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A Weak Positive Orthant Dependence Concept

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Abstract

In this paper, we introduce a new concept of the multivariate positive dependence. This concept is weaker than the positive orthant dependence. Some basic properties and preservation results are presented.

1. Introduction

A bivariate random variable (X,Y) is said to be positively quadrant dependent(PQD) if $P(X\le x,\ Y\le y)\ge P(X\le x)P(Y\le y)$ (or $P(X> x,\ Y> y)\ge P(X> x)P(Y> y)$) holds for all real numbers x and y. See Lehmann(1966). Ahmed et al.(1978) have extended this notion to the multivariate random variables : The random variables $X_1,\cdots,X_n$ are said to be positively upper(lower) orthant dependent(PUOD(PLOD)) if

$$P[\bigcap_{i=1}^{n}(X_i>x_i)]\ge\prod_{i=1}^{n}P(X_i>x_i)\,(P[\bigcap_{i=1}^{n}(X_i\le x_i)]\ge\prod_{i=1}^{n}P(X_i\le x_i))$$

holds for all real numbers $x_1,\cdots,x_n$. $X_1,\cdots,X_n$ are said to be positively orthant dependent(POD) if they are PUOD and PLOD. Various multivariate positive dependence concepts have been investigated. For review of some multivariate positive dependence concepts one may consult, Barlow and Proschan(1981), and Tong(1980). But most of these multivarite positive dependence concepts introduced in the literature are only stronger than POD. One may face problem if one wishes to investigate a new positive dependence concept weaker than POD(cf. Example 2.2).

For bivariate random variables Alzaid(1990) introduced the notion of positive dependence weaker than positive quadrant dependence(PQD): The bivariate random variable $(X,\ Y)$(or

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the joint distribution function F) is said to be weakly positive quadrant dependent of the first(second) type(WPQD1(WPQD2)) if

$$\int_x^\infty \int_y^\infty [P(X>s,\ Y>t)-P(X>s)P(Y>t)]\,\mathrm{dt\,ds} \geq 0. \tag{1.1}$$

$$\left(\int_{-\infty}^x \int_{-\infty}^y [P(X>s,\ Y>t)-P(X>s)P(Y>t)]\,\mathrm{dt\,ds} \geq 0.\right)$$

$(X,\ Y)$ is called weakly positive quadrant dependent(WPQD) if it is WPQD1 and WPQD2.

In this paper we introduce the notion of weak positive orthant dependence, which is a multivariate version of weak positive quadrant dependence, and investigate some basic properties and preservation results. The importance of this concept of positive dependence lies in the fact that it is weaker than the positive orthant dependence and enjoys most of the properties and preservation results of the positive orthant dependence.

In Section 2, the concepts of weak positive orthant dependence and preliminary results are given. Basic properties of POD which are enjoyed in WPOD are presented in Section 3. In Section 4, some preservation results are also developed with an application.

2. Preliminaries

We extend the notion of weak positive quadrant dependence to the multivariate case:

Definition 2.1 The random variables $X_1, \cdots, X_n$ are called weakly positive upper(lower) orthant dependent of the first type (WPUOD1(WPLOD1)) if

$$\int_{x_1}^\infty \cdots \int_{x_n}^\infty \{P(\bigcap_{i=1}^n X_i > s_i) - \prod_{i=1}^n P(X_i > s_i)\}\, ds_n \cdots ds_1 \geq 0. \tag{2.1}$$

$$\left(\int_{x_1}^\infty \cdots \int_{x_n}^\infty \{P(\bigcap_{i=1}^n X_i \leq s_i) - \prod_{i=1}^n P(X_i \leq s_i)\}\, ds_n \cdots ds_1 \geq 0.\right)$$

and they are called weakly positive upper(lower) orthant dependent of the second type (WPUOD2(WPLOD2)) if

$$\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \{P(\bigcap_{i=1}^n X_i > s_i) - \prod_{i=1}^n P(X_i > s_i)\}\, ds_n \cdots ds_1 \geq 0. \tag{2.2}$$

$$\left(\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \{P(\bigcap_{i=1}^n X_i \leq s_i) - \prod_{i=1}^n P(X_i \leq s_i)\}\, ds_n \cdots ds_1 \geq 0.\right)$$

$X_1, \cdots, X_n$ are called weakly positive upper(lower) orthant dependent (WPUOD(WPLOD)) if they are WPUOD1(WPLOD1) and WPUOD2(WPLOD2) and $X_1, \cdots, X_n$ are called weakly positive orthant dependent(WPOD) if they are WPUOD and WPLOD.

