots, z_n^{(0)}]$ in (2.7) using Modified algorithm 2.1. And
$z_i \sim [z_i | \beta_0^{(1)}, \beta_1^{(1)}, \cdots, \beta_{p-1}^{(1)}, \sigma^{2(1)}, z_{m+1}^{(1)}, \cdots, z_{i-1}^{(1)}, z_{i+1}^{(0)}, \cdots, z_n^{(0)}]$ in (2.8) using Modified algorithm 2.2.

The above two steps form an iteration which updates $\beta_0^{(0)}, \cdots, \beta_{p-1}^{(0)}, \sigma^{2(0)}, z^{(0)}$ to $\beta_0^{(1)}, \cdots, \beta_{p-1}^{(1)}, \sigma^{2(1)}, z^{(1)}$. Thus t such iterations produce a "one-string run". Also, n parallel strings are run with different starting positions to make sure that the samples converge to the whole posterior distribution, instead of a local maximum of the posterior distribution. Each cnvergence of Gibbs sampler and Metropolis algorithm is checked using Gelman and Rubin's (1992) method.

3. Illustrative Example

Schmee and Hahn(1979) used the results of temperature accelerated life tests on electrical insulation in 40 motorettes in order to illustrate the iterative least square method. Ten motorettes were tested at each of the four temperatures: $150^{o}, 170^{o}, 190^{o}$, and 220^{o} in degrees ^{o}c. The time to failure in hours is given in Table 3.1. A star indicated that a motorette was taken off the study without failing at the event time indicated. For these data, assume that

$t_i = \beta_0 + \beta_1 \nu_i + \sigma \varepsilon_i$ for $i = 1, \cdots, 40$ where ε_i distributed to generalized log gamma distribution, $\nu_i = \frac{1000}{temperature + 273.2}$ and $t_i = \log_{10}(i\text{th}$ failure time). Then there are 23-right censored data. Reorder the data so that the first 17 observations are uncensored (i.e. a failure is observed at t_i) and the remaining 23 observations are censored (c_i denotes a censored event time).

Table 3.1. Motorette Data (source : Schmee and Hahn(1979))

150^0	8064*	8064*	8064*	8064*	8064*
	8064*	8064*	8064*	8064*	8064*
170^0	1764	2773	3442	3542	3780
	4860	5196	5448*	5448*	5448*
190^0	408	408	1344	1344	1440
	1680*	1680*	1680*	1680*	1680*
220^0	408	408	504	504	504
	528*	528*	528*	528*	528*

Table 3.2 Posterior means and standard derivation

	k=1.1	k=1.5	k=1.8	k=2.0	k=7.0	k=8.0
$E\beta_o$	2.89	2.25	3.31	4.81	0.71	-0.044
$(S.D.)$	(2.49)	(2.57)	(2.53)	(2.87)	(1.03)	(1.02)
$E\beta_1$	0.53	0.76	0.32	-0.21	1.68	1.77
$(S.D.)$	(0.99)	(0.86)	(1.06)	(1.29)	(0.46)	(0.45)
$E\sigma^2$	0.42	0.42	0.59	1.38	0.35	0.043
$(S.D.)$	(0.27)	(0.27)	(0.49)	(0.61)	(0.01)	(0.02)

Table 3.2 shows the posterior mean and posterior standard deviation of β_0, β_1 and σ^2 for different values of k. In Table 3.2, their estimators are separated into two groups such as small values(k = 1.1, 1.5, 1.8, 2.0) and large values(k= 7.0, 8.0). And the variance of estimators shows a tendency to be small as k increase. In particular, their estimates are very slightly different when k=7.0 and 8.0. Actually the generalized log-gamma distribution does not change greatly as k gets large.

