The Korean Communications
in Statistics Vol. 5, No. 2, 1998
pp. 531-538

Some Results of Non-Central Wishart Distribution

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Abstract

This paper first examines the skewness of Wishart distribution, using Tracy and Sultan(1993)'s results. Second, it investigates the variance-covariance matrix of random matrix $S_Y = YY'$ which has a non-central Wishart distribution. Third, it proposes the exact form of the third moment of the random matrix with non-central Wishart distribution.

1. Introduction

Let random matrix $Y\ (p \times n)$ be distributed as $N_{p,n}(M, \Sigma, \Phi)$ $(M \neq 0)$, where Σ and Φ are $(p \times p)$ and $(n \times n)$ positive semidefinite matrices, respectively (in fact $\mathrm{E}(Y) = M$ and $\mathrm{Cov}(\mathrm{vec}\, Y, \mathrm{vec}\, Y) = \Phi \otimes \Sigma$). For $\Phi = I_n$, YY' is said to have a non-central Wishart distribution with scale matrix Σ and degrees of freedom parameter n , where I_n be an $(n \times n)$ identity matrix. Let Y be X when $M = 0$. Then XX' is said to have Wishart distribution.

Von Rosen (1988) obtained the moments of arbitrary order of matrix X, and using these calculated the second order moment of the quadratic form where A is an $(n \times n)$ arbitrary non-random matrix. Neudecker and Wansbeek(1987)'s three results involved X: [1] the expectation of $XAX'CXBX'$; [2] the covariance of $\mathrm{vec}\, XAX'$ and $\mathrm{vec}\, XBX'$; and [3] the expectation of $X \otimes X \otimes X \otimes X$. Tracy and Sultan (1993) found the third order moment of XAX' using the sixth moment of the matrix normal distribution and applying the properties of commutation matrices and Kronecker products. Kang and Kim(1996a) proposed the N-th moment of matrix quadratic form and the kurtosis of Wishart distribution. Kang and Kim(1996b) obtained the vectorizing form of the N-th moment of non-central Wishart distribution.

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Section 2 summarizes some preliminary results for commutation matrix, Kronecker product and vec-operator. Section 3 describes the third moment of the matrix quadratic form XAX' of Tracy and Sultan(1993), its properties, and some related notations. Finally, section 4 proposes the skewness of Wishart distribution using Tracy and Sultan(1993)'s results, the variance-covariance matrix of random matrix $S_Y = YY'$ with non-central Wishart distribution, and the exact form of the third moment of non-central Wishart distribution.

2. Preliminary results

This section examines the definitions of Kronecker product and vec-operator and their important properties, which play very important and basic role in multivariate statistics. It also reviews commutation matrix which often appears in multivariate statistics. The definitions and their properties introduced in this section would be the fundamental knowledge which is necessary for the development of section 3 and 4.

Definition 1. Let $A=(a_{ij})$ be an $(m \times n)$ matrix, and $B=(b_{kl})$ be a $(p \times q)$ matrix. Then the **Kronecker product** $A \otimes B$ of A and B is defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

It is an $(mp \times nq)$ matrix.

Definition 2. For a $(p \times n)$ matrix A, let $\text{vec}\,A$ denote the pn-vector obtained by '**vectorizing**' of A ; that is, $\text{vec}\,A = [a_1', a_2', \cdots, a_n']'$ if $A = [a_1, a_2, \cdots, a_n]$, where a_i is a p-vector.

Definition 3. Commutation matrix $I_{m,n}$ is an $(mn \times mn)$ matrix containing mn blocks of order $(m \times n)$ such that the (ij)th block has an 1 in its (ji)th position and zeroes elsewhere. One has $I_{m,n} = \sum_{i=1}^{n} \sum_{j=1}^{m} (H_{ij} \otimes H_{ij}')$, where H_{ij} is an $(n \times m)$ matrix with a 1 in its (ij)th position and zeroes elsewhere, and can be written as $H_{ij} = e_i e_j'$, where $e_i(e_j)$ is the $i(j)$th unit column vector of order $n(m)$. For example,

$$I_{3,2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Some preliminary property of commutation matrix, Kronecker product and vec-operator are the following:

Property 1. $I_m \otimes I_n = I_{mn}$, where I_m is an $(m \times m)$ identity matrix.

Property 2. $I_{m,n}^{-1} = I_{m,n}' = I_{n,m}$.

Property 3. $(I_n \otimes I_{n,n} \otimes I_n)(I_n \otimes I_n \otimes I_{n,n})(I_n \otimes I_{n,n} \otimes I_n)$
$= (I_n \otimes I_n \otimes I_{n,n})(I_n \otimes I_{n,n} \otimes I_n)(I_n \otimes I_n \otimes I_{n,n})$.

