The Korean Communications
in Statistics Vol. 5, No. 1, 1998
pp. 225-230

Approximate MLE for Rayleigh Distribution in Singly Right Censored Samples

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Abstract

By assuming a singly right cenosred sample, we propose the approximate maximum likelihood estimator (AMLE) of the scale parameter of the p-dimensional Rayleigh distribution. We compare the proposed estimator in terms of the mean squared error through Monte Carlo methods.

1. Introduction

The Rayleigh distribution may be known as the distribution of distance X from the origin to a point $(Y_1, Y_2, \ldots, Y_p)$ in a p-dimensional Euclidean space, where the components Y are independent random variables, each of which is normally distributed $N(0, \sigma^2)$.

The pdf of the Rayleigh distribution(i.e., the pdf of X) follows as

$$f(x; p, \sigma) = \begin{cases} \dfrac{2x^{p-1}\exp(-x^2/2\sigma^2)}{(2\sigma^2)^{p/2}\Gamma(p/2)}, & 0 < x < \infty, \\ = 0, & \text{otherwise}. \end{cases} \tag{1}$$

With $\sigma = 1$ is the pdf of chi-square distribution with p degrees of freedom.

This is very useful distribution and the special cases in which p=1, 2, and 3 are important in various scientific applications. The one-dimensional Rayleigh distribution is sometimes known as the folded normal, the folded Gaussian, or the half normal distribution. And the case of two-dimensional may be not only the most important of all the Rayleigh distributions, it is also a special case of the Weibull distribution. Also three-dimensional Rayleigh distributionmany is of special interest to engineers and physicists. It was originally derived by Maxwell (1860) as the distribution of the velocity of gas particles in three-dimensional space.

The maximum likelihood estimator (MLE) in singly right censored samples from Rayleigh distributon were considered by Cohen (1991). But the MLE can not be exactly calculated, when $p \geq 3$.

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The approximate maximum likelihood estimating method was first developed by Balakrishnan (1989a,b) for the purpose of providing the explicit estimators of the scale parameter in the Rayleigh distribution. He derived approximate maximum likelihood estimator of the scale parameter from Type-II double censored sample in two-dimensional Rayleigh distribution. Kang (1996) obtained the AMLE for the scale parameter of the double exponential distribution based on Type-II censored samples and he showed that the proposed estimator is generally more efficienct than the BLUE and the optimum unbiased absolute estimator. Kang and Cho (1997a,b) derived the minimum risk estimator and the AMLE of the scale parameter of the one-parameter exponential distribution under general Type-II censored and progressive Type-II censored sample, and the proposed estimators were compared in terms of the mean squared error through Nelson's data.

We consider the AMLE of the scale parameter from singly right censored samples in p -dimensional Rayleigh distributon. Here we shall consider the AMLE and asymptotic variance of the AMLE, and compare the AMLE with MLE of the scale parameter for the Rayleigh distribution in singly right censored samples through Monte Carlo method.

2. Parameter Estimation in Singly Right Censored

The singly right censored samples consist of a total of N observations of which n are fully measured while $c=N-n$ are censored. For each of the censored observations, it is known only that $T<x$, where as for each the measured observations, $x\leq T$. In Type-I samples T is fixed constant.

The likelihood function of a singly right censoring sample as described from p-dimensional Rayleigh distribution with pdf (1) is

$$L=2^{-n(p-2)/2}\sigma^{-np}\left[\Gamma\left(\frac{p}{2}\right)\right]^{-n}\left[\prod_{i=1}^{n}x_i^{\,p-1}\right]\left[\exp\left(-\frac{1}{2}\sigma^2\sum_{i=1}^{n}x_i^2\right)\right][1-F_p(T)]^cK, \quad (2)$$

where K is an ordering constant and $F_p(X)$ is the cdf of X.

Let $Z=X/\sigma$. The pdf of (1) becomes

$$\begin{aligned} g_p(z)&=\frac{2^{-(p-2)/2}}{\Gamma(p/2)}z^{p-1}\exp\left(-\frac{z^2}{2}\right), && 0<z<\infty, \\ &=0, && \text{otherwise}. \end{aligned} \quad (3)$$

When $X=T$, then $Z=T/\sigma=\xi$ and

$$F_p(T)=G_p(\xi)=\int_0^{\xi}g_p(z)\ dz. \quad (4)$$

And the cdf (4) is calculated follows as

$$G_p(\xi)=\int_0^{\xi} g_p(z)\ dz=\int_0^{\xi} \alpha(p) z^{p-1} \exp\left(-\frac{z^2}{2}\right) dz$$
$$=\alpha(p)2^{\frac{p}{2}-1}\int_0^{\frac{\xi^2}{2}} y^{\frac{p}{2}-1} \exp(-y)\ dy$$
$$=\alpha(p)2^{\frac{p}{2}-1}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\xi^2}{2}\right)^{\frac{p}{2}-1}}{n!\left(\frac{p}{2}+n\right)}\equiv R_p,$$

where $\alpha(p)=2^{-(p-2)/2}/\Gamma(p/2)$.