Some basic properties of WPUOD (WPLOD) random variables are introduced.

(P_1) Any subset of WPUOD(WPLOD) random variables is WPUOD(WPLOD).

(P_2) The set consisting of a single random variable is WPUOD(WPLOD).

(P_3) If $(X_1, \cdots, X_n)$ is WPUOD(WPLOD) then $(a_1X_1 + b_1, \cdots, a_nX_n + b_n)$ is WPUOD (WPLOD) for $a_i > 0$, $i = 1, 2, \cdots, n$.

(P_4) The union of independent sets of WPUOD(WPLOD) random variables is WPUOD (WPLOD).

Proof We only show the WPUOD case of (P_4). Let $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_m)$ be WPUOD respectively and independent each other. Then

$$\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} [P(\bigcap_{i=1}^{n} (X_i > s_i), \bigcap_{j=1}^{m} (Y_j > t_j)) - \prod_{i=1}^{n} P(X_i > s_i) \prod_{j=1}^{m} P(Y_j > t_j)] dt_m \cdots dt_1 ds_n \cdots ds_1 \tag{2.3}$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} [P(\bigcap_{i=1}^{n} (X_i > s_i)) P(\bigcap_{j=1}^{m} (Y_j > t_j)) - \prod_{i=1}^{n} P(X_i > s_i) \prod_{j=1}^{m} P(Y_j > t_j)] dt_m \cdots dt_1 ds_n \cdots ds_1 \geq 0 .$$

The first equality follows from independence of $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_m)$, and nonnegativity of the right hand side of (2.3) follows from WPUOD1 assumption. Hence, the union $(X_1, \cdots, X_n, Y_1, \cdots, Y_m)$ is WPUOD1. Similarly, we can prove that $(X_1, \cdots, X_n, Y_1, \cdots, Y_m)$ is WPUOD2. The proof is complete.

Remark and Example 2.2 will show the relation between POD and WPOD:

Remark Note that if the random variables $X_1, \cdots, X_n$ are POD then they are WPOD.

Example 2.2. Let X_1, X_2, X_3 be random variables with the following probabilities (all probabilities are multipled by 40) $P(X_1 = x_1, X_2 = x_2, X_3 = x_3)$:

		$X_3 = 0$			$X_3 = 1$		
		X_2			X_2		
		0	1	2	0	1	2
X_1	0	2	0	3	2	0	3
	1	6	3	1	6	3	1
	2	0	1	4	0	1	4

Trivially, (X_1, X_2, X_3) is WPUOD2 and it is easy to check that (X_1, X_2, X_3) is WPUOD1 but not PUOD. Hence WPUOD does not imply PUOD.

Definition 2.3(Shaked, Shanthikumar, 1994) Let $\underline{X} = (X_1, \cdots, X_n)$ and $\underline{Y} = (Y_1, \cdots, Y_n)$ be random vectors. $\underline{X}$ is smaller than $\underline{Y}$ in the upper(lower) orthant-convex (cocave) order ($\underline{X} \le_{uo-cx} \underline{Y}$($\underline{X} \le_{lo-cv} \underline{Y}$)) if and only if

$$\begin{aligned} &\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} P(X_1 > s_1, \cdots, X_n > s_n)\, ds_n \cdots ds_1 \\ &\le \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} P(Y_1 > s_1, \cdots, Y_n > s_n)\, ds_n \cdots ds_1 . \end{aligned} \tag{2.4}$$

$$\left(\begin{aligned} &\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} P(X_1 \le s_1, \cdots, X_n \le s_n)\, ds_n \cdots ds_1 \\ &\ge \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} P(Y_1 \le s_1, \cdots, Y_n \le s_n)\, ds_n \cdots ds_1 . \end{aligned} \right)$$

Theorem 2.4(Shaked, Shanthikumar, 1994) Let $\underline{X} = (X_1, \cdots, X_n)$ and $\underline{Y} = (Y_1, \cdots, Y_n)$ be random vectors. Then $\underline{X} \le_{uo-cx} \underline{Y}$($\underline{X} \le_{lo-cv} \underline{Y}$) if and only if $E[\prod_{i=1}^n g_i(X_i)] \le E[\prod_{i=1}^n g_i(Y_i)]$($E[\prod_{i=1}^n h_i(X_i)] \le E[\prod_{i=1}^n h_i(Y_i)]$) for all nonnegative increasing convex(concave) functions $g_1, \cdots, g_n$($h_1, \cdots, h_n$).

