The Korean Communications
in Statistics Vol. 5, No. 2, 1998
pp. 503-515

Monotone Likelihood Ratio Property of the Poisson Signal Distribution with Three Sources of Errors in the Parameter[1]

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Abstract

When a neutral particle beam(NPB) aimed at the object and receive a small number of neutron signals at the detector, it follows approximately Poisson distribution. Under the four assumptions in the presence of errors and uncertainties for the Poisson parameters, an exact probability distribution of neutral particles have been derived.

The probability distribution for the neutron signals received by a detector averaged over the three sources of errors is expressed as a four-dimensional integral of certain data. Two of the four integrals can be evaluated analytically and thereby the integral is reduced to a two-dimensional integral. The monotone likelihood ratio(MLR) property of the distribution is proved by using the Cauchy mean value theorem for the univariate distribution and multivariate distribution. Its MLR property can be used to find a criteria for the hypothesis testing problem related to the distribution.

1. Introduction

A beam of neutral particles can be used to estimate the density or mass of an object (Feller (1970)). A method of discrimination proposed here is to use a neutral particle beam(NPB) aimed at the object, and a small number of neutron signals are counted at the detector. Beyer and Qualls (1987) showed that the number of return neutron particles from an object interrogation for a given dwell time follow Poisson distributon.

The mean return neutron signal λ is computed by

$$\lambda = (2\pi)^{-1} \cdot \frac{A\varepsilon\tau I}{R^2 r^2 \sigma_1^2} \cdot f(\theta, \phi, \psi, E) \tag{1.1}$$

where A is the detector area in m^2, ε is the detector efficiency, I is the probe current in amperes divided by 1.602×10^{-19} coulombs, τ is the dwell time in seconds, R is the probe to object distance in m, $\sqrt{2}\sigma_1$ is the beam half divergence angle, r is the object to detector distance in m, $f(\theta,\phi,\psi,E)$ is the mean number of neutrons leaked from the object per

1) 본 연구는 1998년 동국대학교 전문학술지 논문게제연구비 지원으로 수행되었습니다.

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incident particle and it depends on the mass of the object, E is the probe particle energy in election volts, θ is the scattering angle, and ϕ, ψ are two angels in the orientation of the object. (1.1) is a parameter of the Poisson distribution. We will prove the basic theory for discrimination of object with NPB where certain parameters of the distribution are not known precisely, but their probability distributions are given.

We may consider three sources of errors or uncertainties in the interrogation of object. The geometry of the interrogation requires the specification of three angles θ, ϕ, and ψ in (1.1). Two of these three sources are the two angles involved in the orientation of the object relative to the platforms and detectors: the azimuthal angle ϕ of the detector and the beam angle of incidence ψ. The polar angle θ of the detector may be measured directly by the interrogation system. The third source of errors in measurement is the uncertainty about the location of the axis of the beam relative to the object, i.e., tracking and pointing errors(or aiming errors).

Kim (1995a) made two assumptions for the tracking and pointing errors:

(1) The beam has a circular Gaussian distribution of intensity with standard deviation σ_1. This distribution is on a plane perpendicular to the beam axis.

(2) Tracking and pointing errors yield a circular Gaussian distribution of the beam axis relative to the object center. The standard deviation of the distribution is σ_2.

Wehner (1987) studied the aiming error distribution of NPB, and the results are applied to the assumption (2).

Under the two assumptions that neutral particle scattering distribution and aiming errors have a circular Gaussian distribution respectively, the exact probability distribution of neutral particles becomes a Poisson-power function distribution. Kim (1995b) proved some properties, such as limiting distribution, unimodality, stochastical ordering, computational recursion formula, of the distribution. Kim (1996) also proved monotone likelihood ratio(MLR) property of this distribution, and studied error rate for the limiting Poisson-power function distribution. Its MLR property can be used to find a criteria for the hypothesis testing problem. Kim (1997) also studied the minimum dwell time which is satisfied the specified error rate for the distribution and an algorithm is developed.

In this paper we add two more assumptions about these three sources of errors.

(3) The angles ψ and ϕ are distributed over the rectangle $[0,\pi]\times[0,2\pi]$ with density $h(\phi,\psi)d\phi d\psi$, where h is a smooth function. If the points determined by ψ and ϕ are uniformly distributed over the sphere, then $h(\phi,\psi) = \sin(\psi)/4\pi$.

(4) The random errors in (2) and (3) are independent.

