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Notes on Locally Convex Fuzzy Topological Vector Spaces

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ABSTRACT

The main goal of this paper is to investigate some properties of the locally convex fuzzy topological vector space.

1. Introduction

The concept of fuzzy topologies was defined by Chang [1]. It was extended to that of a fuzzy vector space and fuzzy topological vector spaces by Katsaras and Liu [2]. Also, in [3] Katsaras introduced the fuzzy topological vector space by using the fuzzy topology which Lowen defined firstly in [7]. One of the main reasons to use the new definition defined by Lowen was to make sure that the fuzzy topological space is translation invariant. Another change made in [3] was to consider on the corresponding scalar field $\mathbb{K}$ the fuzzy usual topology consisting of all lower semi-continuous functions from $\mathbb{K}$ into the unit interval.

Our goal of this paper is to investigate some of properties of the locally convex fuzzy topological vector space and related topics. The main result is: if (X, τ) is a locally convex topological vector space, then $(X, \omega(\tau))$ is a locally convex fuzzy topological vector space and $\omega(B_P)=\{f\in I^X \mid f(0)>0,\ f$ is lower semi-continuous, convex and balanced$\}$ is a local base at zero for the fuzzy topology $\omega(\tau)$ on X.

2. Preliminaries

In this section, we explain some basic definitions and results from [2], [4], and [7] for reference purposes.

Let X be a non empty set. A fuzzy set in X is an element of the set I^X of all functions from X into the unit interval I. χ_A denotes the characteristic function of the set A. If f is a function from X into Y and $\mu\in I^Y=\{\mu \mid \mu: Y\to[0,1]\}$, then $f^{-1}(\mu)$ is the fuzzy set in X defined by $f^{-1}(\mu)=\mu\circ f$. Also, for $\rho\in I^X$, $f(\rho)$ is the member of I^Y which is defined by

$$f(\rho)(y)=\begin{cases}\vee\{\rho(x)\mid x\in f^{-1}(y)\}, & \text{if } f^{-1}(y)\neq\varnothing\\ 0, & \text{otherwise}.\end{cases}$$

The symbols $\vee$ and $\wedge$ are used for the supremum and infimum of the family respectively.

Definition 2.1 [7]. A subfamily τ of I^X *is said to be a fuzzy topology* on a set X if,

(a) τ contains every constant fuzzy set in X,

(b) if $\mu_1, \mu_2\in\tau$, then $\mu_1\wedge\mu_2\in\tau$,

(c) if for each $\{\mu_i\}_i\subset\tau$, then $\vee_i\mu_i\in\tau$.

A fuzzy topological space is a set X equipped with a fuzzy topology. The elements of τ are called the open fuzzy sets in X.

Definition 2.2 [8]. A fuzzy set μ in X is called a *neighborhood of* x if there exists an open fuzzy set ρ with $\rho\le\mu$, and $\rho(x)=\mu(x)>0$.

Theorem 2.3 [8]. A fuzzy set μ in X is open if and only if μ is a neighborhood of x for each $x\in X$ with $\mu(x)>0$.

Let X be a vector space over $\mathbb{K}$, where $\mathbb{K}$ denotes either the set of all the real or the complex numbers. Let $\mu_1, \mu_2, \cdots, \mu_n\in I^X$. The fuzzy set $\mu=\mu_1\times\mu_2\times\cdots\times\mu_n$ in X^n, is defined by

$$\mu(x_1, x_2, \cdots, x_n)=\{\mu_1(x_1)\wedge\mu_2(x_2)\wedge\cdots\wedge\mu_n(x_n)\}.$$

Definition 2.4 [2]. If $f: X^n\to X$, given by $f(x_1, x_2, \cdots, x_n)=x_1+x_2+\cdots+x_n$, then the fuzzy set $f(\mu)$ in X is called *the sum of the fuzzy sets* $\mu_1, \mu_2, \cdots, \mu_n$, and it is denoted by $\mu_1+\mu_2+\cdots+\mu_n$. That is

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$(\mu_1+\mu_2+\cdots+\mu_n)(x)=\vee\{\mu_1(x_1)\wedge\cdots\wedge\mu_n(x_n)\mid x=x_1+x_2+\cdots+x_n\}$.