In making the transformation from X to Z, ξ has become the parameter to be estimated. After $\hat{\xi}$ has been calculated, it then follows that $\hat{\sigma}=T/\hat{\xi}$.

The maximum likelihood estimating equation for σ is

$$\frac{\partial \ln L}{\partial\sigma}=-\frac{np}{\sigma}+\frac{1}{\sigma^3}\sum_{i=1}^{n}x_i^2+\left(\frac{cT}{\sigma^2}\right)\frac{g_p(\xi)}{1-G_p(\xi)}=0. \tag{5}$$

Cohen (1991) obtained this equation;

$$\frac{1}{nT^2}\sum_{i=1}^{n}x_i^2=\frac{1}{\xi}\left[\frac{p}{\xi}-\left(\frac{c}{n}\right)\frac{g_p(\xi)}{1-G_p(\xi)}\right]\equiv H_p\left(\xi,\frac{c}{n}\right), \tag{6}$$

where as a result of the transformation from X to Z, ξ has become the parameter to be estimated. He calculate $\hat{\xi}$ as the solution of (6), and $\hat{\sigma}$ follows as

$$\hat{\sigma}=\frac{T}{\hat{\xi}}. \tag{7}$$

To evaluate $H_p(\xi, c/n)$ as defined in (6), the cdf $G_p(z)$ and the pdf $g_p(z)$ had to be calculated. For $p=2$ and 3,

$$G_2(z)=\sqrt{2\pi}[\phi(0)-\phi(z)],\qquad G_3(z)=2[\Phi(0)-z\phi(z)]-1, \tag{8}$$

and

$$g_2(z)=\sqrt{2\pi}z\phi(z),\qquad g_3(z)=2z^2\phi(z), \tag{9}$$

where $\phi(z)$ and $\Phi(z)$ are the pdf and the cdf of the standard normal distribution, respectively.

For $p=2$, Cohen (1991) substituted $g(\cdot)$ and $G(\cdot)$ from (8) and (9) into (6), and simply to obtain

$$H_2\left(\xi,\frac{c}{n}\right)=\frac{2}{\xi^2}-\frac{c}{n}=\frac{2\sigma^2}{T^2}-\frac{c}{n} \tag{10}$$

then the MLE is

$$\hat{\sigma}_{MLE}=\sqrt{\frac{1}{2n}\left(\sum_{i=1}^{n} X_i^2+cT^2\right)}. \tag{11}$$

For $p=3$, he was able to obtain the MLE by same method as

$$H_3\left(\xi, \frac{c}{n}\right)=\frac{1}{nT^2}\sum_{i=1}^{n} x_i^2=\frac{3}{\xi^2}-\frac{c}{n}\left(\frac{\xi\phi(\xi)}{1-\Phi(\xi)+\xi\phi(\xi)}\right). \tag{12}$$

But the MLEs do not always have an explicit solution for σ, and if $p\geq 3$, then this method is very complexd.

So we can expand the function $g_p(\xi)/(1-G_p(\xi))$, in (6) to Taylor series around the point $\tau= G_p^{-1}(R_p)=\left[\alpha(p)^{-1}R_p\left(\sum_{n=0}^{\infty}(-1)^n/\left(n!\left(\frac{p}{2}+n\right)\right)\right)^{-1}\right]^{\frac{1}{p-2}}$ and then approximate it by

$$\frac{g_p(\xi)}{(1-G_p(\xi))}\simeq\alpha+\beta\xi, \tag{13}$$

$$\alpha=\frac{q_p(1-R_p)-\tau[t_p(1-R_p)+q_p^2]}{(1-R_p)^2},$$

and

$$\beta=\frac{t_p(1-R_p)+q_p^2}{(1-R_p)^2}.$$

where $q_p\equiv g_p(\tau)$, $t_p\equiv g_p'(\tau)$.