2. An Exact Distribution with Three Sources of Errors

We first consider the case of one detector and assume there are no background neutron signals. We wish to calculate the probability that exactly x neutron counts, $x=0,1,2,\cdots$, are received by the single detector in presence of errors. Let us assume that mean return neutron signal does depend on the tracking and pointing error as well as the orientation of the object (i.e., ϕ and ψ) and that the orientation of the object is uniformly distributed on the unit sphere. Then the mean return neutron signal λ becomes

$$\lambda = (2\pi)^{-1} \cdot S \cdot f(\theta, \phi, \psi, E) \cdot e^{-(\omega_1^2+\omega_2^2)/(2\sigma_1^2)}, \tag{2.1}$$

where

$$S = \frac{A\varepsilon\tau I}{R^2 r^2 \sigma_1^2},$$

and (ω_1, ω_2) are coordinates of points on beam cross section. A more detailed description of this formula is given in the report of the American Physical Society report (1987).

The interrogation of object requires the true value of the parameters in (2.1) to compute the mean of the Poisson statistics. We assume that the structures of the object are given. However, the general problem treated in this paper does not require detailed knowledge about the object. This permits the interrogation problem to be posed in terms of hypothesis testing. We thus can apply the theory of the Neyman-Pearson test of hypothesis to the problem of using these return signals.

The probability distribution for the neutron signal received by a detector averaged over the above mentioned three sources of errors and under the assumption of a Poisson distribution of counts is expressed as a four-dimensional integral of certain data.

$$P(x \mid \lambda) = \frac{1}{x!}\int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{\omega_2=-\infty}^{\infty}\int_{\omega_1=-\infty}^{\infty} e^{-\lambda}(\lambda)^x e^{-(\omega_1^2+\omega_2^2)/(2\sigma_2^2)} h(\phi, \psi)\frac{d\omega_1\, d\omega_2}{2\pi\sigma_2^2}\, d\phi\, d\psi, \tag{2.2}$$

where λ is defined in (2.1) and σ_2 is a standard deviation of the circular Gaussian aiming error distribution of the beam relative to the object and h is defined in section 1.

We average over the aiming error and uncertainties of angles distribution in (2.1) to modify discrimination for this errors. In repeated sequential interrogation, the probability in (2.2) leads to a reasonable and correct modification.

When f in (2.1) does not depend on the orientation of the object and in presence of aiming errors only, Kim (1995a) studied the probability distribution of exactly x neutron particles are received by the single detector. Under the two assumptions that neutral particle scattering distribution and aiming errors have a circular Gaussian distribution respectively, and from the change to polar coordinates and change of variables, (2.2) becomes

$$P(x \mid \lambda) = \frac{\ell}{k^{\ell} x!}\gamma(x+\ell\,; k), \tag{2.3}$$

where

$$\gamma(\nu; k) = \int_0^k t^{\nu-1} e^{-t} dt$$

is the incomplete gamma function.

It follows that the exact probability distribution of neutral particles in (2.3) becomes a Poisson-power function distribution (Johnson (1970)). Note that k in (2.3) be the mean number of return neutron signals counted with the assumption that no errors are made in the measurement of the parameters and that the beam is perfectly centered on the object.

Beckman and Johnson (1987) give evidence from an experiment that the beam has a Pearson Type Ⅶ distribution of intensity instead of a circular Gaussian distribution of intensity in assumption (1). This distribution is much heavier in the tails than is the Gaussian. Kim (1995a) compared a circular Gaussian distribution with a Pearson Type Ⅶ distribution for scattering distribution of the NPB.

Under the four assumptions, (2.2) becomes

$$f(x|\lambda) = \frac{\ell}{x!} \int_{\psi=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{\gamma(\nu; k)}{k^{\ell}} h(\psi, \phi)\, d\phi\, d\psi \ . \tag{2.4}$$

Thus two of the four integrals in (2.2) can be evaluated analytically and thereby the integral is reduced to a two-dimensional integral.

3. Monotone Likelihood Ratio Property of the Distribution

In this section we will show that the distribution in (2.4) has a MLR property. It can be used to find decision rules for hypothesis testing problem (Karlin and Rubin (1956)).

First, we need the Cauchy extension of the mean value theorem given in Buck (1985) which allows one to use the same mean value for the integrals appearing in a quotient of integrals. Because this theorem may not be widely known, it is useful to state the special case of it that we will use.