Property 4. For $(m \times n)$ matrix A , $I_{n,m} \operatorname{vec} A = \operatorname{vec} A'$.

Property 5. For A and B conformable matrices, $(A \otimes B)' = A' \otimes B'$.

Property 6. For same sized matrices A, B, C and D,
$$(A+B) \otimes (C+D) = (A \otimes B) + (A \otimes D) + (B \otimes C) + (B \otimes D).$$

Property 7. For A and B conformable matrices,
$$\operatorname{vec}' A' \operatorname{vec} B = \operatorname{tr}(AB).$$

Property 8. For $(m \times n)$ matrix A and $(p \times q)$ matrix B,
$$\operatorname{vec}(A \otimes B) = (I_n \otimes I_{m,q} \otimes I_p)(\operatorname{vec} A \otimes \operatorname{vec} B).$$

Property 9. For A, B, and C conformable matrices,
$$\operatorname{vec}(ACB) = (B' \otimes A) \operatorname{vec} C.$$

Property 10. For A, B, C, and D conformable matrices,
$$(AB) \otimes (CD) = (A \otimes C)(B \otimes D).$$

Property 11. For $(m \times n)$ matrix A and $(p \times q)$ matrix B,
$$I_{m,p}(A \otimes B) = (B \otimes A) I_{n,q} , \quad I_{m,p}(A \otimes B) I_{q,n} = B \otimes A.$$

Property 12. For $(m \times n)$ matrix A, $(p \times q)$ matrix B, and $(r \times s)$ matrix C,

$$A \otimes B \otimes C = I_{r,mp}(C \otimes A \otimes B) I_{qn,s} = I_{pr,s}(B \otimes C \otimes A) I_{n,sq} .$$

Property 13. For $(m \times n)$ matrix A, $(p \times q)$ matrix B, and $(r \times s)$ matrix C,

$$\mathrm{vec}(A \otimes B \otimes C) = (I_n \otimes I_{m,qs} \otimes I_{pr})(I_{mnq} \otimes I_{p,s} \otimes I_r)(\mathrm{vec} A \otimes \mathrm{vec} B \otimes \mathrm{vec}\, C).$$

For detailed proofs, see Magnus and Neudecker(1979), Neudecker and Wansbeek(1983), and Tracy and Sultan(1993).

Property 14. Neudecker and Wansbeek(1983) have discussed higher order commutation matrices, for which

$$I_{xy,z} = (I_{x,z} \otimes I_y)(I_x \otimes I_{y,z}) = (I_{y,z} \otimes I_x)(I_y \otimes I_{x,z}),$$

$$I_{x,yz} = (I_y \otimes I_{x,z})(I_{x,y} \otimes I_z) = (I_z \otimes I_{x,y})(I_{x,z} \otimes I_y).$$

$$I_{p,p^2} I_{p,p^2} = I_{p^2,p} , \quad I_{p,p^2} I_{p,p^2} = I_{p^2,p} .$$

Property 15. For $X \sim N_{p,n}(0, \Sigma, \Phi)$, the N-th moment is (von Rosen, 1988)

$$\mathrm{E}(\otimes^r X) = \sum_{i=2}^{r} P(p, i, r)(\mathrm{vec}\Sigma\, \mathrm{vec}'\Phi \otimes \mathrm{E}(\otimes^{r-2} X)) P(n, i, r)' , \quad r \geq 2$$

where $\mathrm{E}(\otimes^0 X) = 1$ and $P(a, b, c) = [(I_{a^{b-2},a} \otimes I_a) I_{a^2, a^{b-2}}] \otimes I_{a^{c-b}}$.

3. The third moments of XAX' and Wishart distribution

This section investigates some notations and the XAX' which have an important role in multivariate statistics. There have not been many studies on the moment of XAX'. The followings are some important studies on it. Von Rosen(1988) obtained the moments of arbitrary order of matrix X, and using these calculated the second order moment of the quadratic form where A is an $(n \times n)$ arbitrary non-random matrix. Neudecker and Wansbeek(1987)'s three results involved X: (1) the expectation of $XAX'CXBX'$; (2) the covariance of $\mathrm{vec}\, XAX'$ and $\mathrm{vec}\, XBX'$; and (3) the expectation of $X \otimes X \otimes X \otimes X$. Tracy and Sultan(1993) obtained the third order moment of XAX' using the sixth moment of the matrix normal distribution and applying the properties of commutation matrices and Kronecker products. Kang and Kim(1996a) obtained the N-th moment of matrix quadratic form and the kurtosis of Wishart distribution. Kang and Kim(1996b) obtained the vectorizing form of the N-th moment of non-central Wishart distribution.

