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A Note on the Strong Laws of Large Numbers for Fuzzy Random Sets[1]

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Abstract

In this paper we introduce the definition for fuzzy random set by Puri and Ralescu(1985) and show strong laws of large numbers for fuzzy random sets on Banach spaces.

1. Introduction

The concept of a fuzzy random variables was defined as a tool for representing relationships between the outcomes of a random experiment and inexact data. By inexactness here we mean nonstatistical inexactness that is due to subjectivity and imprecision of human knowledge rather than to the occurrence of random events. The notion of fuzzy random variables was first introduced by Kwakernaak(1978) and some fundamental properties were investigated. Kwakernaak defined a fuzzy random variable as a function F: $\Omega \rightarrow \mathcal{F}(\mathbb{R})$ (subject to certain measurability conditions), where (Ω, A, P) is a probability space, and $\mathcal{F}(\mathbb{R})$ denotes all piecewise continuous functions u: $\mathbb{R} \rightarrow [0,1]$. Stein and Talati(1981) and Puri and Ralescu(1986) give different definitions of fuzzy random variables. However, the definitions and properties developed by Puri and Ralescu(1986) provide a natural generalization of random vectors and random set.

The concept of a random set, though vaguely known for a long time, did not develop until Robins(1944,1945) provided for the first time a solid mathematical formulation of this concept and investigated relationships between random sets and geometric probabilities. Later, Kendall(1974) and Matheron(1975) provided a comprehensive mathematical theory of random sets which was greatly influenced by the geometric probability prospective. With respect to laws of large numbers, since the pioneering works of Artstein and Vitale(1975), many useful results for random sets have been developed(cf., Taylor and Inoue)

There are many recent results for laws of large numbers for fuzzy random variables and fuzzy random sets, see for example, Miyakoshi and Shimbo(1984), Puri and Ralescu(1986), Klement, Puri and Ralescu(1986), Inoue(1991). In this note we introduce the definition by Puri

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and Ralescu and study strong laws of large numbers on a real separable Banach space.

2. Fuzzy random sets

We consider a fuzzy set in a real separable Banach space E. Let K(E) denote all nonempty compact subsets of E and let Kc(E) denote all nonempty convex, compact subsets of E. For $A_1, A_2 \in K(E)$ (or $\in Kc(E)$) and $\lambda \in \mathbb{R}$, Minkowski addition and scalar multiplication are given by

$$A_1+A_2=\{ a_1+a_2 : a_1 \in A_1, a_2 \in A_2 \}$$

and

$$\lambda A_1 = \{\lambda a_1 : a_1 \in A_1\}.$$

Neither K(E) nor Kc(E) are linear space even when $E=\mathbb{R}$. For example, let $\lambda_1=1$, $\lambda_2 = -1$ and $A=[0,1]$, and observe that

$$\{0\}=0A=(\lambda_1+\lambda_2)A \neq \lambda_1 A+\lambda_2 A=[-1,1].$$

The Hausdorff distance between two sets A_1 , $A_2 \in K(E)$ can be defined as

$$d_H(A_1,A_2)= \inf\{ r>0 : A_1 \subset A_2 + rU , A_2 \subset A_1 + rU \}$$
$$= \max\{ \sup_{a\in A1} \inf_{b\in A2} \| a-b \| , \sup_{b\in A2} \inf_{a\in A1} \| a-b \| \}$$

where $U=\{x: x\in E \text{ and } \|x\|\leq 1 \}$. With the Hausdorff metric, K(E) (and Kc(E)) is a complete, separable metric space. Also, denote $\| A_1 \| = d_H(A_1,\{0\}) = \sup \{ \| a_1 \| : a_1\in A_1\}$ and coA_1 =convex hull of $A_1\in Kc(E)$. Let $(\Omega, \mathcal{A}, P)$ be a probability space. A random set(random compact set) is a Borel measurable function $X : \Omega \to K(E)$. If X is a random compact convex set (i.e. coK(E) valued), then EX is defined as

$$EX=\{ Ef : f\in L^1(\Omega, \mathcal{A}, P), f(\omega)\in X(\omega) \text{ a.s } \}$$

where $f : \Omega\to E$ is a selection of X and Ef denotes the classical expection (via the Bochner integral). In general EX may be empty, but if $E\|X\|< \infty$, then $EX \in coK(E)$.