Lemma 3.1. (the Cauchy extension of the mean value theorem) Let $F(x)$ and $G(x)$ be continuous functions on the closed finite interval $[a, b]$. Suppose F and G do not have a common zero. Assume that $\int_a^b G(x)dx \neq 0$. Then there exists μ with $a < \mu < b$ such that

$$\frac{\int_a^b F(x)\, dx}{\int_a^b G(x)\, dx} = \frac{F(\mu)}{G(\mu)} \ . \tag{3.1}$$

We also have need of an expression from Abramowitz (1964) for the incomplete gamma function as a finite series.

Lemma 3.2.

$$\gamma(\nu; x) = \frac{e^{-x}x^{\nu}}{\nu} \sum_{j=0}^{n} \frac{x^j}{(\nu+1)_j} + \frac{1}{(\nu)_{n+1}} \int_0^x y^{n+\nu} e^{-y} dy \tag{3.2}$$

where $(x)_j$ is the usual raising factorial:

$$(x)_j = \begin{cases} x(x+1)(x+2)\cdots(x+j-1), & j>0 \\ 1, & j=0. \end{cases}$$

The expression in Lemma 3.2 is easily obtained by successive integration by parts. On letting $n\to\infty$, (3.2) becomes

$$\gamma(\nu; x) = \frac{e^{-x}x^{\nu}}{\nu} \sum_{j=0}^{\infty} \frac{x^j}{(\nu+1)_j}, \tag{3.3}$$

which is convergent for each $\nu>0$ and for all x. It turns out to be convenient to write (3.2) in the form

$$\frac{\nu\gamma(\nu;x)}{e^{-x}x^{\nu}} = 1 + \frac{x}{(\nu+1)} + \frac{x^2}{(\nu+1)(\nu+2)} + \cdots + \frac{x^n}{(\nu+1)(\nu+2)\cdots(\nu+n)}$$
$$+ \frac{\nu e^x x^{-\nu}}{(\nu)_{n+1}} \int_0^x y^{n+\nu} e^{-y} dy. \tag{3.4}$$

Theorem 3.1. The likelihood ratio of the distribution in (2.4) is monotone and converges to 0 as $x\to\infty$; and the Neyman-Pearson test for the hypotheses of $H_0 : k=t$ vs. $H_1 : k=d$, when $d<t$, is a left-tail test.

Proof. For the case of one detector when f depends on the angles ϕ and ψ for fixed polar angle θ, the Neyman-Pearson likelihood ratio of the distribution (2.4) is

$$L(x) = \frac{\int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\gamma(\nu; d)}{d^{\ell}} h(\psi,\phi)\, d\phi\, d\psi}{\int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\gamma(\nu; t)}{t^{\ell}} h(\psi,\phi)\, d\phi\, d\psi}, \tag{3.5}$$

where $\nu = x+\ell$. Applying (3.1) to (3.5) once for each integral, we obtain

$$L(x) = \left(\frac{t^*}{d^*}\right)^{\ell} \frac{\gamma(\nu; d^*)}{\gamma(\nu; t^*)}, \tag{3.6}$$

where d^* and t^* are the Cauchy mean values for the integrals in (3.5). On substituting (3.3) into (3.6), we have

$$L(x) = e^{t^*-d^*}\left(\frac{d^*}{t^*}\right)^x W, \tag{3.7}$$

where

$$W= \frac{\sum_{j=0}^{\infty} \frac{(d^*)^j}{(\nu+1)_j}}{\sum_{j=0}^{\infty} \frac{(t^*)^j}{(\nu+1)_j}} . \tag{3.8}$$

Because each term in the numerator of W is less than or equal to the corresponding term in the denominator and there is equality only for the first term, we have $W < 1$. Then, under the reasonable condition that $\max_{\phi,\psi}(d/t) < 1$, we have $\lim L(x) \rightarrow 0$ as $x \rightarrow \infty$.

Equation (3.7) can be used to determine the region of x-space to be searched in the Neyman-Pearson test. Note that (3.7) involves both $t^* - d^*$ and d^*/t^*.