Definition 2.5 [2]. For $\mu\in I^X$ and t a scalar, the fuzzy set $t\mu$ is it the image of μ under the map $g: X\to X$, $g(x)=tx$, that is if $\mu\in I^X$ and $t\in\mathbb{K}$, then

$$(t\mu)(x)=\begin{cases}\mu(x/t) & ,\text{ if } t\neq 0\\ 0 & ,\text{ if } t=0 \text{ and } x\neq 0\\ \vee\{\mu(y)\mid y\in X\} & ,\text{ if } t=0 \text{ and } x=0.\end{cases}$$

Definition 2.6 [2]. $\mu\in I^X$ is said to be

(a) *convex* if $t\mu+(1-t)\mu\subseteq\mu$ for each $t\in[0,1]$

(b) *balanced* if $t\mu\subseteq\mu$ for each $t\in\mathbb{K}$ with $|t|\leq 1$

(c) *absolutely convex* if μ is convex and balanced

(d) *absorbing* if $\vee\{t\mu(x)\mid t>0\}=1$ for all $x\in X$.

Definition 2.7 [7]. Let (X, τ) be a topological space and $\omega(\tau)$ the set of all lower semi-continuous maps $f:(X,\tau)\to[0,1]$. Then $\omega(\tau)$ is a fuzzy topology on X. This topology is called *the fuzzy topology generated by* τ on X. The *fuzzy usual topology* on $\mathbb{K}$ is the fuzzy topology generated by the usual topology of $\mathbb{K}$.

Definition 2.8 [3]. A *fuzzy linear topology* on a vector space X over $\mathbb{K}$ is a fuzzy topology on X such that the two mappings

$+: X\times X\to X,\ (x,y)\to x+y$

$\cdot: \mathbb{K}\times X\to X,\ (t,x)\to tx$

are continuous when $\mathbb{K}$ has the fuzzy usual topology and $\mathbb{K}\times X$ and $X\times X$ have the corresponding product fuzzy topologies. A linear space with a fuzzy linear topology is called a *fuzzy linear space* or a *fuzzy topological vector space.*

Definition 2.9 [3]. Let x be a point in a fuzzy topological space X. A family F of neighborhoods of x is called a *base for the system of all neighborhoods* of x if for each neighborhood μ of x and each $0<\theta<\mu(x)$ there exists $\mu_1\in F$ with $\mu_1\leq\mu$ and $\mu_1(x)>\theta$.

Definition 2.10 [4]. A *fuzzy seminorm* on X is a fuzzy set ρ in X which is absolutely convex and absorbing. If in addition $\wedge\{(t\rho)(x)\mid t>0\}=0$ for $x\neq 0$, then ρ is called a *fuzzy norm.*

3. Locally Convex Fuzzy Topological Vector Space

The results presented in this section indicated that many of the properties of the fuzzy neighborhood systems are similar to those of the neighborhood systems in ordinary topological space. The following properties explain the neighborhood system, the zero neighborhood system in a fuzzy topological space.

Theorem 3.1 (Theorem 4.2 in [3]). The system N_0 of all neighborhoods of zero in a fuzzy topological vector space X over $\mathbb{K}$ has the following properties:

(1) every non zero constant fuzzy set in X belongs to N_0 and $\mu(0)>0$ for $\mu\in N_0$,

(2) if $\mu_1,\mu_2\in N_0$, then $\mu_1\wedge\mu_2\in N_0$,

(3) if $\mu\in N_0$, then $t\mu\in N_0$ for each nonzero scalar t,

(4) for each $\mu\in N_0$ there exists $\rho\in N_0$ balanced with $\rho\leq\mu$ and $\rho(0)=\mu(0)$,

(5) let $\mu\in N_0$ and $0<\theta<\mu(0)$. Then, there exists $\mu_1\in N_0$ with $\mu_1+\mu_1\leq\mu$ and $\mu_1(0)>\theta$,

(6) let $\mu\in I^X$ with $\mu(0)>0$. If for each $0<\theta<\mu(0)$, there exists $\rho\in N_0$ with $\rho\leq\mu$ and $\rho(0)>\theta$, then $\mu\in N_0$,

(7) let $\mu\in N_0$ and $x_0\in X$. For each $0<\theta<\mu(0)$, there exists a positive number δ such that $\mu(tx_0)>\theta$ for all $t\in\mathbb{K}$ with $|t|\leq\delta$,

(8) for each $\mu\in N_0$ there exists $\rho\in N_0$, $\rho\leq\mu$ and $\rho(0)=\mu(0)$ such that $-x+\rho\in N_0$ for any $x\in X$ for which $\mu(x)>0$.