A fuzzy set is a subset whose boundaries may not be identifiable with certainty. More technically, a fuzzy set in a space E is defined as $\{(x,u(x)) : x\in E, 0\leq u(x)\leq 1 \}$. The function $u(:E \to[0,1])$ is referred to as the membership function. For each u, we denote by $L_\alpha(u) =\{x\in E : u(x) \geq \alpha\}$, $0\leq\alpha\leq 1$, its α-level set. By supp u, we denote the support of u, i.e. the closure of the set $\{x\in E : u(x)>0 \}$. Let $\mathcal{F}(E)$ denote the space of fuzzy sets $u : E \to[0,1]$ with the following properties :

(i) u is upper semicontinuous,

(ii) $\{x\in E : u(x)\geq\alpha\}$ is compact for each $\alpha>0$,

(iii) $\{x\in E : u(x)=1\}\neq\emptyset$

Similarly, denote by $\mathcal{F}c(E)$ the subspace of $\mathcal{F}(E)$ satisfying that $\{x\in E : u(x)\geq\alpha\}$ is compact convex for each $\alpha>0$. A linear structure in $\mathcal{F}(E)$ is defined via the following operations :

$$(u+v)(x)= \sup_{y+z=x} \min [u(y),v(z)],$$

$$(\lambda u)(x) = \begin{cases} u(\lambda^{-1}x) & \text{if } \lambda \neq 0 \\ I_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases}$$

for $u, v \in \mathcal{F}(E)$, $\lambda \in \mathbb{R}$. Note that $L_\alpha(u+v) = L_\alpha(u) + L_\alpha(v)$ and $L_\alpha(\lambda u) = \lambda L_\alpha(u)$ for every $0 \le \alpha \le 1$

There is no unique metric in $\mathcal{F}(E)$ which extends the Hausdorff distance. In this note we will be concerned with the metric defined as

$$d(u,v) = \int d_H(L_\alpha(u), L_\alpha(v))\, d\alpha.$$

Then, the space $(\mathcal{F}(E), d)$ is a complete separable metric space (cf. Klement, Puri and Ralescu(1986)). The concept of a random set was generalized by Puri and Ralescu(1986). Let $X : \Omega \rightarrow \mathcal{F}(E)$ be a function such that for each $\alpha \in (0,1]$ and each $\omega \in \Omega$,

$$X_\alpha(\omega) = \{x \in E : X(\omega)(x) \ge \alpha\} \in K(E).$$

A fuzzy random set is a Borel measurable function $X : \Omega \rightarrow \mathcal{F}(E)$ such taht $X_\alpha(\omega)$ (as a function of ω) is a random set in $K(E)$ for each $\alpha \in (0,1]$. Similarly, coX can be defined as a function $\Omega \rightarrow \mathcal{F}(E)$ such that

$$L_\alpha(coX) = co\{x \in E : X(\omega)(x) \ge \alpha\} \quad \text{for each } \alpha.$$

Tightness and moment conditions can be used to obtain laws of large numbers. A sequence of fuzzy random sets $\{X_k\}$ is said to be tight if for fixed $\alpha \in (0,1]$ and each $\varepsilon > 0$ there exists a compact subset of $K(E)$, K_ε, such that $P[X_{k,\alpha} \notin K_\varepsilon] < \varepsilon$ for all k. Since ($K(E)$, d_H) is a complete separable metric space, each random set $X_{k,\alpha}$, $X_{k,\alpha}(\omega) = \{x \in E : X_k(\omega)(x) \ge \alpha\}$, is tight(Billingsley(1968), p.234). A fuzzy random set X_k is called integrably bounded if for any $\alpha \in (0,1]$ there exists $h_\alpha : \Omega \rightarrow \mathbb{R}$, $h_\alpha \in L^1(R)$ such that $\|x\| \le h_\alpha$ for all x, ω with $x \in X_{k,\alpha}(\omega)$. A sequence of fuzzy random sets $\{X_k\}$ is said to be compactly uniformly integrable if for fixed α and each $\varepsilon > 0$ there exists a compact set K_ε such that $E\|X_{k,\alpha} I_{[X_{k,\alpha} \notin K_\varepsilon]}\| < \varepsilon$ uniformly in k. Klement, Puri and Ralescu(1986) showed that there exists a unique fuzzy set $EX \in \mathcal{F}(E)$ such that $L_\alpha(EcoX) = E(L_\alpha coX)$ for each $\alpha \in (0,1]$ if X is an integrably bounded fuzzy random set.