We now show that $L(x)$ is monotone decreasing for all x. We work with the quotient $L(x+1)/L(x)$. In this quotient there are four integrals. Let d^o be the Cauchy mean value for the " d" integrals and let t^o be the Cauchy mean value for the " t" integrals. We assume that $\max_{\phi,\psi} d < \min_{\phi,\psi} t$ and hence that

$$\max_{\phi,\psi} d^o < \min_{\phi,\psi} t^o. \tag{3.9}$$

Then for every $\nu = x + \ell$, we have:

$$\frac{L(x+1)}{L(x)} = \frac{\int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\gamma(\nu+1;d)}{d^\ell} h(\phi,\psi)\, d\phi d\psi}{\int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\gamma(\nu;d)}{d^\ell} h(\phi,\psi)\, d\phi d\psi} \frac{\int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\gamma(\nu;t)}{t^\ell} h(\phi,\psi)\, d\phi d\psi}{\int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\gamma(\nu+1;t)}{t^\ell} h(\phi,\psi)\, d\phi d\psi}$$

$$= \frac{\gamma(\nu+1;d^o)}{\gamma(\nu;d^o)} \frac{\gamma(\nu;t^o)}{\gamma(\nu+1;t^o)} \quad \text{from (3.6).}$$

By the integration by parts for the incomplete function, $\gamma(\nu+1;x) = \nu\gamma(\nu;x) - x^\nu e^{-x}$, and hence

$$\frac{L(x+1)}{L(x)} = \frac{\nu\gamma(\nu;d^o) - (d^o)^\nu e^{-d^o}}{\gamma(\nu;d^o)} \frac{\gamma(\nu;t^o)}{\nu\gamma(\nu;t^o) - (t^o)^\nu e^{-t^o}}$$

$$= \frac{1 - \dfrac{(d^o)^{\nu} e^{-d^o}}{\nu\gamma(\nu; d^o)}}{1 - \dfrac{(t^o)^{\nu} e^{-t^o}}{\nu\gamma(\nu; t^o)}} < 1 \qquad \text{from (3.4).} \tag{3.10}$$

Thus $L(x)$ is everywhere monotone decreasing under assumption (3.9).

4. Monotone Likelihood Ratio for the Joint Distribution of the Vector of Neutron Signals

This section is an extension of section 3 to the general problem of discrimination. The accomplishment of this section provides algorithms for discrimination with multiple signals and/or in the presence of three sources of errors. The errors of uncertainties are the object orientation and tracking and pointing errors are object position relative to the beam.

The return neutron signals observed from k different time intervals or places are formed into a vector of neutron signals. Suppose one is given a vector $X=(X_1, X_2, \cdots, X_k)$ of a finite set of quantities X_j which are Poisson distributed random variables with different means. Suppose the mean of the jth signal is

$$\lambda_j = k_j \cdot e^{-(\omega_1^2+\omega_2^2)/2\sigma_1^2}, \qquad j=1,2,\cdots,k \tag{4.1}$$

where k_j can be computed by the bistatic radar formula in (1.1). The joint distribution of the signal vector X with three sources of errors is given by

$$P(\mathrm{x}) = \int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\prod_{j=1}^{k} \frac{e^{-k_j u_1}(k_j u_1)^{x_j}}{x_j!}\right) u_2\, h(\phi,\psi) \frac{d\omega_1 d\omega_2}{2\pi\sigma_2^2}\, d\phi d\psi \tag{4.2}$$

where $u_i = e^{-(\omega_1^2+\omega_2^2)/2\sigma_i^2}, \quad i=1,2.$

Let $u=e^{-(\omega_1^2+\omega_2^2)/2\sigma_1^2}$ and $\ell = (\sigma_1/\sigma_2)^2$. Then by the same calculation procedure we used for the univariate distribution, (4.2) becomes

$$P(\mathrm{x}) = \int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\ell}{(\Sigma k_i)^{\ell}}\left(\prod_{j=1}^{k}\left(\frac{k_j}{\Sigma k_i}\right)^{x_j}\frac{1}{x_j!}\right)\gamma(\Sigma x_i + \ell; \Sigma k_i)\, h(\phi,\psi)\, d\phi d\psi, \tag{4.3}$$

where γ is the incomplete gamma function.

Consider the two hypothesis that H_0: "Object is heavy" versus H_1: "Object is light". Denote $k_j = t_j$ under H_0 and $k_j = d_j$ $(d_j < t_j)$ under H_1. Put $\nu = \sum x_i + \ell$ in (4.3). Then the likelihood ratio of the joint distribution in (4.3) is

$$L(\mathrm{x})=\frac{P(\mathrm{x}|H_1)}{P(\mathrm{x}|H_0)}=\frac{\int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi}(\prod_{j=1}^{k}(d_j)^{x_j})(\Sigma d_i)^{-\nu}\gamma(\nu;\Sigma d_i)h(\phi,\psi)d\phi d\psi}{\int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi}(\prod_{j=1}^{k}(t_j)^{x_j})(\Sigma t_i)^{-\nu}\gamma(\nu;\Sigma t_i)h(\phi,\psi)d\phi d\psi} \tag{4.4}$$