Next, we want to estimate the posterior marginal density, say $[\beta_a | T]$. But since the full conditional densities are not in closed form we can not use the Rao-Blackwellized estimation proposed by Gelfand and Smith(1990) as follows;

$$\widehat{[\beta_a|T]} = \frac{1}{m}\sum_{i=1}^{m}[\beta_a|\beta_0^{(i)},\cdots,\beta_{a-1}^{(i)},\beta_{a+1}^{(i)},\cdots,\beta_{p-1}^{(i)},\sigma^{2(i)},T] \tag{3.1}$$

where $1 \le a \le p-1$ and $\{\beta^{(i)},\sigma^{2(i)}\}_{i=1}^{m}$ are Gibbs output. In this time, without using kernel density estimation, we estimate the posterior marginal density $[\beta_a | T]$ using the Monte Carlo method proposed by Chen(1994) as follows; for fixed β_a^*,

$$\begin{aligned}\widehat{[\beta_a^*|T]} \sim\ & \frac{1}{m}\sum_{i=1}^{m} q(\beta_a^{(i)}|\beta_0^{(i)},\cdots,\beta_{a-1}^{(i)},\beta_{a+1}^{(i)},\cdots,\beta_{p-1}^{(i)},\sigma^{2(i)}) \\ \times\ & \frac{g(\beta_a^*,\beta_0^{(i)},\cdots,\beta_{a-1}^{(i)},\beta_{a+1}^{(i)},,\cdots,\beta_{p-1}^{(i)},\sigma^{2(i)})}{g(\beta_0^{(i)},\cdots,\beta_{a-1}^{(i)},\beta_a^{(i)},\beta_{a+1}^{(i)},,\cdots,\beta_{p-1}^{(i)},\sigma^{2(i)})}\end{aligned} \tag{3.2}$$

where $g(\beta_0,\cdots,\beta_a,\cdots,\beta_{p-1},\sigma^2)$ and $q(\beta_a|\beta_0,\cdots,\beta_{a-1},\beta_{a+1},\cdots,\beta_{p-1},\sigma^2)$ and are the joint posterior density in (2.4) and the conditional density of β_a given $\beta_0,\cdots,\beta_{a-1},\beta_{a+1},\cdots,\beta_{p-1},\sigma^2$ obtained from $q(\beta_0,\cdots,\beta_a,\cdots,\beta_{p-1},\sigma^2)$, respectively and $\{\beta^{(i)},\sigma^{2(i)}\}_{i=1}^{m}$ are Gibbs output. Choosing a good function q can be quite difficult. In our case, a reasonable choice of q is to use a normal density whose mean and variance are based on the sample mean and sample covariance of Gibbs output $\{\beta^{(i)},\sigma^{2(i)}\}_{i=1}^{m}$. The grid points β_a^*'s need not be uniformly spaced. Then using the formula in (3.2), the estimated posterior marginal densities are in Figure 3.1 and 3.2. According to the properties of generalized log-gamma distribution, the distributional shape strongly depends on the value of k in Figure 3.1 and 3.2.

References

[1] Abramowitz, M. and Stegun, I.(1965). *Handbook of Mathematical functions,* New York.

[2] Chen, M. H. (1994). Importance-weighted marginal Bayesian posterior density estimation, *Journal of American Statistical Association,* Vol. 89, 818-824.

[3] Chib, S.(1992). Bayes Inference in the Tobit censored regression model. *Journal of Econometrics,* Vol. 51, 79-99.

[4] Chung, Y.(1997). Sampling based approach to the Bayesian analysis of binary regression model with incomplete data, *Journal of Korean Statistical Society,* Vol. 26, 493-506.

[5] Dempster, A.P., Laird, N., and Rudin(1977). Maximum likelihood from incomplete data via the EM algorithm, *Journal of the Royal Statistical Society B,* Vol. 39, 1-38.

[6] Farewell, V.T. and Prentice, R.L.(1977). A study of distributional shape in life testng, *Technometrics*, Vol. 19, 69-75.

[7] Gelfand, A.E. and Smith, A.F.M.(1990). Sampling Based Approaches to Calculating Marginal Densities, *Journal of the American Statistical Association*, Vol. 85, 389-409.

[8] Gelman, A. and Rubin, D.B. (1992). Inference from iterative simulation using multiple sequences, *Statistical Science,* Vol. 7, 457-472.

[9]] Ibrahim, J.G.(1990). Incomplete data in Generalized Linear Model, *Journal of the American Statistical Association*, Vol. 85, 765-769.

[10] Lawless, J.F.(1982). *Statistical models and Methods for lifetime Data.* New York : John-Wiley.