The following result provides relation between WPUOD1(WPLOD2) and the upper(lower) orthant convex(concave) order.

Lemma 2.5 Let $\underline{X} = (X_1, \cdots, X_n)$ be a random vector and let $\underline{X}^* = (X_1^*, X_2^*, \cdots, X_n^*)$ have independent components such that $X_i^* =^d X_i$($=^d$ stands for equality in distribution). Then $\underset{\sim}{X}$ is WPUOD1 if and only if $\underset{\sim}{X} \geq_{uo-cx} \underset{\sim}{X}^*$, $\underset{\sim}{X}$ is WPLOD2 if and only if $\underset{\sim}{X} \geq_{lo-cv} \underset{\sim}{X}^*$.

Proof We only prove WPUOD1 case ($\Rightarrow$). Assume $\underset{\sim}{X}$ is WPUOD1. Then from assumptions we have

$$\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} P\left(\bigcap_{i=1}^{n} (X_i > s_i)\right) ds_n \cdots ds_1 \geq \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{i=1}^{n} P(X_i > s_i)\, ds_n \cdots ds_1$$
$$= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{i=1}^{n} P(X_i^* > s_i)\, ds_n \cdots ds_1 = \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} P\left(\bigcap_{i=1}^{n} (X_i^* > s_i)\right) ds_n \cdots ds_1.$$

Hence $\underset{\sim}{X} \geq_{uo-cx} \underset{\sim}{X}^*$. ($\Leftarrow$). It follows from assumptions $\underset{\sim}{X} \geq_{uo-cx} \underset{\sim}{X}^*$ and $X_i^* =^d X_i$ that

$$\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \left[P\left(\bigcap_{i=1}^{n} (X_i > s_i)\right) - \prod_{i=1}^{n} P(X_i > s_i)\right] ds_n \cdots ds_1$$
$$\geq \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \left[P\left(\bigcap_{i=1}^{n} (X_i^* > s_i)\right) - \prod_{i=1}^{n} P(X_i^* > s_i)\right] ds_n \cdots ds_1 = 0.$$

The zero follows from assumption that $\underset{\sim}{X}^*$ has independent components. Hence $\underset{\sim}{X}$ is WPUOD1. Similarly, we can prove WPUOD2 case.

From Theorem 2.4 and Lemma 2.5 we obtain following theorem.

Theorem 2.6 $\underset{\sim}{X} = (X_1, \cdots, X_n)$ is WPUOD1(WPLOD2) if and only if $E[\prod_{i=1}^{n} f_i(X_i)] \geq (\leq) \prod_{i=1}^{n} [Ef_i(X_i)]$ for all nonnegative increasing convex(concave) functions $f_1, \cdots, f_n$.

Theorem 2.7 $\underset{\sim}{X} = (X_1, \cdots, X_n)$ is WPUOD1(WPLOD2) if and only if $(f_1(X_1), \cdots, f_n(X_n))$ is WPUOD1(WPLOD2) for all increasing convex(concave) functions $f_1, \cdots, f_n$.

Proof It is sufficient to show only if part. Assume $\underset{\sim}{X}$ is WPUOD1(WPLOD2). Then for all nonnegative increasing convex(concave) functions $F_1, \cdots, F_n$,

$$E\left[\prod_{i=1}^{n} F_i(f_i(X_i))\right] \geq (\leq) \prod_{i=1}^{n} E[F_i(f_i(X_i))]$$

since $F_i \cdot f_i$'s are nonnegative increasing convex(concave) functions. Hence $(f_1(X_1), \cdots, f_n(X_n))$ is WPUOD1(WPLOD2) according to Theorem 2.6.

3. Some Properties of Weak Positive Orthant Dependence

In this section we show that WPOD enjoys some properties of POD.