Applying (3.1) to (4.4), we obtain with d's replaced by the appropriate intermediate values d^*'s and the t's replaced by the appropriate intermediate values t^*'s:

$$L(\mathrm{x})=\left(\frac{\Sigma t_i^*}{\Sigma d_i^*}\right)^{\nu}\left(\prod_{j=1}^{k}\left(\frac{d_j^*}{t_j^*}\right)^{x_j}\right)\frac{\gamma(\nu;\Sigma d_i^*)}{\gamma(\nu;\Sigma t_i^*)}. \tag{4.5}$$

Using (3.3) in (4.5), we obtain

$$L(\mathrm{x})=\left(\prod_{j=1}^{k}(d_j^*/t_j^*)^{x_j}\right)e^{\Sigma t_i^*-\Sigma d_i^*}W \tag{4.6}$$

where

$$W=\frac{\sum_{j=0}^{\infty}\frac{(\Sigma d_i^*)^j}{(\nu+1)_j}}{\sum_{j=0}^{\infty}\frac{(\Sigma t_i^*)^j}{(\nu+1)_j}}. \tag{4.7}$$

Again, because each term in the numerator of W is less than or equal to the corresponding term in the denominator and there is equality only for the first term, we have $W<1$. Put $A_j=\max_{\phi,\psi}(d_j/t_j)$, and $A=\max_j A_j$. Thus, (4.6) becomes

$$L(\mathrm{x})\le A^{\Sigma x_j}e^{\Sigma t_i^*-\Sigma d_i^*}W. \tag{4.8}$$

We make the assumption that $A<1$. If for any j, $A_j>1$, we do not use the reading from that signal. Hence we have

$$\lim_{\|\mathrm{x}\|\to\infty}L(\mathrm{x})=0. \tag{4.9}$$

Equation (4.8) can be used to determine the region of x-space to be searched in the Neyman-Pearson test.

We now show that $L(\mathrm{x})$ is monotone decreasing for sufficiently large $\|\mathrm{x}\|$. Let e_m be a k-dimensional vector with the mth component one and the other components zero. From (4.4), we have

$$\frac{L(\mathrm{x}+e_m)}{L(\mathrm{x})}=\frac{\int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\prod_{j=1}^{k}(d_j)^{x_j}\right)d_m(\Sigma d_i)^{-(\nu+1)}\gamma(\nu+1;\Sigma d_i)h(\phi,\psi)\,d\phi d\psi}{\int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\prod_{j=1}^{k}(t_j)^{x_j}\right)t_m(\Sigma t_i)^{-(\nu+1)}\gamma(\nu+1;\Sigma t_i)h(\phi,\psi)\,d\phi d\psi}\times\frac{\int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\prod_{j=1}^{k}(t_j)^{x_j}\right)(\Sigma t_i)^{-\nu}\gamma(\nu;\Sigma t_i)h(\phi,\psi)\,d\phi d\psi}{\int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\prod_{j=1}^{k}(d_j)^{x_j}\right)(\Sigma d_i)^{-\nu}\gamma(\nu;\Sigma d_i)h(\phi,\psi)\,d\phi d\psi}. \tag{4.10}$$

To make (4.10) easier to work with, let us set

$$N_1 = \int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\prod_{j=1}^{k}(d_j)^{x_j}\right)d_m(\Sigma d_i)^{-(\nu+1)}\gamma(\nu+1;\Sigma d_i)\,h(\phi,\psi)\,d\phi d\psi,$$

$$D_1 = \int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\prod_{j=1}^{k}(t_j)^{x_j}\right)t_m(\Sigma t_i)^{-(\nu+1)}\gamma(\nu+1;\Sigma t_i)\,h(\phi,\psi)\,d\phi d\psi,$$

$$N_2 = \int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\prod_{j=1}^{k}(t_j)^{x_j}\right)(\Sigma t_i)^{-\nu}\gamma(\nu;\Sigma t_i)\,h(\phi,\psi)\,d\phi d\psi,$$

$$D_2 = \int_{\psi=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\prod_{j=1}^{k}(d_j)^{x_j}\right)(\Sigma d_i)^{-\nu}\gamma(\nu;\Sigma d_i)\,h(\phi,\psi)\,d\phi d\psi,$$

and therefore

$$\frac{L(\mathrm{x}+e_m)}{L(\mathrm{x})} = \frac{N_1}{D_1}\,\frac{N_2}{D_2}\,. \tag{4.11}$$