Conversely, if N_0 is a family of fuzzy sets in a vector space X over $\mathbb{K}$ satisfying (1) to (8), then there exists a unique fuzzy linear topology τ on X such that N_0 conincides with the family of all neighborhoods of zero.

Theorem 3.2 (Theorem 4.3 in [3]). Let B be a family of balanced fuzzy sets in a vector space X over $\mathbb{K}$. Then B is a base at zero for a fuzzy linear topology on X if and only if B satisfies the following properties.

(1) $\mu(0)>0$ for each $\mu\in B$,

(2) for each non zero constant fuzzy set c in X and $0<\theta<c$ there exists $\mu\in B$ with $\mu\leq c$ and $\mu(0)>\theta$,

(3) if $\mu_1,\mu_2\in B$ and $0<\theta<\{\mu_1(0)\wedge\mu_2(0)\}$, then there exists $\mu\in B$ with $\mu\leq\mu_1\wedge\mu_2$ and $\mu(0)>\theta$,

(4) if $\mu\in B$ and t a non zero scalar, then for each $0<\theta<\mu(0)$ there exists $\mu_1\in B$ with $\mu_1\leq t\mu$ and $\mu_1(0)>\theta$,

(5) let $\mu\in B$ and $0<\theta<\mu(0)$. Then, there exists $\mu_1\in B$ such that $\mu_1(0)>\theta$ and $\mu_1+\mu_1\leq\mu$,

(6) let $\mu\in B$ and $x_0\in X$. If $0<\theta<\mu(0)$, then there exists a positive number δ such that $\mu(tx_0)>\theta$ for all

scalars t with $|t| \leq \delta$,

(7) for each $\mu \in B$ there exists a fuzzy set ρ in X with $\rho \leq \mu$, $\rho(0)=\mu(0)$ and such that for each $x \in X$ for which $\rho(x)>0$ and each $0<\theta<\rho(x)$ there exists $\mu_1 \in B$ with $\mu_1 \leq -x+\rho$ and $\mu_1(0)>\theta$.

Theorem 3.3 (Theorem 3.4 in [5]). Suppose that P is a seminorm on X and τ the induced topology. Let B_P be the local base at zero of τ consisting of open balls centred at zero. Define $\omega(B_P)=\{f \in I^X | f(0)>0$, f is lower semi-continuous, convex and balanced$\}$. Then $\omega(B_P)$ is the local base at zero for some fuzzy vector topology on X.

The followings explain that if (X, τ) is a locally convex topological vector space then $(X, \omega(\tau))$, where $\omega(\tau)$ is the fuzzy topology generated by τ on X, is a locally convex fuzzy topological vector space and $\omega(\tau)$ has the local base at zero. From these results, the local convexity may be concretely applied to the field relative to the generated fuzzy linear topology.

Definition 3.4 [3]. A fuzzy topological vector space X is called *locally convex* if it has a base at zero consisting of convex fuzzy sets.

Theorem 3.5 (Theorem 6.4 in [4]). If X is a locally convex fuzzy linear space, then X has a base at zero consisting of absolutely convex fuzzy sets.

Theorem 3.6 (Theorem 6.5 in [4]). Let $\{\rho_\alpha | \alpha \in A\}$ be a family of fuzzy seminorms on X. For each finite subset S of A, let $\rho_S = \wedge \{\rho_\alpha | \alpha \in S\}$. Then the family $B=\{\theta \wedge (t\rho_S) | 0<\theta \leq 1,\ S \subset A$ finite, $t>0\}$, is a base at zero for a locally convex fuzzy linear topology τ. Moreover τ is the weakest of all fuzzy linear topologies on X which are finer than each τ_{ρ_α}.

Theorem 3.7. Let (X, τ) be a locally convex topological vector space. Then $(X, \omega(\tau))$ is a locally convex fuzzy topological vector space.