Definition 3.1(Esary, Proschan, Walkup,1967) The random variables $X_1, \cdots, X_n$ are said to be associated if $\mathrm{Cov}[f(X_1, \cdots, X_n), g(X_1, \cdots, X_n)] \geq 0$ for all increasing real valued functions f, g for which the covariance exists.

Lemma 3.2(Ahmed et al., 1978) If $Y_1, \cdots, Y_m$ are associated and if $g_i(Y_1, \cdots, Y_m)$ are nonnegative increasing for $i = 1, 2, \cdots, k$, then

$$E[\prod_{i=1}^{k} g_i(Y_1, \cdots, Y_m)] \geq \prod_{i=1}^{k} E[g_i(Y_1, \cdots, Y_m)].$$

Definition 3.3(Ebrahimi, Ghosh, 1981) A random vector $\underline{Y}$ is said to be stochastically increasing in the random vector $\underline{X}$($\underline{Y} \uparrow$ st. in $\underline{X}$) if $E[f(\underline{Y})|\underline{X} = \underline{x}]$is increasing in $\underline{x}$ for every increasing real valued integrable function f.

Theorem 3.4 Let $\underline{X} = (X_1, \cdots, X_n)$ and $\underline{Y} = (Y_1, \cdots, Y_m)$. Assume (i) $\underline{X}$, given $\underline{Y}$, is conditionally WPUOD1(WPUOD2), (ii) $X_i \uparrow$ st. in $\underline{Y}$ for $i = 1, 2, \cdots, n$, and (iii) $\underline{Y}$ is associated. Then (a) $(\underline{X}, \underline{Y})$ is WPUOD1(WPUOD2), (b) $\underline{X}$ is WPUOD1(WPUOD2).

Proof of (a). Observe that

$$\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} P[\bigcap_{i=1}^{n} (X_i > s_i), \bigcap_{i=1}^{m} (Y_i > t_i)]\, dt_m \cdots dt_1\, ds_n \cdots ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} E_{\underline{Y}}\{P[\bigcap_{i=1}^{n} (X_i > s_i) | \underline{Y}]\, I_{\{\bigcap_{i=1}^{m} (Y_i > t_i)\}}\}\, dt_m \cdots dt_1\, ds_n \cdots ds_1$$

$$\geq \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} E_{\underline{Y}}\{\prod_{i=1}^{n} P[(X_i > s_i) | \underline{Y}]\, I_{\{\bigcap_{i=1}^{m} (Y_i > t_i)\}}\} dt_m \cdots dt_1\, ds_n \cdots ds_1$$

$$\geq \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} \prod_{i=1}^{n} E_{\underline{Y}}\{P[(X_i > s_i) | \underline{Y}]\} E_{\underline{Y}}\{I_{\{\bigcap_{i=1}^{m} (Y_i > t_i)\}}\} dt_m \cdots dt_1\, ds_n \cdots ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} \prod_{i=1}^{n} P(X_i > s_i) P(\bigcap_{i=1}^{m} (Y_i > t_i))\, dt_m \cdots dt_1\, ds_n \cdots ds_1$$

$$\geq \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} \prod_{i=1}^{n} P(X_i > s_i) \prod_{i=1}^{m} P(Y_i > t_i)\, dt_m \cdots dt_1\, ds_n \cdots ds_1 .$$

The first inequality follows from assumption (i). The second and the third inequalities follow from assumptions (ii) and (iii) together with Lemma 3.2. Thus $(\underline{X}, \underline{Y})$ is WPUOD1. Similarly, $(\underline{X}, \underline{Y})$ is WPUOD2. Hence $(\underline{X}, \underline{Y})$ is WPUOD. (b). Since $(\underline{X}, \underline{Y})$ is WPUOD by the property (P_1) of WPUOD $\underline{X}$ is also WPUOD. The proof is complete.

Theorem 3.5 Let $\underline{X} = (X_1, \cdots, X_n)$ be WPUOD and Z be independent of X_i . Define $Y_i = a_i X_i + Z$, $a_i > 0$. Then $\underline{Y} = (Y_1, \cdots, Y_n)$ is WPUOD.

Proof Since (i) $\underline{Y} = (Y_1, Y_2, \cdots, Y_n)$, given Z, is WPUOD by property (P_3), (ii) $Y_i \uparrow$ in Z and (iii) Z is associated $\underline{Y}$ is WPUOD according to Theorem 3.4.