We choose to write the quotient on the right-hand side of (4.11) in a different form:

$$\frac{L(\mathrm{x}+e_m)}{L(\mathrm{x})} = \frac{N_1}{D_2}\,\frac{N_2}{D_1}\,. \tag{4.12}$$

We work first with the quotient N_1/D_2 in (4.12). Let d_j^o be the Cauchy mean values in this quotient and put $d^o = \Sigma d_j^o$. Then

$$\frac{N_1}{D_2} = \frac{d_m^o}{d^o}\,\frac{\gamma(\nu+1;d^o)}{\gamma(\nu;d^o)} = \frac{d_m^o}{d^o}\,\nu\left(1-\frac{(d^o)^\nu e^{-d^o}}{\nu\gamma(\nu;d^o)}\right) \tag{4.13}$$

Let t_j^o be the corresponding Cauchy mean value for the quotient N_2/D_1 and put $t^o = \Sigma t_j^o$. Then

$$\frac{N_2}{D_1} = \frac{t^o}{t_m^o}\,\frac{\gamma(\nu;t^o)}{\gamma(\nu+1;t^o)} = \frac{t^o}{t_m^o}\,\frac{1}{\nu\left(1-\dfrac{(t^o)^\nu e^{-t^o}}{\nu\gamma(\nu;t^o)}\right)}\,. \tag{4.14}$$

Thus, from (4.12), (4.13), and (4.14) we have

$$\frac{L(\mathrm{x}+e_m)}{L(\mathrm{x})} = \frac{d_m^o}{d^o}\,\frac{t^o}{t_m^o}\,\frac{1-\dfrac{(d^o)^\nu e^{-d^o}}{\nu\gamma(\nu;d^o)}}{1-\dfrac{(t^o)^\nu e^{-t^o}}{\nu\gamma(\nu;t^o)}}\,. \tag{4.15}$$

Let

$$M(x) \equiv \frac{\nu\gamma(\nu;x)}{e^{-x}x^\nu}\,, \tag{4.16}$$

then (4.13) becomes

$$\begin{aligned}\frac{L(\mathrm{x}+e_m)}{L(\mathrm{x})} &= \frac{d_m^o}{d^o}\,\frac{t^o}{t_m^o}\,\frac{1-\dfrac{1}{M(d^o)}}{1-\dfrac{1}{M(t^o)}}\\ &= \frac{d_m^o}{d^o}\,\frac{t^o}{t_m^o}\,\frac{M(t^o)}{M(d^o)}\,\frac{M(d^o)-1}{M(t^o)-1}\\ &= \frac{d_m^o}{d^o}\,\frac{t^o}{t_m^o}\,W_1^n W_2^n\,,\end{aligned} \tag{4.17}$$

where

$$W_1^n = \frac{M(t^o)}{M(d^o)} \quad \text{and} \quad W_2^n = \frac{M(d^o)-1}{M(t^o)-1}, \tag{4.18}$$

and where

$$M(x) = 1 + \frac{x}{(\nu+1)} + \frac{x^2}{(\nu+1)(\nu+2)} + \cdots + \frac{x^n}{(\nu+1)(\nu+2)\cdots(\nu+n)}$$
$$+ \frac{\nu e^x x^{-\nu}}{(\nu)_{n+1}} \int_0^x y^{n+\nu} e^{-y} dy. \tag{4.19}$$

Equations (4.15), (4.17), and (4.19) then yield

$$\frac{L(\mathrm{x}+e_m)}{L(\mathrm{x})} = \frac{d_m^o}{t_m^o} W_1^n W_3^n , \tag{4.20}$$

where

$$W_3^n = \frac{1+\frac{d^o}{\nu+2}+\cdots+\frac{(d^o)^{n-1}}{(\nu+2)\cdots(\nu+n)}+\frac{\nu(\nu+1)e^{d^o}(d^o)^{-\nu-1}}{(\nu)_{n+1}}\int_0^{d^o} y^{n+\nu}e^{-y}dy}{1+\frac{t^o}{\nu+2}+\cdots+\frac{(t^o)^{n-1}}{(\nu+2)\cdots(\nu+n)}+\frac{\nu(\nu+1)e^{t^o}(t^o)^{-\nu-1}}{(\nu)_{n+1}}\int_0^{t^o} y^{n+\nu}e^{-y}dy}. \tag{4.21}$$

We need to estimate the integrals:

$$\int_0^{t^o} y^{n+\nu} e^{-y} dy \quad \text{and} \quad \int_0^{d^o} y^{n+\nu} e^{-y} dy, \tag{4.22}$$

in (4.21). We refer to Bhattacharjee (1970). The integrands in (4.22), $y^{n+\nu}e^{-y}$, are increasing for $y < n+\nu$ and decreasing for $y > n+\nu$. Hence for

$$n+\nu > \max(t^o, d^o), \tag{4.23}$$

the integrals in (4.22) can be estimated by $(t^o)^{n+\nu}e^{-t^o}$ and $(d^o)^{n+\nu+1}e^{-d^o}$, respectively.