Now making use of the approximate expression in (13), we obtain the approximate likelihood equation of (5) as follows;

$$\frac{\partial \ln L}{\partial\sigma}\simeq\frac{\partial \ln L^*}{\partial\sigma}\equiv-\frac{np}{s}+\frac{1}{\sigma^3}\sum_{i=1}^{n} x_i^2+\left(\frac{cT}{\sigma^2}\right)(\alpha+\beta\xi)=0. \tag{14}$$

Upon solving equation (14) for σ, we obtain the AMLE of σ as follows;

$$\hat{\sigma}_{AMLE}=\frac{c\alpha T+\sqrt{(c\alpha T)^2+4np\left(c\beta T^2+\sum_{i=1}^{n} X_i^2\right)}}{2np}. \tag{15}$$

Since $\hat{\sigma}_{AMLE}$ is the approximate maximum likelihood estimator, it follows that $\hat{\sigma}_{MLE}$ is asymptotically normal distributed with mean σ and variance $1/E[-\partial^2\ln L^*/\partial\sigma^2]$. The asymptotic variance of maximum likelihood estimate $\hat{\sigma}$ is given by

$$\mathrm{AsyVar}(\hat{\sigma})=-\left[E\left(\frac{\partial^2\ln L}{\partial\sigma^2}\right)\right]^{-1}. \tag{16}$$

In the censored sample the expected value of the second partial of the loglikelihood function is

$$E\left(\frac{\partial^2 \ln L}{\partial \sigma^2} \right) = - \frac{2pA(n)}{\sigma^2} \left[1 - \frac{\xi g_p(\xi)}{2pG_p(\xi)} \left\{ \xi^2 - (p-2) - \frac{\xi g_p(\xi)}{1-G_p(\xi)} \right\} \right], \tag{17}$$

where $A(n) = NG_p(\xi)$ in Type-I censored samples and $A(n) = n$ in Type-II censored samples.

$$E\left(\frac{\partial^2 \ln L^*}{\partial \sigma^2} \right) = - \frac{2pA(n)}{\sigma^2} \left[1 - \frac{\xi}{2p} (\alpha^* + \beta^* \xi)\{ \xi^2 - (p-2) - \xi(\alpha + \beta\xi)\} \right], \tag{18}$$

where

$$\alpha^* = \frac{q_p R_p - \tau(t_p R_p - q_p^2)}{R_p^2}$$

and

$$\beta^* = \frac{t_p R_p - q_p^2}{R_p^2}.$$

So we can obtain the asymptotic variance of the AMLE of σ as

$$\mathrm{AsyVar}(\hat{\sigma}_{AMLE}) = \frac{\sigma^2}{2pA(n)} \left[1 - \frac{\xi}{2p} (\alpha^* + \beta^* \xi)\{ \xi^2 - (p-2) - \xi(\alpha + \beta\xi)\} \right]^{-1}. \tag{19}$$

When two-dimensional distribution, parameter $\alpha \simeq 0$ and parameter $\beta \simeq 1$. So approximate variance of AMLE of σ is

$$\mathrm{AsyVar}(\hat{\sigma}_{AMLE}) \simeq \mathrm{AsyVar}(\hat{\sigma}_{MLE}) = \frac{\sigma^2}{4A(n)}. \tag{20}$$

3. The Simulated Result

The Rayleigh random numbers were generated by IMSL subroutine RNGAM and transformed $\sigma\sqrt{2 * \mathrm{RNGAM}}$. The MLE of σ was obtained by Fortran program and Newton-Raphson method.

We investigated the MSE's of the $\hat{\sigma}_{MLE}$ and the $\hat{\sigma}_{AMLE}$ for each $\sigma = 0.5$, 1.0 and censord time T_i in $p = 3$. The simulation procedure is repeated 1,000 times for each sample sizes $n = 10(100)30$. The censored time T_1 is censored about 25 pesentile and T_2 is about 15 percentile from complete samples and T_3 is not have censoring samples.

For the case of the right censored samples at T_1, T_1, the MSE of the MLE is a little smaller than the MSE of the AMLE, but the MSE's of two estimators are not much different.

For the case of the complete samples, the MSE's of two estimators are same. Although the

MSE is a little better than the AMLE in the sense of MES, the proposed estimator $\hat{\sigma}_{AMLE}$ provide the explict estimator of the scale parameter.

Table 1. MSE's of the $\hat{\sigma}_{MLE}$ and $\hat{\sigma}_{AMLE}$, I =MSE ($\hat{\sigma}_{MLE}$), II =MSE ($\hat{\sigma}_{AMLE}$).

σ=0.5						
n	T_1=1.0		T_2=1.5		T_3=3.0	
	I	II	I	II	I	II
10	.00678	.00770	.00448	.00486	.00422	.00422
40	.00148	.00186	.00112	.00123	.00107	.00107
70	.00079	.00105	.00060	.00066	.00058	.00058
100	.00054	.00079	.00044	.00050	.00044	.00044
σ=1.0						
n	T_1=1.0		T_2=1.5		T_3=3.0	
	I	II	I	II	I	II
10	.01995	.02264	.01790	.01946	.01687	.01687
40	.00487	.00576	.00447	.00491	.00427	.00427
70	.00250	.00301	.00238	.00263	.00231	.00231
100	.00188	.00238	.00177	.00200	.00175	.00175

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