Definition 3.6(Ebrahimi, Ghosh, 1981) A random vector $\underline{Y}$ is said to be stochastically right tail increasing in the random vector $\underline{X}$ if $E[f(\underline{Y}) | \underline{X} > \underline{x}]$ is increasing on $\underline{x}$ for every real valued increasing function f .

Theorem 3.7 Assume (i) $\underline{X} = (X_1, \cdots, X_n)$ is WPUOD, (ii) $\underline{Y} = (Y_1, \cdots, Y_m)$ is conditionally independent given $\underline{X}$, (iii) Y_j is stochastically right tail increasing in $\underline{X}$ for all $j = 1, \cdots, m$. Then (a) $(\underline{X}, \underline{Y})$ is WPUOD, (b) $\underline{Y}$ is WPUOD.

Proof (a) First we show the WPUOD1 case :

$$\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} P(\bigcap_{i=1}^{n} (X_i > s_i), \bigcap_{i=1}^{m} (Y_i > t_i)) dt_m \cdots dt_1 ds_n \cdots ds_1$$
$$= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} P(\bigcap_{i=1}^{m} (Y_i > t_i) | \bigcap_{i=1}^{n} (X_i > s_i)) P(\bigcap_{i=1}^{n} (X_i > s_i)) dt_m \cdots dt_1 ds_n \cdots ds_1$$
$$= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} \prod_{i=1}^{m} P((Y_i > t_i) | \bigcap_{i=1}^{n} (X_i > s_i)) P(\bigcap_{i=1}^{n} (X_i > s_i)) dt_m \cdots dt_1 ds_n \cdots ds_1$$
$$\geq \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} \prod_{i=1}^{n} P(X_i > s_i) \prod_{i=1}^{m} P(Y_i > t_i) dt_m \cdots dt_1 ds_n \cdots ds_1.$$

The second equality follows from assumption (ii) and the inequality follows from assumptions (i) and (iii). Similarly, the WPUOD2 case can be proved. (b) follows from (a) immediately by property (P_1).

From the property (P_2) and Theorem 3.7, Corollary 3.8 follows immediately.

Corollary 3.8 Let $(X_1, \cdots, X_n)$ be conditionally independent given a scalar random variable Y and X_j be stochastically right tail increasing in Y for all $j = 1, \cdots, n$. Then $(X_1,$

$\cdots, X_n)$ is WPUOD.

4. Some Preservation Results with Application

Using Theorem 2.6 we partially extend Theorem 5 of Alzaid(1990).

Theorem 4.1 Let $(X_{11}, \cdots, X_{1p}), \cdots, (X_{n1}, \cdots, X_{np})$ be independent vectors. Let f_i be a function of n variables and let $Y_1 = f_1(X_{11}, \cdots, X_{n1}), \cdots, Y_p = f_p(X_{1p}, \cdots, X_{np})$.
Assume that for each i, $(X_{i1}, \cdots, X_{ip})$ is WPUOD1 and $f_1, \cdots, f_p$ are nonnegative increasing convex functions for the ith coordinate, $i = 1, \cdots, n$ then $(Y_1, \cdots, Y_p)$ is WPUOD1.

Proof Define for $k = 2, \cdots, n$, $j = 1, \cdots, p$,

$$h_j^{(k)}(X_{(k+1)j}, \cdots, X_{nj}) = E\{h_j^{(k-1)}(X_{kj}, \cdots, X_{nj}) \mid X_{(k+1)j}, \cdots, X_{nj}\}. \quad (4.1)$$

We also define $h_j^{(1)}(X_{2j}, \cdots, X_{nj}) = E\{h_j(X_{1j}, \cdots, X_{nj}) \mid X_{2j}, \cdots, X_{nj}\}$ for any function $h_j(X_{1j}, \cdots, X_{nj})$ having property of f_j. Then we obtain

$$Eh_j(X_{1j}, \cdots, X_{nj}) = Eh_j^{(1)}(X_{2j}, \cdots, X_{nj}) = \cdots = Eh_j^{(n-1)}(X_{nj}). \quad (4.2)$$

In view of Theorem 2.6 it is sufficient to prove that for any functions $h_1, \cdots, h_p$ having properties of $f_1, \cdots, f_p$, respectively,

$$E\Big[\prod_{j=1}^{p}(h_j(X_{1j}, \cdots, X_{nj}))\Big] \geq \prod_{j=1}^{p} E(h_j(X_{1j}, \cdots, X_{nj})). \quad (4.3)$$