Now define

$$A = \frac{\max_{\phi,\psi} d_m}{\min_{\phi,\psi} t_m} . \tag{4.24}$$

Then if (4.23) holds and by use of (4.24) we have

$$0 < \frac{L(\mathrm{x}+e_m)}{L(\mathrm{x})} \le A\, W_4^n W_5^n, \tag{4.25}$$

where

$$W_4^n = \frac{1+\frac{t^o}{\nu+1}+\frac{(t^o)^2}{(\nu+1)(\nu+2)}+\cdots+\frac{(t^o)^n}{(\nu+1)(\nu+2)\cdots(\nu+n)}+\frac{\nu(\nu+1)}{(\nu)_{n+1}t^o}}{1+\frac{d^o}{\nu+1}+\frac{(d^o)^2}{(\nu+1)(\nu+2)}+\cdots+\frac{(d^o)^n}{(\nu+1)(\nu+2)\cdots(\nu+n)}}, \tag{4.26}$$

and

$$W_5^n = \frac{1+\frac{d^o}{\nu+2}+\cdots+\frac{(d^o)^{n-1}}{(\nu+2)\cdots(\nu+n)}+\frac{\nu(\nu+1)}{(\nu)_{n+1}d^o}}{1+\frac{t^o}{\nu+2}+\cdots+\frac{(t^o)^{n-1}}{(\nu+2)\cdots(\nu+n)}}, \tag{4.27}$$

W_4^n is an upper bound for W_1^n and W_5^n is an upper bound for W_3^n provided (4.23) holds.

It is assumed that $A<1$. If it is not, then the reading of the mth detector is not used. At this time, our objective is to determine a ν_0 so that for

$$\nu > \max(\nu_0,\ \max(t^o, d^o) - n), \tag{4.28}$$

we have

$$W_1^n W_3^n \leq W_4^n W_5^n < A^{-1}. \tag{4.29}$$

This is sufficient to ensure that $L(x)$ is eventually decreasing in the coordinate x_m. Equation (4.29) implies

$$\begin{aligned} &\left(1+\frac{t^o}{\nu+1}+\frac{(t^o)^2}{(\nu+1)(\nu+2)}+\cdots+\frac{(t^o)^n}{(\nu+1)(\nu+2)\cdots(\nu+n)}+\frac{\nu(\nu+1)}{(\nu)_{n+1}t^o}\right)\\ &\times\left(1+\frac{d^o}{\nu+2}+\cdots+\frac{(d^o)^{n-1}}{(\nu+2)\cdots(\nu+n)}+\frac{\nu(\nu+1)}{(\nu)_{n+1}d^o}\right)\\ &< A^{-1}\left(1+\frac{d^o}{\nu+1}+\frac{(d^o)^2}{(\nu+1)(\nu+2)}+\cdots+\frac{(d^o)^n}{(\nu+1)(\nu+2)\cdots(\nu+n)}\right)\\ &\times\left(1+\frac{t^o}{\nu+2}+\cdots+\frac{(t^o)^{n-1}}{(\nu+2)\cdots(\nu+n)}\right). \end{aligned} \tag{4.30}$$

On multiplying the inequality (4.30) by $((\nu)_{n+1})^2/\nu^2(\nu+1)$, we obtain a polynomial inequality in ν of degree $2n-1$. If we replace the inequality by an equality, we obtain a polynomial equation. This equation has a real root since its degree is odd. The largest real root for ν may or may not be the desired ν_0 in (4.28). Each example requires investigation.

It is useful to record two special cases of (4.30): $n=1$ and $n=2$.