Notation. $P_{k,l;m} = I_{p^k, p^l} \otimes I_{p^m}$, $N_{k,l;m} = I_{n^k, n^l} \otimes I_{n^m}$

$$P_{k;l,m} = I_{p^k} \otimes I_{p^l,p^m}\ ,\quad N_{k;l,m} = I_{n^k} \otimes I_{n^l,n^m}\ .$$

We now turn to the XAX' and the third moment of Wishart Distribution of Tracy and Sultan(1993) are considered. Tracy and Sultan(1993) describes the following expectation of $XAX' \otimes XBX' \otimes XCX'$, where $X \sim N_{p,n}(0, \Sigma, \Phi)$.

$$\begin{aligned}&\mathrm{vec}\,\mathrm{E}(XAX' \otimes XBX' \otimes 'XCX')\\ &\qquad = (P_{1;1,2} \otimes I_{p^2})(P_{3;1,1} \otimes I_p)\,\mathrm{E}(\otimes^6 X)(\mathrm{vec}\,A\ \mathrm{vec}\,B\ \mathrm{vec}\,C).\end{aligned}$$

This equation can be represented again with the following form which is useful for matrix computation.

$$\begin{aligned}&\mathrm{E}(XAX' \otimes XBX' \otimes XCX') = \\ &\mathrm{tr}(\Phi A)\,\mathrm{tr}(\Phi B)\,\mathrm{tr}(\Phi C)(\otimes^3\Sigma) + \mathrm{tr}(\Phi A'\Phi B)\,\mathrm{tr}(\Phi C)\,V + \mathrm{tr}(\Phi B\Phi A)\,\mathrm{tr}(\Phi C)\,Q(\otimes^3\Sigma)\\ &+ \mathrm{tr}(\Phi C'\Phi B\Phi A)QPVP + \mathrm{tr}(\Phi C\Phi B\Phi A)PQ(\otimes^3\Sigma) + \mathrm{tr}(\Phi C'\Phi B)\,\mathrm{tr}(\Phi A)QPVPQ\\ &+ \mathrm{tr}(\Phi B'\Phi C\Phi A)QPV + \mathrm{tr}(\Phi B\Phi C\Phi A)QP(\otimes^3\Sigma) + \mathrm{tr}(\Phi C'\Phi B\Phi A)\,VP \qquad (1)\\ &+ \mathrm{tr}(\Phi C\Phi B\Phi A)\,VPQ + \mathrm{tr}(\Phi B\Phi C)\mathrm{tr}(\Phi A)P(\otimes^3\Sigma) + \mathrm{tr}(\Phi A'\Phi C\Phi B)PV\\ &+ \mathrm{tr}(\Phi B\Phi C'\Phi A)PVPQ + \mathrm{tr}(\Phi C'\Phi A)\,\mathrm{tr}(\Phi B)PVP + \mathrm{tr}(\Phi C'\Phi A)\,\mathrm{tr}(\Phi B)QPQ(\otimes^3\Sigma),\end{aligned}$$

where A, B and C are $(n\times n)$ arbitrary constant matrices and $V = (\mathrm{vec}\,\Sigma\,\mathrm{vec}'\Sigma) \otimes \Sigma$, $P = P_{1;1,1} = I_p \otimes I_{p,p}$, $Q = P_{1,1;1} = I_{p,p} \otimes I_p$ (and hence $I_{p,p^2} = PQ$ and $I_{p^2,p} = QP$).

4. Some results of non-central Wishart distribution

With Equation (1), we seek the third moment of Wishart distribution.

Theorem 4. 1. If $V = (\mathrm{vec}\,\Sigma\,\mathrm{vec}'\Sigma) \otimes \Sigma$, $P = P_{1;1,1} = I_p \otimes I_{p,p}$ and $Q = P_{1,1;1} = I_{p,p} \otimes I_p$, then the third moment of Wishart distribution is given by

$$\begin{aligned}\mathrm{E}(\otimes^3 S_X) &= n\left[(P+QP)V + (I_{p^3}+QP)VP + (I_{p^3}+P)VPQ + (PQ+QP)(\otimes^3\Sigma)\right]\\ &\quad + n^2[V + PVP + QPVPQ + (P+Q+QPQ)(\otimes^3\Sigma)] + n^3(\otimes^3\Sigma).\end{aligned}$$