Proof. Since τ is locally convex there exists a separating family of seminorms $P=\{P_\alpha | \alpha \in A\}$ which induced τ. Associate to each P_α and to each $\varepsilon>0$, the set $V_{\alpha,\varepsilon}=\{x \in X | P_\alpha(x)<\varepsilon\}$. Then for each $P_\alpha \in P$, χ_{V_α} is a fuzzy seminorm, where V_α stands for $V_{\alpha,1}$. By Theorem 3.6 the family consisting of absolutely convex sets $B=\{\theta \wedge (t\chi_{V_S}) | 0<\theta \leq 1,\ S \subset A$ finite, $t>0\}$ is a base at zero for a fuzzy linear topology, where $V_S = \cap \{V_\alpha | \alpha \in S\}$.

Now, we will show that B is a base of $\omega(\tau)$. Let $\mu \in \omega(\tau)$ with $\mu(0)>0$ and $\theta \in (0, \mu(0))$. If $\alpha \in (\theta, \mu(0))$, the set $\{x \in X | \mu(x)>\alpha\}$ is a τ-open set, whence there exist $t^{-1}>0$ and a finite subset S of A such that $t^{-1}V_S \subset \{x \in X | \mu(x)>\alpha\}$, and so $\alpha \wedge t\chi_{V_S} \leq \mu$. Therefore, $(X, \omega(\tau))$ is a locally convex fuzzy topological vector space.

Since the quotient space $(X/W, \tau_W)$ of a locally convex topological vector space (X, τ) is a locally convex topological vector space where W is a closed subspace of X, we get the following corollary.

Corollary 3.8. Let (X, τ) be a locally convex topological vector space and W a closed subspace of X. Then $(X/W, \omega(\tau_W))$ is also a locally convex fuzzy topological vector space where τ_W is the quotient topology of τ.

Now we prove the main theorem of this paper. It gives the concrete form of the local base of the generated fuzzy linear topology by a locally convex topological vector space.

Theorem 3.9. Let (X, τ) be a locally convex topological vector space. Then $\omega(B_P)=\{f \in I^X | f(0)>0$, f is lower semi-continuous, convex and balanced$\}$ is a local base at zero for the fuzzy topology $\omega(\tau)$ on X.

Proof. Since τ is locally convex there exists a separating family of semi-norms $P=\{P_\alpha | \alpha \in A\}$ which induced τ. To prove this theorem, it is sufficient to show that satisfies the conditions in Theorem 3.2. (1) to (5) are almost trivial.

To prove (6), let $\mu \in \omega(B_P)$, $x_0 \in X$ and $\theta \in (0, \mu(0))$. From the continuity of the scalar multiplication g and the lower semi-continuity of μ, $\mu \circ g$ is lower semi-continuous at $(0, x_0)$. Hence for any $\varepsilon>0$ there exists $\delta>0$ and $\alpha_1, \cdots, \alpha_n \in A$ such that for each (t, x) with $|t|<\delta$ and

$$x \in \bigcap_{i=1}^{n} V_{\alpha_i,\delta},\ (\mu \circ g)(t, x) > (\mu \circ g)(0, x_0) - \varepsilon.$$

Setting $\varepsilon=\mu(0)-\theta$ and $x=x_0$. We find that there exists a $\delta>0$ such that $|t|<\delta$ implies $\mu(tx_0)>\theta$.

To prove (7), let $\mu \in \omega(B_P)$ and $\rho=\mu$. Let $x \in X$ be such that $\mu(x)>0$ and $\theta \in (0, \mu(x))$. Put $\theta' = \frac{1}{2}(\theta+\mu(x))$. From the lower semi-continuity of μ, $N=\{x \in X | \mu(x)>\theta'\}$ is an open neighborhood of x and hence $-x+N$ is an open neighborhood of zero. And there exists $\delta>0$, $\alpha_1, \cdots, \alpha_n \in A$ such that

$$V_x = \bigcap_{i=1}^{n} V_{\alpha_i,\delta} \subseteq -x + N.$$

We now set $\mu_1 = \theta' \wedge \chi_{V_x}$, then one can easily show that $\mu_1 \in \omega(B_P)$, $\mu_1(0) > \theta$ and $\mu_1 \leq -x + \mu$. Therefore $\omega(B_P)$ is a local base for some fuzzy vector topology on X.

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