For $n=1$, (4.30) becomes:

$$\left(1+\frac{t^o}{(\nu+1)}+\frac{1}{t^o}\right)\left(1+\frac{1}{d^o}\right) < A^{-1}\left(1+\frac{d^o}{(\nu+1)}\right). \tag{4.31}$$

We rewrite (4.31) in the form

$$g(\nu) \equiv A^{-1}\left(1+\frac{d^o}{\nu+1}\right)-\left(1+\frac{t^o}{\nu+1}+\frac{1}{t^o}\right)\left(1+\frac{1}{d^o}\right) > 0. \tag{4.32}$$

$g(\nu)$ has a simple pole at $\nu=-1$. The zero of $g(\nu)$ is

$$\nu^* = \frac{A^{-1}(1+d^o)-\left(1+t^o+\frac{1}{t^o}\right)\left(1+\frac{1}{d^o}\right)}{\left(1+\frac{1}{t^o}\right)\left(1+\frac{1}{d^o}\right)-A^{-1}}, \tag{4.33}$$

provided $A^{-1} \neq (1+1/t^o)(1+1/d^o)$.

If we have equality, $g(\nu)$ has no zero. If $A^{-1}<(1+1/t^o)(1+1/d^o)$, then since $g(+\infty)<0$, there is no ν_0 for (4.31). If $A^{-1}>(1+1/t^o)(1+1/d^o)$, then since $g(+\infty)>0$, the appropriate ν_0 is the ν^* given by (4.33). This ν_0 is usually positive.

For $n=2$, (4.30) become

$$\left(1+\frac{t^o}{\nu+1}+\frac{(t^o)^2}{(\nu+1)(\nu+2)}+\frac{1}{(\nu+2)t^o}\right)\left(1+\frac{d^o}{\nu+2}+\frac{1}{(\nu+2)d^o}\right)$$
$$< A^{-1}\left(1+\frac{d^o}{\nu+1}+\frac{(d^o)^2}{(\nu+1)(\nu+2)}\right)\left(1+\frac{t^o}{\nu+2}\right). \tag{4.34}$$

Because $A^{-1}>1$, there exists a ν_0 so that for $\nu>\nu_0$, the inequality (4.34) is satisfied. This ν_0 is determined by replacing inequality by $=$ in (4.34) and solving the resulting cubic equation for ν. The cubic equation is

$$g(\nu)\equiv a_0\nu^3+a_1\nu^2+a_2\nu+a_3=0, \tag{4.35}$$

where

$$\begin{aligned}
a_0 &=(A^{-1}-1)d^o t^o,\\
a_1 &=(A^{-1}-1)d^o(t^o)^2+[(A^{-1}-1)(d^o)^2+5(A^{-1}-1)d^o-1]t^o-d^o,\\
a_2 &=-d^o(t^o)^3+[(A^{-1}-1)(d^o)^2+(3A^{-1}-4)d^o-1](t^o)^2\\
&\quad+[A^{-1}(d^o)^3+(4A^{-1}-3)(d^o)^2+8(A^{-1}-1)d^o-3]t^o-(d^o)^2-3d^o-1,\\
a_3 &=-(d^o+1)^2(t^o)^3+[A^{-1}(d^o)^3+2(A^{-1}-1)(d^o)^2+2(A^{-1}-2)d^o-2](t^o)^2\\
&\quad+2[A^{-1}(d^o)^3+(2A^{-1}-1)(d^o)^2+2(A^{-1}-1)d^o-1]t^o-(d^o+1)^2.
\end{aligned} \tag{4.36}$$

The Cardano formula for the solutions to (4.35) is not enlightening. We note that $g(+\infty)=+\infty$. We do not know whether the largest real zero of $g(\nu)$ is positive, negative, or zero. We simply record for example that for $A=0.5$, $t^o=20$, and $d^o=10$, the largest real solution to (4.35) is $\nu=-0.2195\cdots$.

This completes our discussion of the monotonicity of $L(x)$. Relation (4.30) provides the basis of an algorithm for determining for each coordinate a $\nu_0=\sum x_i+\ell$ beyond which $L(x)$ is monotonic decreasing in that coordinate. We should remark that we could have applied the Cauchy mean value theorem to the right-hand side of (4.8) as it stands. However the estimate obtained does not seem to be very useful.

5. Summary

We study the followings:

1) The probability distribution for the neutron signal received by a detector averaged over the above mentioned three sources of errors is expressed as a four-dimensional integral of certain data. We show that two of the four integrals can be evaluated analytically and thereby the integral is reduced to a two-dimensional integral.

2) We calculate and investigate the likelihood ratio of the distribution for univariate signal

and multiple signals. Under the reasonable condition, the distribution of univariate signal has a monotone likelihood ratio.

3) For multiple signals, (4.8) measures the "size" of $L(x)$ and (4.30) measures the monotonicity of $L(x)$. The resulting algorithm is then used to determine bounds on the rejection region in Neyman-Pearson test of hypothesis.

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