Proof. When $A = B = C = I_n = \Phi$, Equation (1) gives the third moment $\mathrm{E}(\otimes^3 XX')$ of Wishart distribution $\mathrm{W}(\Sigma, n)$, in the square matrix form, the 15 terms being

$$\begin{aligned}&\mathrm{tr}(I_n)\,\mathrm{tr}(I_n)\,\mathrm{tr}(I_n)(\otimes^3\Sigma) + \mathrm{tr}(I_n)\,\mathrm{tr}(I_n)\,V + \mathrm{tr}(I_n)\,\mathrm{tr}(I_n)\,Q(\otimes^3\Sigma)\\ &+ \mathrm{tr}(I_n)QPVP + \mathrm{tr}(I_n)PQ(\otimes^3\Sigma) + \mathrm{tr}(I_n)\,\mathrm{tr}(I_n)QPVPQ\end{aligned}$$

$$+ \operatorname{tr}(I_n)QPV + \operatorname{tr}(I_n)QP(\otimes^3\Sigma) + \operatorname{tr}(I_n)VP$$
$$+ \operatorname{tr}(I_n)VPQ + \operatorname{tr}(I_n)\operatorname{tr}(I_n)P(\otimes^3\Sigma) + \operatorname{tr}(I_n)PV$$
$$+ \operatorname{tr}(I_n)PVPQ + \operatorname{tr}(I_n)\operatorname{tr}(I_n)PVP + \operatorname{tr}(I_n)\operatorname{tr}(I_n)QPQ(\otimes^3\Sigma).$$

Since $\operatorname{tr}(I_n) = n$, the above equation is represented as follows:

$$\mathrm{E}(\otimes^3 XX') = n[(P+QP)V + (I_{p^3}+QP)VP + (I_{p^3}+P)VPQ + (PQ+QP)(\otimes^3\Sigma)]$$
$$+ n^2[V + PVP + QPVPQ + (P+Q+QPQ)(\otimes^3\Sigma)] + n^3(\otimes^3\Sigma).$$

Now we seek the skewness of Wishart distribution and the third moment of non-central Wishart distribution from some properties of Kronecker product and vec-operator and the results mentioned in section 3. In order to do this, if we assume that $\Phi = I_n$, $S_X = XX'$, and $S_Y = YY'$, then S_X has a Wishart distribution while S_Y a non-central Wishart distribution. In addition, since $Y = X + M$, S_Y is $S_Y = S_X + XM' + MX' + MM'$ by Property 6. Then, we look for the skewness of random matrix S_X and the third moment of random matrix S_Y.

Lemma 4. 2. $\mathrm{E}S_Y = n\Sigma + MM'$.

Proof. See chapter 3 of Mardia, Kent and Bibby (1979).

Lemma 4. 3. If $U = \operatorname{vec}\Sigma \operatorname{vec}' I_n$, then

$$\mathrm{E}(\otimes^2 S_Y) = n \operatorname{vec}\Sigma \operatorname{vec}'\Sigma + U(M\otimes M)' + (M\otimes M)U' + \otimes^2 MM'$$
$$+ n^2(\otimes^2\Sigma) + nI_{p,p}(\otimes^2\Sigma).$$

Proof. It can be proved by Property 7 and chapter 5 of Kang (1996).

Theorem 4. 4. If $Y \sim N_{p,n}(M, \Sigma, I_n)$, $S_Y = YY'$ and $U = \operatorname{vec}\Sigma \operatorname{vec}' I_n$, then the variance-covariance matrix of non-central Wishart distribution is following;

$$\mathrm{E}[\otimes^2(YY' - \mathrm{E}YY')] = \mathrm{E}[\otimes^2(S_Y - \mathrm{E}S_Y)] =$$
$$n \operatorname{vec}\Sigma \operatorname{vec}'\Sigma + U(M\otimes M)' + (M\otimes M)U' + nI_{p,p}(\Sigma\otimes\Sigma) - n(MM'\otimes\Sigma + \Sigma\otimes MM').$$

Proof. Based on the fact that $\mathrm{E}S_Y$ is constant matrix, the property of expectation, and Property 6, the variance-covariance matrix of non-central Wishart distribution can be represented as follows:

$$\mathrm{E}[\otimes^2(S_Y - \mathrm{E}S_Y)] = \mathrm{E}(\otimes^2 S_Y - S_Y\otimes \mathrm{E}S_Y - \mathrm{E}S_Y\otimes S_Y + \otimes^2 \mathrm{E}S_Y)$$
$$= \mathrm{E}(\otimes^2 S_Y) - \mathrm{E}(S_Y\otimes \mathrm{E}S_Y) - \mathrm{E}(\mathrm{E}S_Y\otimes S_Y) + \mathrm{E}(\otimes^2 \mathrm{E}S_Y)$$

$$= \mathrm{E}(\otimes^2 S_Y) - \mathrm{E}S_Y \otimes \mathrm{E}S_Y - \mathrm{E}S_Y \otimes \mathrm{E}S_Y + \otimes^2 \mathrm{E}S_Y \qquad (2)$$
$$= \mathrm{E}(\otimes^2 S_Y) - \otimes^2 \mathrm{E}S_Y.$$