3. Strong laws of large numbers for fuzzy random sets.

In this section we are concerned with strong laws of large numbers for fuzzy random sets developed by Puri and Ralescu(1986). A strong laws of large numbers and a central limit theorem for fuzzy random sets on $\mathbb{R}^p$ were driven in Klement, Puri and Ralescu(1986), Inoue(1991) showed a strong law of large number for tight fuzzy random sets in a real separable Banach space. The one method of approach in proving limit theorems for fuzzy random sets is to apply limit results in random sets(see, for example, Theorem 3.4 and Theorem 3.6). The another method of approach is similar to the approach for random sets initiated by Artstein and Vitale(1975)(see, for example, Theorem 3.3). The approach is as follows :

(a) Consider first fuzzy random convex sets and embed $\mathcal{F}c(E)$ into a Banach space.
(b) Use a limit theorem for Banach space valued random elements.
(c) Drop the convexity condition.

Several embedding theorems exist for embedding the convex, compact subset of E into Banach spaces. The main tool in proving strong Law of large numbers is a result due to Radstrom(1952) which states that the collection of compact convex subsets of a Banach space can be embedded as a convex cone in a normed space.

Lemma 3.1 (Puri and Ralescu(1983 a) Let E be a reflexive Banach space. Then there exists a normed space $\mathfrak{X}$ such that $\mathcal{F}c(E)$ can be embedded isometrically into $\mathfrak{X}$.

Theorem 3.2 (Klement, Puri and Ralescu(1986)) Let $\{X_k\}$ be independent and identically distributed fuzzy random variables(sets) on $\mathbb{R}^p$ such that $E\|suppX_1\| < \infty$. Then,

$$d\left(\frac{1}{n}\sum_{k=1}^{n} X_k,\ E(coX_1)\right) \to 0 \ a.s.$$

The above theorem can be easily applied to a real reflexive separable Banach space E by Lemma 3.1 A short proof will be given in Theorem 3.3 for the completeness. A more general result was obtained by Inoue(1991), which is shown in Theorem 3.4.

Theorem 3.3 Let $\{X_k\}$ be independent and identically distributed fuzzy random sets in $\mathcal{F}(E)$ such that $E\|suppX_1\| < \infty$. Then,

$$d\left(\frac{1}{n}\sum_{k=1}^{n} X_k,\ E(coX_1)\right) \to 0 \ a.s.$$

proof. (i) Consider first $X_k : \Omega \to \mathcal{F}c(E)$. Let $j : \mathcal{F}c(E) \to \mathfrak{X}$ be isometry provided by Lemma 3.1. Since $\mathcal{F}c(E)$ is separable, it is easy to show that $\mathfrak{X}$ is separable. By a standard SLLN for i.i.d. random elements in Banach space, it follows that $\frac{1}{n}\sum_{k=1}^{n}(j \circ X_k) \to E(j \circ X_1)$ $a.s.$ By the similar arguments in Klement, Puri and Ralescu(1986), $E(j \circ X_1) = j(EX_1)$ if $E\|suppX_1\| < \infty$. Thus,

$$\left\|\frac{1}{n}\sum_{k=1}^{n}(j \circ X_k) - j(EX_1)\right\| \to 0 \ a.s.$$

and hence

$$d\left(\frac{1}{n}\sum_{k=1}^{n} X_k\ ,\ E(X_1)\right) \rightarrow 0\ a.s.$$

(ii) In the general case, i.e. $X_k : \Omega \rightarrow \mathcal{F}(E)$.

Applying the theorem in Arstein and Hansen(1985) and dominated convergence theorem to the case (i), the result follows.

Identically distributed assumption and reflexivity can be relaxed as follows. In the following theorems E is a separable Banach space.

Theorem 3.4 (Inoue (1991)) Let $\{X_k\}$ be a sequence of independent, tight fuzzy random sets in $\mathcal{F}(E)$ and $\|X_{k,\alpha}\|^r\ (1 \leq r \leq 2)$ is bounded by $h_\alpha(\omega)$ which is integrable. Then

$$d\left(\frac{1}{n}\sum_{k=1}^{n} X_k\ ,\ \frac{1}{n}\sum_{k=1}^{n} E(coX_k)\right) \rightarrow 0\ a.s.$$

Corollary 3.5 Let $\{X_k\}$ be independent and identically distributed fuzzy random sets in $\mathcal{F}(E)$ such that $E\|suppX_1\| < \infty$. Then,

$$d\left(\frac{1}{n}\sum_{k=1}^{n} X_k\ ,\ E(coX_1)\right) \rightarrow 0\ a.s.$$

proof. Since $\|X_{1,\alpha}\| \leq \|suppX_1\|$ and $E\|suppX_1\| < \infty$, By the above theorem the result follows.