This is so since for any nonnegative increasing convex functions $k_1, \cdots, k_p$ the functions $k_1 f_1, \cdots, k_p f_p$ have the same properties as do $f_1, \cdots, f_p$. To show that (4.3) is valid, we follow an iteration argument.

$$E\Big[\prod_{j=1}^{p} h_j(X_{1j}, \cdots, X_{nj})\Big] = E\Big[E\Big\{\prod_{j=1}^{p} h_j(X_{1j}, \cdots, X_{nj}) \mid X_{2j}, \cdots, X_{nj}\Big\}\Big]$$

$$\geq E\Big[\prod_{j=1}^{p} E\{h_j(X_{1j}, \cdots, X_{nj}) \mid X_{2j}, \cdots, X_{nj}\}\Big] = E\Big[\prod_{j=1}^{p} h_j^{(1)}(X_{2j}, \cdots, X_{nj})\Big]$$

(by proceeding with the interation argument used above)

$$= E\Big[\prod_{j=1}^{p} h_j^{(n-1)}(X_{nj})\Big] \geq \prod_{j=1}^{p} Eh_j^{(n-1)}(X_{nj}) = \prod_{j=1}^{p} Eh_j(X_{1j}, \cdots, X_{nj}).$$

Since $(X_{11},\cdots,X_{1p})$ is WPUOD1 and h_j's are nonnegative increasing convex functions the

first above inequality holds. From the facts that $(X_{n1},\cdots,X_{np})$ is WPUOD1 and $h_j^{(n-1)}$'s are nonnegative increasing convex functions the second above inequality follows and the last equality follows from (4.2). The proof is complete. A similar result holds for the WPLOD2 property.

The following theorem is an application of Theorem 4.1 which is very important in recognizing WPUOD1 in compound distributions which arise naturally in stochastic processes.

Theorem 4.2 Let $(N_1,\cdots,N_p)$ be a p-variate variable with components assuming values in the set $\{1,2,\cdots\}$ and let $\{(X_{i1},\cdots,X_{ip}): i\geq 1\}$ be a sequence of nonnegative independent p-variate random variables independent of $(N_1,\cdots,N_p)$. Suppose that $(N_1,\cdots,N_p)$ is WPUOD1 and that $(X_{i1},\cdots,X_{ip})$ is WPUOD1. Define $\underline{Y}=(Y_1,\cdots,Y_p)$ by $Y_1=\sum_{j=1}^{N_1}X_{j1}$, $\cdots, Y_p=\sum_{j=1}^{N_p}X_{jp}$. Then $\underline{Y}=(Y_1,\cdots,Y_p)$ is WPUOD1.

Proof Let $f_1,\cdots,f_p$ be nonnegative increasing convex functions. Then

$$E[\prod_{i=1}^{p}f_i(Y_i)]=E[E\prod_{i=1}^{p}\{f_i(\sum_{j=1}^{N_i}X_{ji})|N_i=n_i\}]=E[E\{\prod_{i=1}^{p}f_i(\sum_{j=1}^{n_i}X_{ji})\}]$$
$$\geq E\prod_{i=1}^{p}E\{f_i(\sum_{j=1}^{n_i}X_{ji})\}\geq\prod_{i=1}^{p}E[E\{f_i(\sum_{j=1}^{n_i}X_{ji})\}]=\prod_{i=1}^{p}Ef_i(Y_i).$$

The first inequality follows from Theorem 4.1 and assumption and the second inequality follows from the fact that $Ef_i(\sum_{j=1}^{n_i}X_{ji})$'s are increasing convex functions in n_i. This completes the proof.