And based on Lemma 4. 2, Lemma 4. 3, and Property 6, the equation (2) can be represented as follows:

$$\begin{aligned}
&\mathrm{E}[\otimes^2 (S_Y - \mathrm{E}S_Y)] \\
&= n \operatorname{vec} \Sigma \operatorname{vec}' \Sigma + U(M \otimes M)' + (M \otimes M)U' + \otimes^2 MM' \\
&\quad + n^2(\otimes^2 \Sigma) + nI_{p,p}(\otimes^2 \Sigma) - (n\Sigma + MM') \otimes (n\Sigma + MM') \\
&= n \operatorname{vec} \Sigma \operatorname{vec}' \Sigma + U(M \otimes M)' + (M \otimes M)U' + \otimes^2 MM' \\
&\quad + n^2(\otimes^2 \Sigma) + nI_{p,p}(\otimes^2 \Sigma) - n^2 \Sigma \otimes \Sigma - n\Sigma \otimes MM' - n\, MM' \otimes \Sigma - \otimes^2 MM' \\
&= n \operatorname{vec} \Sigma \operatorname{vec}' \Sigma + U(M \otimes M)' + (M \otimes M)U' \\
&\qquad + nI_{p,p}(\otimes^2 \Sigma) - n(\Sigma \otimes MM' + MM' \otimes \Sigma).
\end{aligned}$$

Theorem 4. 5. If $Y \sim N_{p,n}(M, \Sigma, I_n)$ and $S_Y = YY'$, then the third moment of non-central Wishart distribution,

$$\begin{aligned}
\mathrm{E}(\otimes^3 S_Y) &= n[(P+QP)V + (I_{p^3}+QP)VP + (I_{p^3}+P)VPQ + (PQ+QP)(\otimes^3 \Sigma)] \\
&\quad + n^2[V + PVP + QPVPQ + (P+Q+QPQ)(\otimes^3 \Sigma)] + n^3(\otimes^3 \Sigma) + (\otimes^3 M)(\otimes^3 M)'.
\end{aligned}$$

Proof. Since $S_Y = S_X + XM' + MX' + MM'$, $\mathrm{E}(\otimes^3 S_Y)$ can be represented like the Equation (3) below:

$$\begin{aligned}
\mathrm{E}(\otimes^3 S_Y) &= \mathrm{E}[\otimes^3 (S_X + XM' + MX' + MM')] \\
&= \mathrm{E}(\otimes^3 S_X) + \mathrm{E}[\otimes^3 (XM')] + \mathrm{E}[\otimes^3 (MX')] + \mathrm{E}[\otimes^3 (MM')]. \qquad (3)
\end{aligned}$$

Each of four terms in Equation (3) can be respectively expressed by **a**, **b**, **c**, **d** below.

a. The first term is clearly expressed in Theorem 4.1.

b. The second term $\mathrm{E}[\otimes^3 (XM')]$ is $\mathrm{E}[(\otimes^3 X)(\otimes^3 M')]$ by Property 10, and since $\mathrm{E}(\otimes^3 X) = 0$ by Property 15, the second term $\mathrm{E}[\otimes^3 (XM')]$ becomes zero.

c. Likewise, the third term $\mathrm{E}[\otimes^3 (MX')]$ also becomes zero.

d. Since $\otimes^3(MM')$ is constant matrix, the fourth term $\mathrm{E}[\otimes^3 (MM')]$ becomes $\otimes^3(MM')$ by the property of expectation.

From **a**, **b**, **c**, **d** above, the third moment of non-central Wishart distribution can be calculated as follows:

$$\begin{aligned}
\mathrm{E}(\otimes^3 S_Y) &= n[(P+QP)V + (I_{p^3}+QP)VP + (I_{p^3}+P)VPQ + (PQ+QP)(\otimes^3 \Sigma)] \\
&\quad + n^2[V + PVP + QPVPQ + (P+Q+QPQ)(\otimes^3 \Sigma)] + n^3(\otimes^3 \Sigma) + (\otimes^3 MM').
\end{aligned}$$

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