[11] Metropolis, N., Rosenbluth, A.W., Rosenbluth, M.N., Teller, A.H., and Teller, E.(1953). Equations of state calculations by fast computing machines, *Journal of Chemical Physics,* Vol. 21, 1087-1091.

[12] Schmee, M.J.and Hahn, G.J.(1979). A simple method for regression analysis with censored data, *Technometrics*, Vol. 21, 417-432.

[13] Tanner, M.A. and Wong. W.(1987). The Calculation of Posterior Distributions by Data Augmentation, *Journal of the American Statistical Association*, Vol. 82, 528-550.1

Figure 3.1 Marginal posterior density of β_0

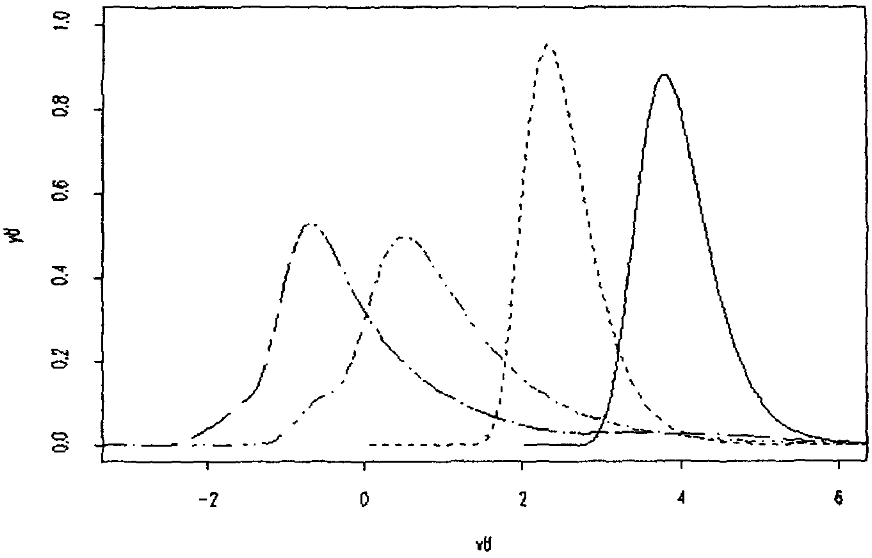
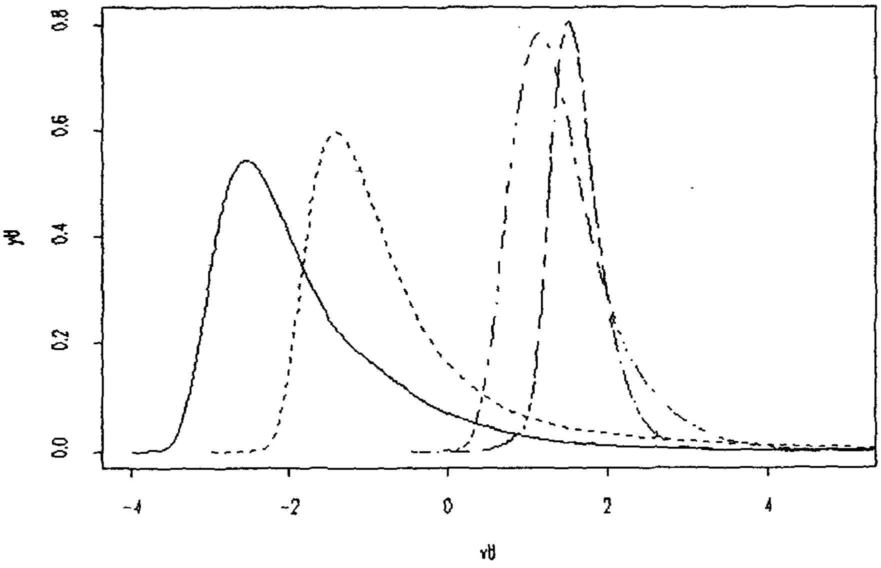


Figure 3.2 Marginal posterior density of β_1



———————— : k = 1.1, ... : k = 1.5

– – – – – – – – – : k = 7.0, –– – –– – –– : k = 8.0