Example 4.3. Let $\{N_1(t),\cdots,N_p(t): t\geq 0\}$ be the p-variate Poisson processes, i.e. $N_1(t)=Z_1(t)+W(t),\cdots,N_p(t)=Z_p(t)+W(t)$ where $Z_1(t),\cdots,Z_p(t)$, and $W(t)$ are independent Poisson processes. Let $\{(X_{n1},\cdots,X_{np}): n=0,1,2,\cdots\}$ be a sequence of independent and identically distributed random variables. Define the p-variate compound Poisson process $\{Y_1(t),\cdots,Y_p(t): t\geq 0\}$ by

$$Y_1(t) = \sum_{n=0}^{N_1(t)} X_{n1}, \cdots, Y_p(t) = \sum_{n=0}^{N_p(t)} X_{np}.$$

Since $\{N_1(t), \cdots, N_p(t)\}$ is WPUOD1 for every $t \geq 0$, consequently an application of Theorem 4.2 implies $(Y_1(t), \cdots, Y_p(t))$ is WPUOD1 for every $t \geq 0$ whenever $(X_{n1}, \cdots, X_{np})$ is WPUOD1.

Theorem 4.4 Let $\{ \underset{\sim}{X}_n = (X_{1n}, \cdots, X_{pn}) : n \geq 1\}$ be a sequence of WPUOD(WPLOD) random vectors with distributions H_n such that $H_n \rightarrow_n H$ where H is the distribution of $\underset{\sim}{X} = (X_1, \cdots, X_p)$. Then $\underset{\sim}{X}$ is WPUOD(WPLOD).

Proof We will only show the WPUOD case. For any real $x_1, \cdots, x_p$,

$$\int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} P(X_1 > s_1, \cdots, X_p > s_p) ds_p \cdots ds_1$$
$$= \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} \lim_{n\to\infty} P(X_{1n} > s_1, \cdots, X_{pn} > s_p) ds_p \cdots ds_1$$

$$\geq \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} \lim_{n\to\infty} \prod_{i=1}^{p} P(X_{in} > s_i) ds_p \cdots ds_1$$
$$= \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} \prod_{i=1}^{p} P(X_i > s_i) ds_p \cdots ds_1.$$

Thus $\underset{\sim}{X}$ is WPUOD1. Similarly, $\underset{\sim}{X}$ is WPUOD2. The proof is complete.

Theorem 4.5 Let H_1 and H_2 be WPUOD distributions both having the same one dimensional marginals. Define $\overline{H_a} = a\overline{H_1} + (1-a)\overline{H_2}$, $a \in (0,1)$ where $\overline{H}_a(x_1, \cdots, x_n) = P_{H_a}(X_1 > x_1, \cdots, X_n > x_n)$, $\overline{H}_i(x_1, \cdots, x_n) = P_{H_i}(X_1 > x_1, \cdots, X_n > x_n)$, $i = 1, 2$. Then H_a is WPUOD.

Proof By definition, the one dimensional marginals of H_a are the same as those of H_1 or H_2. Since

$$\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \overline{H}_a ds_n \cdots ds_1 = \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} a\overline{H}_1 + (1-a)\overline{H}_2 ds_n \cdots ds_1$$
$$\geq \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} [a \prod_{i=1}^{n} P_{H_1}(X_i > s_i) + (1-a) \prod_{i=1}^{n} P_{H_2}(X_i > s_i)] ds_n \cdots ds_1$$

H_a is WPUOD1. Similarly, we can prove that H_a is WPUOD2. The proof is complete.

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References

[1] Ahmed, A. N., Langberg, N. A., Leon, R. V., and Proschan, F.(1978). *Two concepts of positive dependence, with applications in multivarite analysis*, Statistics Report M486 AFOSR Technical Report No. 78-6 The Florida State University.

[2] Alzaid, A. A.(1990). A weak quadrant dependence concept with applications, *Communications in Statistics - Stochastic Models,* 6(2), 353-363.

[3] Barlow, R. E. and Proschan, F.(1981). *Statistical Theory of Reliability and Life Testing : Probability Models*, To Begin with, Silver Spring, Maryland.

[4] Ebrahimi, N and Ghosh, M.(1981). Multivariate negative dependence., *Communications in Statistics A* 10, 307-337.

[5] Esary, J., Proschan, F. and Walkup, D.(1967). Association of random variables, with applications, *Annals of Mathematical Statistics* 38, 1466-1474.

[6] Lehmann, E. L.(1966). Some concepts of dependence, *Annal of Mathematical Statistics* 37, 1137-1153.

[7] Shaked, M. and Shanthikumar, G. T. (1994). *Stochastic Orders and Their Applications,* Academic Press, Inc.

[8] Tong, Y.L.(1980). *Probability inequalities in Multivariate Distributions,* Academic Press, New York.