Finally, we will show a SLLN for a triangular array of fuzzy random sets in $\mathcal{F}(E)$.

Theorem 3.6 Let $\{X_k\}$ be a sequence of independent compactly uniformly integrable fuzzy random sets in $\mathcal{F}(E)$. Let X be a random variable such that $P(\|supp\ X_k\| \geq x) \leq P(X \geq x)$ for all k and $x>0$ and $E(X^{1+\frac{1}{r}}) = \Gamma < \infty,\ r > 0$. If the non-negative array $\{a_{nk} : n \geq 1,\ 1 \leq k \leq n\}$ satisfies the condition

(i) $\sum_{k=1}^{n} a_{nk} \leq 1$,

(ii) $\max_{1 \leq k \leq n} a_{nk} \rightarrow 0$ as $n \rightarrow \infty$,

(iii) $\max_{1 \leq k \leq n} a_{nk} = O(n^{-r})$

Then,

$$d\left(\sum_{k=1}^{n} a_{nk}X_k\ ,\ \sum_{k=1}^{n} a_{nk}E(coX_k)\right) \rightarrow 0\ a.s.$$

Proof.

$$E\|suppX_k\| = \int_{\Omega} P(\|suppX\| \geq x)\ dx$$
$$\leq \int_{\Omega} P(X \geq x)\ dx = EX < \infty$$

and hence $L_\alpha(EX) = E(L_\alpha X)$.

Now,

$$d\left(\sum_{k=1}^{n} a_{nk}X_k,\ \sum_{k=1}^{n} a_{nk}E\,coX_k\right)$$
$$= \int_0^1 d_H\left(L_\alpha\left(\sum_{k=1}^{n} a_{nk}X_k\right),\ L_\alpha\left(\sum_{k=1}^{n} a_{nk}E\,coX_k\right)\right)\ d\alpha,$$

and

$$d_H\left(L_\alpha\left(\sum_{k=1}^{n} a_{nk}X_k\right),\ L_\alpha\left(\sum_{k=1}^{n} a_{nk}E\,coX_k\right)\right)$$
$$= d_H\left(\sum_{k=1}^{n} L_\alpha(a_{nk}X_k),\ \sum_{k=1}^{n} L_\alpha(a_{nk}E\,coX_k)\right)$$
$$= d_H\left(\sum_{k=1}^{n} a_{nk}L_\alpha(X_k),\ \sum_{k=1}^{n} a_{nk}L_\alpha(E\,coX_k)\right)$$
$$= d_H\left(\sum_{k=1}^{n} a_{nk}X_{k,\alpha},\ \sum_{k=1}^{n} a_{nk}E(L_\alpha(coX_k))\right)$$
$$= d_H\left(\sum_{k=1}^{n} a_{nk}X_{k,\alpha},\ \sum_{k=1}^{n} a_{nk}E(coX_{k,\alpha})\right).$$

Applying random set version in Theorem 3.1,Taylor and Inoue(1985), for each $\alpha \in (0,1]$

$$d_H\left(\sum_{k=1}^{n} a_{nk}X_{k,\alpha},\ \sum_{k=1}^{n} a_{nk}E(coX_{k,\alpha})\right) \rightarrow 0\ a.s.$$

Also,

$$d_H\left(\sum_{k=1}^{n} a_{nk}X_{k,\alpha},\ \sum_{k=1}^{n} a_{nk}E(coX_{k,\alpha})\right)$$
$$\leq d_H\left(\sum_{k=1}^{n} a_{nk}X_{k,\alpha}, \{0\}\right) + d_H\left(\sum_{k=1}^{n} a_{nk}E(coX_{k,\alpha}), \{0\}\right),$$

and

$$d_H\left(\sum_{k=1}^{n} a_{nk}X_{k,\alpha}, \{0\}\right) \leq \sup_k \|X_{k,\alpha}\| \leq \sup_k \|supp\,X_k\| < \infty$$

Thus, by the Lebesgue dominated convergence theorem,

$$d\left(\sum_{k=1}^{n} a_{nk}X_k,\ \sum_{k=1}^{n} a_{nk}E(coX_k)\right) \rightarrow 0\ a.s.$$

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